Discrete – Time Markov Chain and Poisson Processes
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Lecture 16
Module: Classification of States
Lecture: Transience and Recurrence of Random Walks

Welcome everyone to the sixteenth lecture of the course Discrete Time Markov Chains and Poison Processes. So, I am Subhamay Saha and I will be teaching the remaining part of the course. So, previously you were being taught by Dr. Ayon Ganguly. So, before starting, just let me make a quick recap of what you have learned till now. So, you have seen the definition of Discrete Time Markov Chains. You have seen, what is its transition probability matrix and you have also seen that the evolution of a Discrete Time Markov Chain is completely specified by its initial distribution and the transition probability matrix. Then you saw the concept of communication. So, what is meant by two states communicating? So, i and j are said to communicate, if you can go from i to j in infinite number of steps and you can also go from j to i in finite number of steps. If this is true, you say i and j are communicating. And all states which are communicating among themselves they form a communicating class. And if the Markov chain has a single communicating class, then you say that the Markov chain is irreducible. Then you saw this classification of states namely a transience and recurrence. So, a state i is said to be recurrent, if starting from i, you come back to i, in finite time with probability 1. And it is said to be transient if there is a positive probability that starting from i you never come back to i. So, that was transience and recurrence. Then you saw a few equivalent character like how to prove whether a state is recurrent or transient. One important result was that every recurrent class is closed that means if you start from that class you will remain in that class forever. And when it is a finite state Markov chain, it becomes an if and only if statement that means a class is closed, if and only if, it is recurrent. But, today we will see that this statement is not if and only if, if it is an infinite state Markov chain. So, one way it is always true that if it is recurrent then it is closed, but if it is finite state Markov chain then closed implies recurrent. But, today we will see that if it is an infinite state Markov chain then closed may not imply that it is recurrent.

So, let us look at that example. So, again recall the definition of a simple random walk. So, what is a simple random walk? So, you, basically, start from 0 and then you at every step, you go to either right or left. So, you go right with probability p and you go left with probability q. So, you start from 0 and then either you will go right with probability p or you go left with probability q. Say suppose, you go right, then you are at 1 then again from 1 you will either go right with probability p or left with probability q. So, that is a simple random walk where this $p + q = 1$ and p is assumed to be strictly between 0 and 1. Now, it is very easy to see that this is irreducible, because you can go from any state to any state. It is easily seen from the transition diagram. So, you like, for example, you take any step, any two states i and j you can go from i to j. For example, if j is to the right of i, you just make moves to the right and you reach j. So, again it is very simple to see that any two states in a simple random walk communicate. So, it is it has a single communicating class or in other words this Markov chain is irreducible. So, now so, basically, now we want to check what? This like, the states are recurrent or transient. Now since you have already seen that this is irreducible, and since this transience recurrence are all class properties, so, if I can show that one state is recurrent, then all states are recurrent. And similarly, if I can show that one state is transient, then all states are transient. So, let us examine the state 0. So, now starting from 0 you can return to 0 only in even number of steps. Why is that? That is because like see, starting from 0, if you have to return to 0 then you will have to make equal number of right moves as well as equal number of right and left moves. The number of times you go right this same number of times you have to go left. Or in other words, the number of times you go left you have to make same number of right moves. So, starting from 0 like, for example, let us see, so, if you start from 0 say in first step you go here then you go here. Now if you have to return then, so, you have made two right moves. Now, if you have to return to 0, you will have to make two left moves. So, starting from 0 if you have to return to 0, then in that path, there should be equal number of right moves and as well as left moves. The number of right moves and left moves should be equal. That means the number of steps should be even. Because the number of right steps should be equal to the number of left steps, only then starting from 0 you can return to 0. So, starting from 0 you can return to 0 only in even number of steps. Thus, you have that $p_{00}^{(2n-1)} = 0$. So, starting from 0 you cannot return to 0 in odd number of steps. So, for $n \geq 1$ to $n-1$ are all odd numbers. So, you cannot start. So, $p_{00}^{(2n-1)}$ is the probability that starting from 0 you are at 0 in $2n-1$ steps. But, since you starting from 0, you can be in 0 only after even number of steps. So, $p_{00}^{(2n-1)} = 0$. Now, $p_{00}^{(2n)} = \binom{2n}{n} p^n q^n$. Now, let me explain, why this is true.

So, as I said, now suppose, you start from 0. Now, if you have, so, you go right, say again you go left, again you go right, and you return right. So, the main thing is the number of right steps should be equal to the number of left steps. So, if you have a path which starts from 0 and at $2n$-th steps, so, the path certainly has to be of even length. So, we take $2n$. So, if you have a path of length $2n$ which starts from 0 and at the $2n$-th step you are at 0 that means in that path, the number of right steps is n and number of left steps is also n. Because the number of left steps should be equal to the number of right steps. So, if starting from 0, you in $2n$ th step you are again at 0, that means in that path there are n right steps and n left steps. So, now we want to count how many such paths of length $2n$ is possible. But, how will you count that? For that you, basically, just need to choose in which steps you made a right move, among $2n$ steps, in n steps you will have to make a right move and in n steps you will have to make a left move. But, if you just choose the steps in which you make a right move then in the remaining steps you make a left move right. So, now, from $2n$ steps, you need to just choose n steps in which you make a right move. How do you do that? How do you choose n objects from $2n$ objects? That is precisely $\binom{2n}{n}$. Now, in that there are n write steps and n left steps. Left step you make with the probability p and so, there are n, so, it is p^n. Remember all these are independent. So, n right steps, probability is p^n. And there are also n left steps whose probability isq^n. So, this gives you the probability that you starting from 0 in the $2n$-th step you are at 0. So, this $\binom{2n}{n}$ is, basically, the number of such paths which starts from 0 and is at 0 in the $2n$th steps and then this is this p^n and q^n is, basically, the probability of each such path. So, the total probability is given by this. So, that is why $p_{00}^{(2n)} = \binom{2n}{n} p^n q^n$. And again this is true for all $n \geq 1$. So, $p_{00}^{(2n-1)} = 0$ and $p_{00}^{(2n)} = \binom{2n}{n} p^n q^n$. Now, in order to proceed further we will need a fact which is called Sterling's formula, which tells you what is $n!$. Like it gives you an approximation or an asymptotic of $n!$. It says that $n! \sim \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}$. Now, as $n \to \infty$, now you do not have to actually, no, like let me not go into the detail of, what this tilde means, but, what this, like only thing you need to understand here is, if in a calculation, you have $n!$, then you can replace it by this quantity, which is $\sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}$. So, if you do that the result you get is correct. That is what Sterling's formula tells you roughly. I do not want to gain into the like, what this tilde actually means mathematically, let me not get into that, but for the purpose of this example what you will need is, you can replace $n!$ by $\sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}$.

Now, using this Sterling's formula, we already know that $p_{00}^{(2n)} = \binom{2n}{n} p^n q^n$. But, $\binom{2n}{n}$ is precisely, so, remember what is 2n choose $\binom{2n}{n}$. What is 2n $\binom{2n}{n}$? It is $\frac{(2n)!}{n!n!}$. So, you get $\frac{(2n)!}{(n!)^2}p^n q^n$. Now, I will use this Sterling's formula. So, this calculation again it is just, it is a pretty simple calculation. So, you have to just replace. So, $n!$ you replace by that and $(2n)!$ also you replace Sterling's formula. So, once you just replace that, what you will get and you do that calculation, I am not doing the calculation here. But it is a very simple calculation. So, that will give you that $p_{00}^{(2n)}$ is this. So, what have I done here? I have just replaced $n!$ by that Sterling's formula. Remember, this you have $(2n)!$. So, basically, if you are going back to this, so, here this is the approximation for $n!$. So, if you have to do $(2n)!$, so, instead of n, you have to take $2n$ in the calculation here. So, in when you do the approximation for $(2n)!$ instead of n, you just put $2n$ and in the denominator is just $n!$. So, what I have done here is, I have used Sterling's formula to replace n factorial and then I get this quantity. Now, there can be two cases. Case 1, where $p = q = \frac{1}{2}$. This is what is called the simple symmetric random walk. Now, if $p = q = \frac{1}{2}$, then obviously $4pq = 1$. Now, then

$$\sum_{n=1}^{\infty} p_{00}^{(n)} = \sum_{n=1}^{\infty} p_{00}^{(2n)} \sim \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty,$$

that is because the odd terms are all 0. So, you get only sum over the even terms but now this $p_{00}^{(2n)}$ is given by this quantity.

So, what I get is this, but now we are using this fact from analysis that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty$. This is a very basic fact from calculus that if you take this series in $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty$. So, you have got that $\sum_{n=1}^{\infty} p_{00}^{(n)} = \infty$, we have already seen that is precisely say, so, if that is true then 0 is recurrent. Because you have already seen this that a state i is recurrent if and only if $\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$ and if it is finite then it is transient. So, here what we have shown is that $\sum_{n=1}^{\infty} p_{00}^{(n)} = \infty$. So, 0 is recurrent but since we have already seen that this Markov chain is irreducible, so, and transience recurrence are all class properties, so, every state in a simple symmetric random walk is recurrent. So, if $p = q = \frac{1}{2}$ or in other words if you, if we are looking at a simple symmetric random walk, then all the states are recurrent. Now, we look at the case, $p \neq q$. In that case$4pq < 1$. So, this you can easily check that when $p \neq q$, $4pq < 1$. Therefore, now again we look at the same

$$\sum_{n=1}^{\infty} p_{00}^{(n)} = \sum_{n=1}^{\infty} p_{00}^{(2n)} \sim \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{(4pq)^n}{\sqrt{n}} \leq \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} (4pq)^n < \infty,$$

that is again, because the odd terms are 0, because you cannot return starting from 0, you cannot be at 0 after odd number of steps. Now, again we are using this we know that $p_{00}^{(2n)}$ is given by this. So, $\frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{(4pq)^n}{\sqrt{n}}$. But, now $4pq$ is some quantity say, you can call it, say $4pq$ equals to some r, $4pq = r < 1$. So, you get this summation. But now, from the denominator if i, you see, $\sqrt{n} \geq 1$. So, if I replace $\sqrt{n}$ by 1 in the denominator this will be less than or equal to this, because I am making the denominator small, so, the term becomes large. So, this sum is less than or equal to this sum that is because $\sqrt{n} \geq 1$ for all $n \geq 1$. But now, this $4pq$ as I said, think of it as some r which is a quantity strictly less than 1. So, this is nothing but a geometric series, right. So, $\sum_{n=1}^{\infty} r^n$, where $r < 1$, this is a geometric series and you know that this is finite. So, $\sum_{n=1}^{\infty} p_{00}^{(n)} < \infty$ thus, in the asymmetric case so, if $p \neq q$, in that case, $\sum_{n=1}^{\infty} p_{00}^{(n)} < \infty$ and hence $\{0\}$ is transient. Again, since the Markov chain is irreducible, so, $\{0\}$ is transient, tells you every state is transient. So, an asymmetric simple random walk is in a transient. So, we saw that as a simple symmetric random walk is recurrent, but an asymmetric simple random walk is transient. So, when $p \neq q$, the asymmetric simple random walk is transient. But now, remember, I said that today we are going to see an example which will show that in the infinite setting, it is not true that closed implies recurrent. In finite setting, it is true that closed implies recurrent. Recurrent implies closed in both the situations. But, like recurrent implies closed in whether it is a finite Markov chain or it is an infinite Markov chain, but when it is a finite Markov chain it is an if and only if statement that recurrent if and only if closed. But, this example of asymmetric simple random walk shows you that in infinite setting that is not true. Why is that? Because, we have already seen that this Markov chain is irreducible. So, it has only single class that and that means that class is closed. So, close means starting from there you always remain there, but since there is only one class, so, that has to be closed. So, but this is an infinite class, because the number of states is infinite, it is, basically, the set of all integers, because the simple random for a simple random walk the state space is the set of all integers which is obviously an infinite set. So, here this is an infinite closed class, but we have shown that this is transient. So, this is an example which shows that an infinite closed class need not be recurrent, because if you look at an asymmetric simple random walk then this whole set is a single class. So, all the whole state space is a single class. So, it is obviously closed, but each state is transient. So, when you are looking at an infinite Markov chain, this recurrent implies, sorry, closed implies recurrent is not true. Another thing let me point out in this example. Now you have seen this result that if you have a finite state Markov chain, then there exists at least one recurrent state. So, you have seen this result that if you have a finite state Markov chain, you have at least one recurrent state. But, again look at asymmetric simple random walk. This is an infinite Markov chain and you see that all states are transient. So, again you see that something that is true for finite state Markov chain is not true for infinite state Markov chain. For a finite state Markov chain there has to be at least one recurrent state. But, if you are in, if you are looking at a Markov chain which is has an infinite state space, then it is possible that all states are transient. Or in other words, it does not have a single recurrent state and an example is provided by this asymmetric simple random walk because you have already shown that in an asymmetric simple random walk all states are transient. So, first you have just shown that $\{0\}$ is transient and since it is irreducible and transience is a class property. So, all states are transient. So, an asymmetric simple random walk is transient. So, this is an example of an infinite state Markov chain where all states are transient, which does not have even a single recurrent state. So, again another result which is true for finite state Markov chain but not for infinite state Markov chain namely that in a finite state Markov chain there exist at least one recurrent state, but when you are in an infinite setting there may not exist even a single recurrent state. All states are transient is possible for an infinite state Markov chain but not for a finite state Markov chain. So, this one example gives you, so, serves two purposes. This is an example where you see that an infinite closed class need not be recurrent and also you see through this example you see that an infinite state Markov chain may not have a single recurrent state, okay. So, we will stop here today. Thank you all.