Discrete-Time Markov Chains and Poisson Processes

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Module: Hitting Times and Strong Markov Property

Lecture 10

Hitting Time II

Welcome to the 10^{th} lecture of the course, Discrete Time Markov Chains and Poisson Processes. Just recall that in the last couple of lectures, we have talked about hitting probability. We define the hitting time.

What we did is that we take a set $A \subseteq S$, S state space and then we define the random variable $T^A = \inf\{n \geq 0 : X_n \in A\}$ and this random variable we called as hitting time to A. And then we have basically understood that this is basically nothing but if I start from outside A, then at what time step for the first time I entered into the set A. The chain enters into the set A or if I am starting inside the set A then this is nothing but 0. We have understood that $T^A = 0$, if starting from A and it is basically nothing but first time after 0 when the Markov chain enters the set A if we are starting outside A. If we are starting from outside A then that was the definition. Then we have seen how we can calculate the probability that $h_i := P_i(T^A < \infty)$. This way we have defined or more formally as we told that we will write it is nothing but hitting probability that hit A i.e $P_i(\text{hit}(A))$. And we have seen that how to calculate these kinds of probabilities at least for two examples, and the basic idea through the theorem was that we can able to write a system of linear equation involving $h_i^{'\text{s}}$ and then we have to solve that system of linear equation and we have to look for the minimal non-negative solution of that system of equations. So, after that we did and then we basically talked about one example, we model the gambling and we generally call it as a gambling model or gambler ruin problem. And we try to find out what is the probability that a gambler who is starting with rupee i go back home empty handed. That probability we try to find out. So, that is basically nothing but what is the probability that hitting 0 starting from i that is basically we try to find out.

Now, we will see another example in this lecture. And the example goes like that, consider the Markov chain which state space S given by these and the transition probability matrix given by these transition probabilities given by this. This is of course, one step transition probabilities. So, what is the probability is given see, $p_{00} = 1$, so the state 0 is basically coming to 0 in one step only. So, once you are in state 0, you cannot come out of the state 0. Whatever the other value?. The other value goes like this for general i except 0 that with probability p_i we move towards left. So, from i to $i+1$, I can go with probability p_i and from from i to $i-1$, I can go with probability q_i. And naturally in this case, we are assuming that $p_i + q_i = 1$ because from i, I can go only to $i+1$ and only to $i-1$ that actually tells us that $p_i + q_i = 1$ or if we know that $p_i + q_i = 1$ that basically means that from i, I can only go to $i+1$ or $i-1$ in one step. Along with we are considering $p_i > 0$ and $p_i < 1$. And so, that basically tells that $0 < p_i < 1$, that basically tells that $0 < q_i < 1$. Now, the thing is that means from i with some positive probability I can go to $i+1$ in one step and that positive probability is p_i and from i, I can go to a $i-1$ in one step with some positive probability and that positive probability is q_i. Similarly, from $i-1$, I can go to i in with probability q_{i+1} and here I have $i+2$ and the probability of moving to this one is p_{i+1}. So, that way the chain process and there is an interesting interpretation of this chain and the interpretation is as follows, this chain can be used to model thus population size of an evolving population, what does this mean?. This means as follows say suppose at some point of time a particular population has i individuals, then this model tells that well from i the population size can increase to $i+1$ in one step or decrease to $i-1$ in one step. So, it will increase if a new baby born in that population and the probability of new born is p_i when the current population size is i. Or it will decrease to $i-1$, so that means one individual will die and then from i, I will go to $i-1$ and the probability of dying is q_i when currently we have i individuals in the population. When currently we have i individuals in the population then probability of die is q_i, probability of born in p_i. And notice that we have written i here. So, this says that probability of born or probability of die may depend on the current size of the population, from i, when I am going to $i+1$ it is p_i, when I am going into $i-1$ it is q_i. Similarly, from $i+1$ when I am moving to $i+2$, it is p_{i+1} and $i+1$ when I am moving to i, it is q_{i+1}. So, the transition probabilities depend on the population size. Just now let us first compare this chain with the gamblers ruin problem. And in case of the gamblers ruin problem, you see that we have the scenario like we have some $i-1$, i, $i+1$ in between and then I have N. So, in both sides basically it was closed and not only that, in case of the gamblers ruin problem that moving from i to $i+1$ or $i+1$ to $i+2$ are same. So, winning probabilities are same in case of the gambler ruin problem but, in this case, this probability depends on the population size. So, in case of the gamblers ruin problem of course, this I have 2 states which are absorbing in both the sides and in between we have some states and in case of the gamblers ruin problem the probability of moving towards right is always same that was p we have taken and moving towards left that we have taken to be q and that end two states are basically observing states. But in this case, we have made some amount of generalization in the sense that, well, this zero is still observing, but in between when we are moving, the probability may depend on what is the current size of the population, what is the current state that the probability depends on that one point is that. And another point is that in this case, we are terminating this chain at the state N or basically maximum value of the state is N, when reach N that basically mean that we achieve our targeted fortune. But in this case, when we are talking about the population that state moves or state can be very very large. So, it can go up to infinity. So, that is basically two changes, we have made over the gamblers ruin problem and when we made that model is called birth and death model, because it can be used to model the population size of an evolving population. In this case, what kind of probabilities we may ask. The probabilities we may ask is that what is the probability that if a population started with i individuals, it will reach the state 0 that means it will extinct. And once the population is extinct, that means no individuals is alive. So, that population will not start again that is one of the assumptions of this model. So, that is why the state 0 is basically returning to 0 with probability 1. Once there is no alive individually in the population, it cannot actually increase the size of the population. But if there is any i, $i > 0$, population is there then in one step it may increase by one individual in one step it may decrease by one individual. So, that natural question now is this one that what is the probability that starting with i individuals, the population becomes extinct. So, starting from here, what is the probability that I hit 0?. And hit 0 means hit 0 for the first time that is it, if I hit 0 in finite time I am done and the population is extinct.

Let us try to solve this. If I try to solve this. Again, I have to write this set of linear equations. And in this case, you will see that h_i is basically as we have denoted by this one. So, in this case, my $A = \{0\}$. So, instead of writing h_i^0, I am just writing h_i in this case. So, using our theorem, if I am already in 0, that means, hitting time is finite, so $h_0 = 1$. So, the probability that hitting time is one where I am starting from 0. So, that is $h_0 = 1$, I have. Now, for any general state what will happen, in one step I can go to $i+1$ or $i-1$. So, h_i can be written as $p_i \times h_{i+1} + q_i \times h_{i-1}$ for $i = 1, 2, 3, \cdots$. So that is what I have written here from i, in one step I can go to $i+1$ or $i-1$. So, going to $i+1$ is p_i, so $p_i \times h_{i+1}$. So as if I am now here, I have to reach 0, I have to find out that probability. Similarly, in another possibility is that from i, I can go to q_i and if I am here, then from here I have to reach 0. So, that contribution comes here. So, that way it proceed and this is true for $i = \{1, 2, 3, \cdots, \}$. Now, the rest of the process is just similar to that of the gamblers ruin problem. We try to write this equation in terms of difference equation. So, we have done exactly that because we have this $p_i + q_i = 1$, what I can write is that I can incorporate here $p_i + q_i$. And then basically that p_i terms we collect in the side and q_i terms we collect in the side and then we have this particular expression. Now, if I call this quantity to be u_i, then this quantity is nothing but u_{i+1}. So, this implies that the same equation I can write as $p_i u_{i+1} = q_i u_i$. This equation I can write in this manner, this is inside the bracket quantities u_i, then basically it is to the inside the bracket quantity is u_{i+1}. So, this particular equation I can write in this particular form.

Exactly, that is what I have written here, taking u_i equals to this and then I can use the recurrence formula to reduce the index i. Now, from here u_i is nothing but $\left(\frac{q_i}{p_i}\right) u_i$. Now, instead of u_i again, I can able to write from this equation itself, if I put $i = i - 1$, what I get is that it turns out to be u_i then $\frac{q_{i-1}}{p_{i-1}} \times u_{i-1}$. So, if I plug in it here, what I get is that $\frac{q_i q_{i-1}}{p_i p_{i-1}} \times u_{i-1}$. Then again u_{i-1} can be written as $\frac{q_{i-2}}{p_{i-2}} u_{i-2}$, and then again, I replace this quantity by this and I can go on this manner still I reach u_1. And then what I get is basically nothing but this product multiplied by u_1. So, now, the thing is that let us call this whole quantity to be γ_i. So, we have this expression. So finally, what we have is that well u_{i+1} can be written as $u_1 \gamma_i$. Now, let us look into this thing that what is $u_1 + u_2 + \cdots + u_i$ that same thing we did in case of the gamblers ruin problem also, we add some equations exactly, that is what I am now going to do. If I add $u_1 + u_2 + \cdots + u_i$, because of this form, this some terms cancels out and finally, I am left with $h_0 - h_i$. So, this is very easy to see because in this case, you see that $u_1 = h_0 - h_1$, $u_2 = h_1 - h_2$ and so on, $u_i = h_{i-1} - h_i$. So, when you add them up basically this h_1 cancels h_1, h_2 cancel in u_3, we have h_2 that cancels with that. And then finally, this h_{i-1} cancels in u_{i-1} there is the h_{i-1} it cancels with that finally, I am left with h_0 here and $-h_i$ here. So, that is what I have written here $h_0 - h_i$ here. That means, basically from here I can write $h_i = h_0 - (u_1 + u_2 \cdots + u_i)$. This way I can able to write. Now, what is the value of h_0, the value of h_0 is given to be 1. So, that means, I just replace this h_0 with 1 here. And then what I did I use this relationship and I write instead of so u_1, I write u_1, u_2, I write $\gamma_1 \times u_1$, then u_3, I write $\gamma_2 \times u_1$, $u_4 = \gamma_3 \times u_1$ and so on so forth. Finally, u_i, I can write $\gamma_{i-1} \times u_1$. So, once I write this thing, I just I can, if I call $\gamma_0 = 0$, then I write that I have γ_0 here. So, $\gamma_0 + \gamma_1 + \gamma_2 + \cdots + \gamma_{i-1}$, I have this sum multiplied by u_1. I have so, basically what we have done we have again write the difference equation and from the difference equation, we get this solution for some u_1. And we know what is the expression of u_1, u_1 is basically $h_0 - h_1$. Now, can I get the value of u_1 to write the explicit solution of h_i, let us discuss that.

To discuss the precise solution of this system of linear equation we reached to this one. Now, suppose what can happen, there are two possibilities now. See here the i can take the value anything like $1, 2, 3, \cdots$, it actually increases. So, i can take any values. And if I have a very large value of i, so, I have a very large summation here. So, that is why we have to look into this summation, $\sum_{i=0}^{\infty} \gamma_i$. Now, this summation there are two possibilities one possibility is this one is infinity and other possibilities this one is less than infinity. Keep in mind this quantity has to be greater than 0 and the reason being that well that γ_i is basically product of some probabilities and then ratio of the product of some probabilities. So, these quantities are all greater than 0. So, product will be greater than 0, and so, division will also be greater than 0. So, each γ_i is greater than 0. So, I have the summation of γ_i is also strictly greater than 0. There is no problem with that. But there is problem with the fact that we that the $\sum_{i=0}^{\infty} \gamma_i$ is finite or it is infinite. What will happen if it is infinite?. If it is infinite then that means that this quantity I can make as large as I want by taking a large value of i, that sum I can make as large as far as I want by taking a large value of i. And if this quantity is very big then this quantity will be negative, $1 - \sum_{i=0}^{\infty} \gamma_i \times u_1$ either will be negative or it will be greater than 1 for any value of u_1 except 0. So, what I am trying to say is that if $u_1 < 0$ and if this quantity is 2 then $h_i > 1$ for very large value of i. And that h_i value is not permissible because h_i finally is a probability. So, h_i has to lies between $[0, 1]$. Similarly, if I take u_1 positive then h_i will be negative for very large i and that is again not permissible. So, what that means can happen that if $\sum_{i=0}^{\infty} \gamma_i < \infty$, then this condition force me to take $u_1 = 0$, u_1 cannot be negative because if u_1 is negative then $h_i > 0$ that is not permissible. If $u_1 > 0$, then $h_i < 0$ that is also not permissible because h_i is a probability it has to lies between $[0, 1]$. So, to infer this one for all values of i, h_i lies between $[0, 1]$, can only choose $u_1 = 0$ if I have that $\sum_{i=0}^{\infty} \gamma_i = \infty$. And that tells that $h_i = 1$ for all i. So, if the sum of this quantity is infinite then $h_i = 1$ with probability i and that says that if sum of these quantities infinite then whatever the population size I start with the population will extinct with time if $\sum_{i=0}^{\infty} \gamma_i = \infty$. Now, what will happen if $\sum_{i=0}^{\infty} \gamma_i < \infty$ that we now need to discuss.

When this $\sum_{i=0}^{\infty} \gamma_i < \infty$, then you see that I can take $u_1 > 0$, of course negative is not possible. The reason being that if it is negative, then this sum is anyway positive if $u_1 < 0$ then this will be greater than 1, because u_1 will be non-negative. So, this can be make positive in this case. So, it will be greater than 1. So, in this particular case u_1 negative implies that $h_i > 1$. So, that is why negative u_1 is not possible. Now, if $u_1 > 0$ that is possible provided that this $h_i \in [0, 1]$. And h_i is always less than 1 because this quantity is positive, $u_1 > 0$, so this part is positive. So, it is 1 minus some positive quantity. So, this h_i will be less than 1, there is no problem if $u_1 > 0$ and this quantity is finite will be less than 1. But the condition I need to satisfy is that h_i strictly greater than or equals to 0. Now, that means that if $\gamma_i < \infty$, I can choose positive u_i, provided this condition is true and this condition is nothing but equivalent to $h_i \geq 0$ that actually give rise this condition. Now, if I simplify it, it basically tells that $u_1 \leq \frac{1}{\sum_{j=0}^{i-1} \gamma_j}$, u_1 is less than equals to this and this is true, has to be true for all $i, i = 1, 2, \cdots,$. That tells me that because it has to be true for all $i, i = 1, 2, \cdots$. So, $u_1 \leq \frac{1}{\sum_{j=0}^{\infty} \gamma_j}$. Because when I add up more values here, this quantity will be small. So, u_i has to be the smaller than the smallest of these quantities for different values of i. And the smallest of this quantity achieved when I take the sum of to infinity because $\gamma_j > 0$. So, u_1 has to be less than that. Now, remember this minimal non-negative solution. Now minimality says that I have to take that h_i which is basically give me the minimal value. And h_i is minimum if u_1 is maximum and u_1 has to be less than equals to this. And so, the maximum value of u_1 permissible in this case is nothing but exactly this quantity, because u_1 is less than equals to this. So, the maximum value possible for u_1 is this one. And when u_1 takes this maximum value, h_i takes his minimum value. So, that is why we can put this minimality condition to have the minimal nonzero solution, we need to take $u_1 = \frac{1}{\sum_{j=0}^{\infty} \gamma_j}$ and this tells that h_i is same as $1 - \frac{(\gamma_0 + \gamma + \cdots + \gamma_{i-1})}{\sum_{j=0}^{\infty} \gamma_j}$. And when you subtract it is basically give me that $h_i = \frac{\sum_{j=i}^{\infty} \gamma_j}{\sum_{j=0}^{\infty} \gamma_j}$. So, finally, we have this particular thing. So, the takeaway from this example again is to that I can solve this kind of problem using a system of linear equation as given by one of the theorems in the last lecture. And then I have to basically solve it and when I am solving it, I have to finally look for minimal non-negative solution of the system of linear equations. And with that I stop here in this particular lecture. Thank you for listening.