Modern Algebra

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Lecture - 01
Groups: Introduction to abstraction

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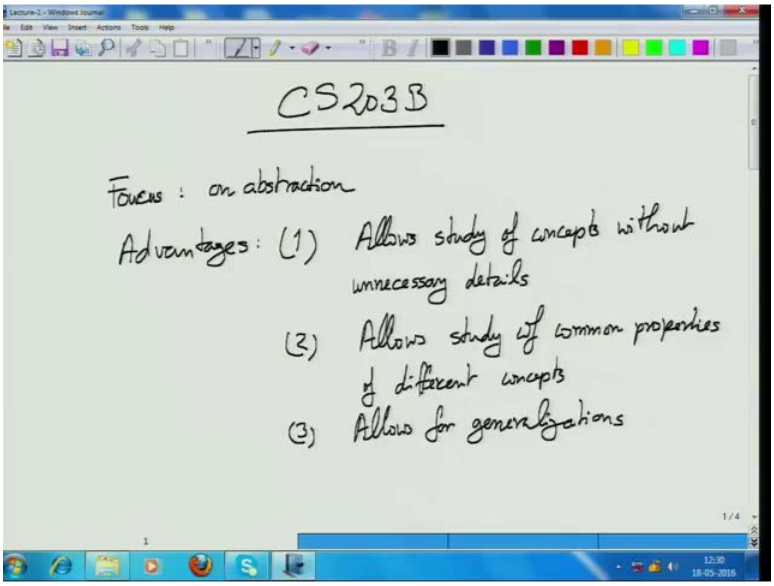

So, this is a course on algebra, the number is 203B. And in this course, we will essentially introduce the algebra again to you. I am sure you have all done algebra in some form of other used algebra in many forms actually. But what I am going to talk about is the modern version of algebra, the way it is used to understand and study different mathematical concepts. And that is something that perhaps many of you have not seen may be partly seen, but not seen as a development from the basics to the higher level and that is what we will do in this course.

The focus will be on abstractions that how do we look at various concepts abstract out the properties of those concepts, and then study properties in the abstract level. What would be the advantage? That we will just write down. The advantages of abstraction, first have the advantage of abstraction is that it allows us to that is very clear that it allows us to study concept without unnecessary details it is get rid of the specific details that are not relevant, and you can just look at it only at the core aspects.

And the second one also it is extremely important is that it allows us to study common properties of different concepts. You have multiple concepts, you abstract their properties out some of there will be common. (Refer Time: 03:13) once you want to study the common property that is applicable across the concepts.

We will see the examples of these, I am sure you have come across examples earlier as well. And third one, which is again very important, is that it allows us to take out the properties that you have studied in abstract, and apply it into a completely different context which is another way of saying that it allows for generalizations. So, we there is a specific concept we abstracted the properties out of study those property. Now this is the abstract world, so we can apply it anywhere that we wish to do.

So, in a science what we study would be much more general version of what that specific thing we started with from that is three very big advantages of abstraction. Of course, there is down side of it also, which is that it becomes perhaps a little boring at rather there are the two abstract the two mathematical. There is the intuition gets lost many times and that you see them you know lot of symbols and manipulation of those symbol that it is not be a why exactly are we doing it. So, it is therefore, it is very important to keep in mind the concrete concepts from which we have abstracted out, so that we never lose sight of what is the effect of our abstract manipulations on those concrete concepts. Any questions?

Well, let us see some example abstractions and these are something that you have already, come across. What are the two most fundamental objects in mathematics, numbers, yes numbers are one, clearly. Operators of that is that you talking of the that is abstraction of when you want to study certain operations and then you use those, but I am saying not yet going into the abstract world in the concrete setting in mathematics, what are the two most fundamental objects it will we like to study. One is numbers. Second?

Student: (Refer Time: 06:00).

Properties of various things sure but object concrete object that you want to study. You have studied it for long in school.

Student: Variables.

Variables are again an abstraction, use variables to study some concrete things. What, geometric objects, curves, planes and so for various surfaces that is also come from the real life, just like numbers in the sense come from real life by count, you want to counts certain thing. So, you introduce a new number. Similarly, in real life the whole world is you know what just the whole collection of geometric objects and you would like to study how these geometric objects behave, what are their properties.

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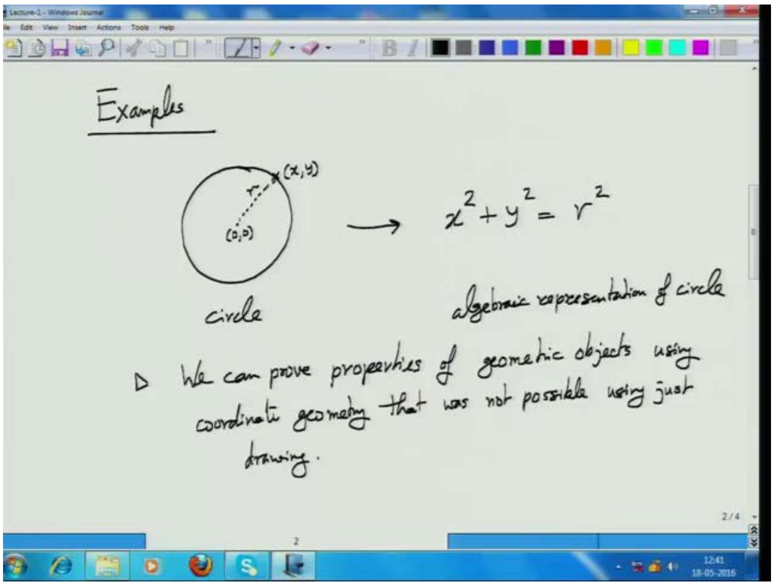

So, with this in my, that is how the mathematics starts in the ancient times; the study was done for numbers or an about numbers and about the geometric objects. And already the abstraction particularly in the case of geometric objects was found extremely you useful. Take for example, a circle as a geometric object, now if I want to study properties of the circle; one way is to draw the circle and see now geometrically various properties by drawing whatever other curves and intersections, you know another way something there been using in the school very.

Yes, so use compass in that is geometric way of studying circle and you draw this circle lines and then see how they interact, learning geometry yes you can that is what written in the school. The all this basic curves you studied using coordinate geometry and what is coordinate geometry give is, for a circle instead of drawing a circle we write this equation x square plus y square equal to r square and this represents a circle. So now, we introduce coordinate here which is center is at point zero and there are any point of the circle is at coordinate are x y. So, the radius of the circle is r and then this circle is represented by this equation x square plus y square is equal to r square.

And then suddenly we can study the circle by using this equation you can differentiate it get the tangent equation of the tangent at various points you can look at multiple two circles and see whether they intersect you can do all that you would do by drawing it pictorially, you can do it algebraically. Now this is already in abstraction, because on its own this equation is just a collection of symbols there is a x, there is a two on top of x, then there is a plus sign, then there is y two on top of y and then there is a (Refer Time: 10:19). What we do is we assign meaning to this and the meaning we assign is that the x represents the x coordinate value of a point on circle, the y represents the corresponding y coordinate of that point and in this is a square x we its top is again a representation of multiplying x with x itself and so on.

So, what we did was that we use this abstraction to represent the circle with an equation and in such a way that all the properties of that circle are inherent in this equation and then we study this equation. And then you as you would recall that coordinate geometry is extremely useful in deriving properties of geometric objects things that you would not be able to prove using just the pictures. You can prove using this abstraction, can you remember or give me an example of a property that you can some nontrivial property that you could prove using the coordinate geometry and then through a hyperbola does not intersect what.

Student: (Refer Time: 12:01).

The curve ok, I will believe you, I do not recall this property, but he saying, so that tangent to hyperbola does not intersect the curve at any other point in it.

Student: (Refer Time: 12:15).

Yeah, the two parts of the hyperbola, and they inter intersect fine that you can prove using this. But if you want to prove it using geometry how would you do it you can put draw of the hyperbola and draw a tangent at one point and observe that it does not intersect, but that does not mean that for no hyperbola for any point on the hyperbola the tangent will not intersect the curve. So, not only you get an easier way of studying property you get a more powerful way of studying properties by applying this abstraction. So, that is the observation that we can prove. So, that is very important observation and that already demonstrate this power of abstraction.

Now, later on in this course, I will show you further abstraction of the same. There is you have a circle we abstracted it out as equation and then I will go one more step. This equation and is study of this properties comes under coordinate geometry I will go one more step and abstract data out even more and that that field is called algebraic geometry. So, we use more algebra to abstract out the circle and similarly, other curves and use that representation to study properties and it turns out that that representation is even more powerful than the coordinate geometry abstraction. And we can prove further or even more properties using that abstraction then with coordinate geometry.

Let us take on this picture for little more while, here you see that equation x square plus y square is equal to r square this seems like an equation over numbers. Now there is addition there is multiplication then here and addition multiplication is what we do for numbers. So, that is a curious things that we have that exist here. Firstly, there is this geometric object that is been seemingly translated into numbers. If I that is what the name coordinate geometry represents that you assign coordinates and coordinates are nothing, but collection of numbers to every point and therefore, represent a geometric object as a collection of points and or collection of numbers and then you see properties or relationship between those numbers that is given by this equation.

Second curious thing about this is that not everything in this is a number. In fact, almost nothing is a number x is not a number, y is not a number, r is not a number, numbers are what; one, two, three, four, one point one. If you want to allow real square root two those are numbers is x, a number y, r they are symbols they are not numbers. So, how can we treat them as a numbers, we have seen to it treat them as a numbers we are multiply x it itself adding it to y r y square. So, how can we treat them as a number, something is not right, with this are we taking too many liberties with our notations.

Student: There is some abstraction.

There is an abstraction that is happening here is yes, but we need to be very careful when we do abstract in precisely defining the meaning of every symbol that we use in the abstraction. So, when we say that x, y and r, here are going to be thought of is numbers is that justified.

Student: Set of numbers.

If any one of, so we are at liberty when we are doing abstraction we had at liberty to assign a meaning to any symbol that we introduce. What we need to ensure is that that meaning is consistent with the operations we carry out. So, if you say x represents say set of numbers collection of numbers, then what is this x square root in what way the collection of numbers of x square related to collection number is x.

And we say that this collection is multiply every pick number in from one collection to every number in other collection and then that overall number is an x are they collected is that what we say. That is not what we intent to mean anyways is you remember doing (Refer Time: 18:58) is there is some num x y represents is specifically one coordinate point it was a many coordinate point. But take one coordinate point then that is a number and then you take a square of that number in x square and y square takes the square over y coordinate.

The thing is that x and y do not represent a single number of they do not stand for a single number they stand for a collection number that is correct. But that collection of number is also not any collection x represents the numbers that correspond to x coordinates of points lying on the circle. So, what is the meaning that we assign to x (Refer Time: 19:49) here do we say that, either we can say just draw the circle look at the x coordinates of all the points and do a represent to the represented by x here. But that is really a kind of messy, but firstly somebody has to draw the circle then assign the meaning or it will imagine a circle in a mind. Is that what we can say here is it x represents any number, y represents any number such that collectively they satisfy this equation, that x square plus y square is equal to r square does that make sense.

So, x is some number here, y is some other number here they need to satisfy this relationship x square plus y square is equal to r square, r is a fixed number r is not does not change. So, we can take r (Refer Time: 20:52) one itself. Then we say this equation corresponds to collection of all pairs of numbers, which satisfy this for x. If you substitute the first of the pair for y, substitute second of the pair, then this equation must be true, so that is the interpretation we assign to this abstraction. But again it is not yet very let say precise, because I said x can be in any number and y can be any number and such that they together satisfy this equation.

But when I say number what do I mean, do I mean integers, do I mean rational numbers, do I mean real numbers or do I mean complex numbers there are different types of numbers are seen. What would you suggest we should treat x as, what kind of number, real numbers yeah goods. That is a natural way of thinking, because this geometric object this circle we are thinking over a real plane. So, we can therefore, think of x and y as a real numbers that is good. But suppose I want to study the rational numbers lying on the circle, that is points with have rational coordinates, which lie on the circle.

Then we can say that the it is all the rational numbers x and y, which satisfy this equation are the points of interest to us, which mean that we are saying that now no longer think of x and y as real number, but think of them as a rational numbers. The question is here the same just that we are saying assigning a different meaning to x and y and we can change the meaning may be some other time we want to study all complex number satisfying this equation. So, we change the meaning to complex or something else. So, that abstraction remains the same, but the meaning we assign to it changes, which is very good thing, because when if you study the properties of this in abstract domain then no matter what meaning we assign to this eventually those property will continue to be true fine, so that that is a important observation.

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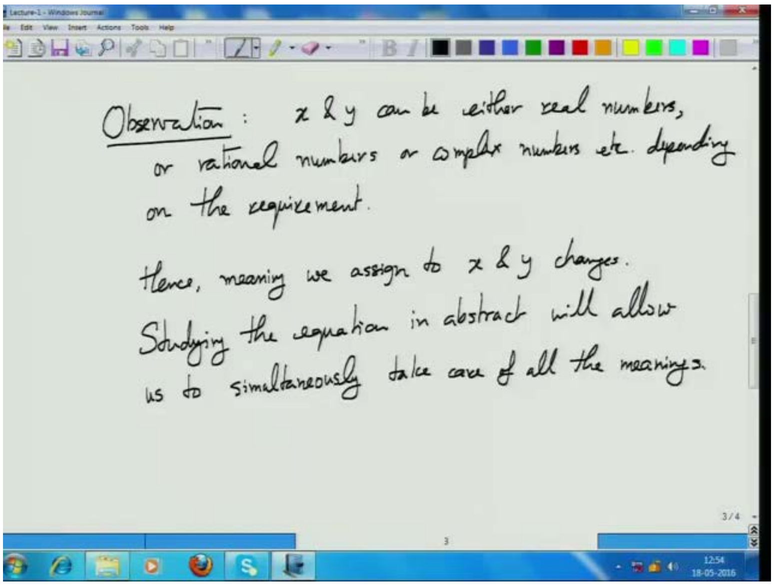

So, we may write it down, this is not hundred percent true because each meaning will have its own (Refer Time: 25:43) also for at least the common properties we can handle simultaneously. So, its mean we would like ideally to study this equation without assigning meaning to x and y that is what you want to do right. I have at least hopefully managed to convince you that that is a good idea. Let us not think of x is a rational number, let us not think of it has real number that we will do later, when we re as and when required. To begin with in a real abstract sense I will not assign any meaning to x and y, are you with me, you agree with me that it is a good idea, because it is a dangerous statement that I making? Because if you do not assign a meaning to x and y, what is it meaning of x times x, how can you add x and y.

Student: (Refer Time: 26:49).

That is one way of doing it, but then you have to do it at least three examples that I give rational reals and complex for each of them you have to then study it separately.

Student: (Refer Time: 27:06).

Yes what, but what if. So, each addition may have slightly different properties. So, then you will have to worry about those peculiarities of each meaning ideally in abstraction we will avoid like to get rid of the specific (Refer Time: 27:32) of the particular use and you just like abstract it out. So, that we can study objects, which are applicable across multiple meanings, which we can this case we would be able to do if we do not assign meaning to x and y. The only hitch there is that and it is a big hitch that then how do we say how do we define addition and multiplication for x for y for a combination of these.

Student: (Refer Time: 28:25).

About.

Student: Addition.

Addition, yes of course that is a solution, yes, you have given it.

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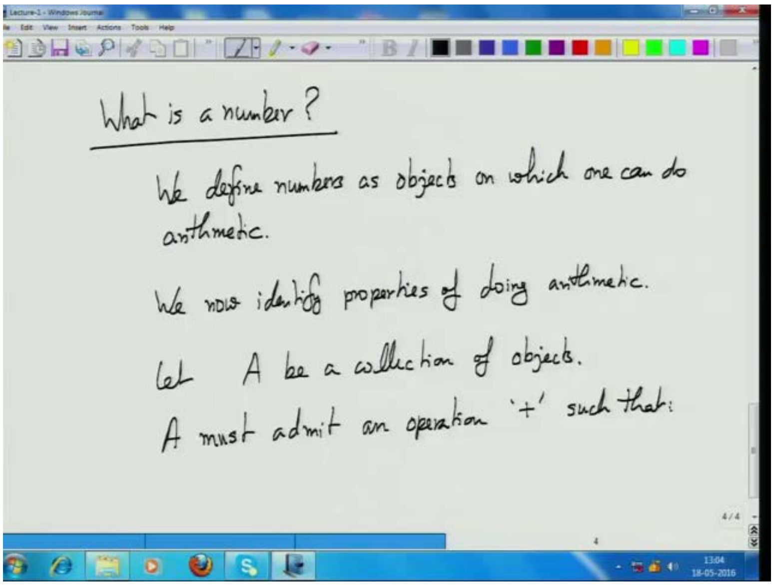

So, let us attack in a more direct fashion, the concept of numbers it is a, what is a number. OK, who can tell what is a number is?

Student: (Refer Time: 29:01).

Any object that is countable that is a number, but there are uncountable real are uncountable, so that is not a good enough definition.

Student: (Refer Time: 29:20).

(Refer Time: 29:24).

Student: (Refer Time: 29:26).

By countable?

Student: (Refer Time: 29:32).

Either power set of count countable (Refer Time: 29:36).

Student: (Refer Time: 29:37).

Ok so, let us may give another argument against this. Let us pick up a collection of countable objects, which is may be collection of all stars in the galaxy are they numbers, yes countable surely are they numbers can you add two stars and get a third star about multiplying two stars that is not a good definition. Let us try to cover with a better definition of a number.

Student: Or it makes a (Refer Time: 30:44).

That is a good start; yes, that is a good start. See what we really want to do with numbers is arithmetic oriented. We have four fundamental arithmetic operations addition, subtraction, multiplication and division. We would like with that is essentially how we play around a number right you do this operations for four numbers. Now of course for certain collection of numbers not all four of these operations are possible. For example, for integers division is not possible addition, subtraction, multiplication is possible, but some others for example, rational numbers division is also possible. So, leaving a side this slight distinction we would like to do arithmetic with numbers.

So, let us define numbers as the collection of objects on which we can do arithmetic, but then we have to now define what does it mean to do arithmetic and you pointed already one important property how doing arithmetic the closure; that is if I add two objects I should get a third object from my collection. Now, we are dealing with the numbers has just at object we are not assigning any other meaning to them. So, we just have we now therefore, have to say what it means to do arithmetic on those objects. So, let us try to write down the properties we want of those objects.

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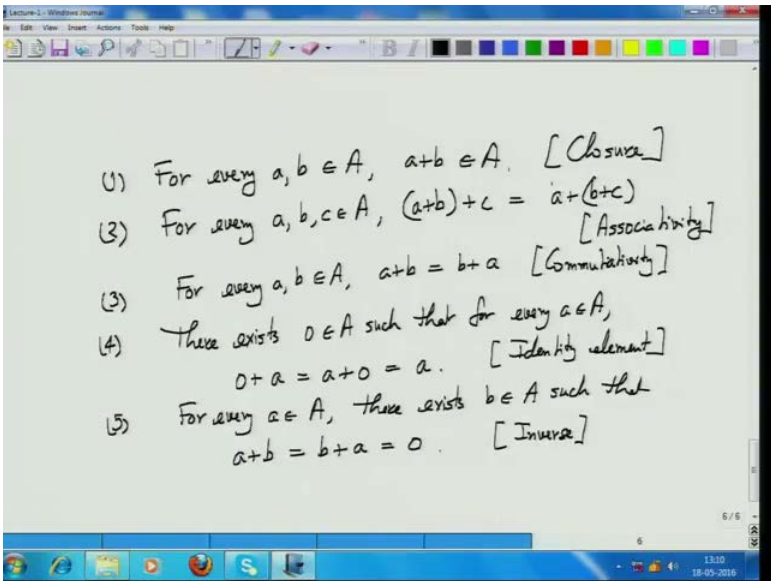

So, this is a formal x definition of the closure property, that capital A is the collection of objects that we are going to think of as numbers. The first requirement is that this collection of objects must admit the addition operation, which I am going to represent with the symbol plus. This satisfies the following properties the first one is the closure property that for any a and b in the collection a plus b must be defined and must be another object in the collection. What other property should addition have?

Student: Associativity.

Associativity, excellent, that for every a, b, c you want to add all three of them. So, it does not matter in which order you add them. Anything else commutative it yes, a plus b is same as b plus a. There are more properties there are required there are required means that does they do whole for typical numbers that we have in mind. So, we should extract those properties out as well one very important of these properties is this very special number 0, it has a property that if you add 0 to any other number the number does not change and the 0 is called the identity of addition.

So, when you add 0 to a, you get itself and finally, that is negative numbers or this property allows you to define subtraction you just addition is not the sufficient to operate a numbers you will also like to do subtraction. Subtraction is simply when you say a minus b we are adding a with minus b and what is minus b minus b is that that number, which you when add to b you get 0. So, the fact that minus b exist is guaranteed by 5th property.

So, we need this property in order to define abstraction in order to define subtraction. In terms of abstraction, we call this property the inverse property the existence of inwards of a real limit. So, these are the 5 properties we require in order to define addition and subtraction. Have I missed out something, then your other rather can you think of some other property that addition and subtraction should satisfy so happens that this is the comprehensive set of properties that the addition operation satisfies for numbers.

Now with this written down we can define numbers to be any collection with an addition operation subtraction comes for free which satisfies all these 5 properties. Of course, this is not quite there because a numbers also have multiplication and division and I am not using that. So, we should also have division and multiplication definition. But for now let us just stay on these properties and I will introduce there were multiplication, division slightly later to have the numbers. And the reason why I am stopping here for now is that this property for a collection is already a very interesting abstraction.

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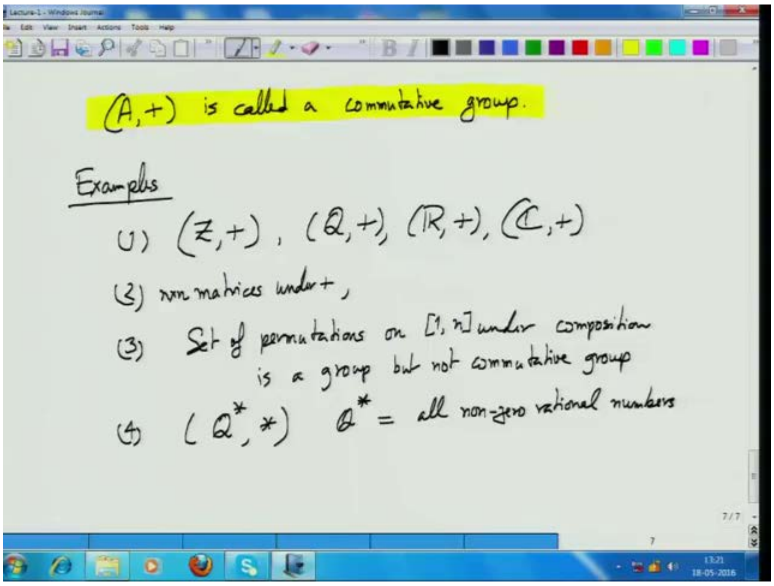

I am sure you probably have seen or this that this collection (Refer Time: 41:38) So, this collection with the addition operation defined as we just did is called a commutative group. The name has its own the history (Refer Time: 41:51) to that why is it called a group, but the key thing or important thing is that groups are in abstraction, which are extremely useful we started with numbers in order to abstract that we to (Refer Time: 42:08) came to groups. But like I initially had suggested that once you do an abstraction and you can apply to completely different domains and that is true with groups as well.

Let me give you an example of course, the obvious examples of groups you already know integers with addition operation that is a group. That is how we started actually not just integers you take and take rational with addition, reals with addition, complex numbers with addition these are all groups for obviously commutative group. That assign by the way I should I have said that that if you drop property three which is the commutatively property in case of numbers we do not need to drop it that this is definitely satisfied by numbers.

But since we have would like to generalize this notion and apply it in other domains also there are situation, where this property does not exist. In that case, we will do away with this property for those situations and if you drop property 3 and the remaining property satisfied by a collection, in that case the collection is called simply a group. So now, come back with some example. So, these are the obvious examples coming from numbers let me give you an examples for groups, which do not come from numbers. You have a suggestion here we come across groups some time earlier.

Student: (Refer Time: 44:10).

Matrices, the excellent example; yes, so let us say n cross n matrices under addition that is a group, we can the identity element is the all 0 metrics you can add it is a commutative group also we can add the all use the other properties are easy to see that they are (Refer Time: 44:43) more examples.

Student: (Refer Time: 44:50).

Permutation, yes, collection of the permutations; let us say look at the permutations on numbers 1 to n each permutation by definition is a mapping from one to n to itself, which is a one, one or two mapping. So, what is the addition operation for the permutations, how do you add to permutations?

Student: (Refer Time: 45:35).

Compose, under composition. So, composition of two permutations is a permutation. So, let us just quickly go through all the property closure is to associativity is true again it is very easy to see. It is not commutative, if we take two permutations, and it matters in which order you apply those permutation 5, 1 composite, 5, 2 is not necessarily equal to 5, 2 compose with 5 1. 5, 1 may map 1 to 2, 5 2 may map 2 to 3. So, if you apply first 5, 1 and then 5, 2 then you will map 1 to 3.

On the other hand, if you apply 5, 2 – first, 5,2 may map 1 to 7; and 5, 1 may map 7 to 20, if you apply first 5, 2 and then 5, 1, then 1 is may goes to 20. So, it is definitely not commutative, but identity exist what is identity for permutation, the identity map 1 to 1, 2 to 2, 3 to 3, because we compose it with any other permutation you get back that same permutation, inverse also exists. Inverse of a permutation is just an inverse map if we compose an inverse map you get the identity permutation.

More examples Q star, which I use to denote all nonzero rational numbers under multiplication operation that is a group, why is that a group closure holds, associative holds, commutative holds identity 1 is the identity, inverse 1 by a. So, a is a rational number, 1 by a is also a rational number and that is the inverse.

So, this is actually a multiplication operation, which is different than addition operation for numbers. That if we view it in that abstract fashion it is same as the addition operation this already a remarkable fact which will not at all evidence and this becomes evident only when we abstracted out the properties of addition, abstract out properties of multiplication see they are the same.

Student: (Refer Time: 49:12).

Multiplication is repeated addition true, but the fact that properties would remain the same is not at all clear. For example, if you start with integers they are also multiplication with repeated addition, but over integers under integers under multiplication do not form a group, because the inverse does not exist, but over rational they do form a group. So, this is a good time to close, because we have defined groups, we have given some examples of groups, and already thrown up certain an unexpected fact.

So, what I would like you to do is go back, think about it. Tomorrow we will meet again at 12, and will continue the discussion.