Discrete – Time Markov Chain and Poisson Processes
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Lecture 18
Lecture: Stationary Distribution II

Hello everyone, welcome to the 18th lecture of the course Discrete-time Markov Chains and Poisson Processes. So, in last class we saw the definition of stationary measure, so a stationary measure is nothing but a row vector whose size is same as the size of the state space, its entries are all non-negative and it should satisfy the condition $\pi P = \pi$. Now, when the sum of the entries is equal to 1, we say that a stationary measure is a stationary distribution. And then, we saw these three examples like, in example one, the Markov chain had a unique stationary distribution. In example two, the Markov chain had infinitely many stationary distributions and in example three, the Markov chain had no stationary distribution. The first two examples were for discrete like finite space Markov chains but the third example was infinite space and that is because we also saw this result that for a Markov chain with a finite state space there always exist at least one stationary distribution. So, if you have to give an example of a Markov chain which does not have any stationary distribution then it has to be an infinite state Markov chain. So, today we will continue with some more properties, some more theorems about stationary distributions. So, let us start.

So, the first theorem which says that let $\{X_n\}_{n\geq 0}$ be an irreducible Markov chain that means it has only a single communicating class. Now if π is an invariant measure such that $\pi_i > 0$ for some $i \in S$, then $\pi_k > 0$ for all $k \in S$. So, remember it is easy to see from the definition of stationary measure that if you look at the 0 vector that is always a stationary measure. So, there always exist at least one stationary measure which is the 0 vector but remember this 0 vector is not a stationary distribution. Now, if you take a non-zero stationary measure find that means if there is at least one component which is strictly greater than 0 and if further the Markov chain is irreducible then each entry should be positive. So, if it is a irreducible Markov chain, either all entries should be 0 or all entries should be positive. The mixture is not possible provided the Markov chain is irreducible, then if you have a stationary, if you take a stationary measure either all its components are 0 or all is its components are strictly positive. Now let us see the proof, the proof is very simple. Now we already know that $\pi_i > 0$ for some $i \in S$. We want to show that $\pi_k > 0$ for all $k \in S$. So, we consider a state $k \neq i$. We want to show that $\pi_k > 0$. Now this is an irreducible Markov chain, so you can go from any state to any other state in finite number of steps. So, there will exist an n such that $p_{ik}^{(n)} > 0$ because k is accessible from i. So, there will exist an n such that $p_{ik}^{(n)} > 0$ and why is this true that is because this is an irreducible Markov chain so any state is accessible from any other state. Now π_k since this is an invariant measure is equal to $\sum_{j\in S} \pi_j p_{jk}^{(n)}$ we have already seen this, so this is not the original definition but the original definition implies this because it is not just $\pi P = \pi$, it is also true that $\pi P^n = \pi$, this is also true. So, that gives you that $\pi_k = \sum_{j\in S} \pi_j p_{jk}^{(n)} \geq \pi_i p_{ik}^{(n)}$ but this is greater than or equal to this because all these terms are non-negative. So, this is $\geq \pi_i p_{ik}^{(n)}$ where i is this particular i for which you know that $\pi_i > 0$, but now $\pi_i > 0$, $p_{ik}^{(n)} > 0$. So, you get that this is strictly greater than 0 hence you finally get that $\pi_k > 0$. So, we have shown that if the Markov chain is irreducible then and you are given a stationary measure where at least one component is strictly greater than 0, then all components should be strictly greater than 0. And the proof is just this two lines prove, it is a very simple proof which uses this irreducibility very crucially.

Now moving ahead to the next theorem, for that now you fixed a state k and now for each

$i \in S$, in the state space. So, for each $i \in S$, the expected time spent in i between visits to k. Now remember what is this T_k. So, recall this definition of T_k which was the first passage time, what was it? It was infimum over $n \geq 1$ such that $X_n = k$. So, the first time after 1 the Markov chain visits the state k. So, now again we are looking at E_k that means you are starting from k. So, what is this quantity? This quantity, so let us first look at this inside thing, so it is $\sum_{n=0}^{T_k-1} \delta_i(X_n)$ remember what is $\delta_i(X_n)$? So, $\delta_i(X_n)$ equals to 1 if $X_n = i$; equals to 0, otherwise. So, this inside summation counts the number of times starting from 0 to $T_k - 1$, the Markov chain is equal to i. And you are taking the expectation of this so this quantity is basically the expected time spent in i between visits to k. So, here because it is a discrete time, so the number of times it visits i is basically the time spent in state i. So, this γ_i^k is the expected time spent in i between successive visits to k or you can also think of the expected number of times the Markov Chain visits the state i between successive visits to k. Now we have this theorem, so what does it say? It says that, let $\{X_n\}_{n \geq 0}$ be an irreducible and recurrent Markov chain. So, it has a single communicating class and $E_k(T_k)$ is finite for all $k \in S$, sorry not $E_k(T_k)$, starting from k you come back in a finite time with probability 1, that is recurrence. So, Xn is irreducible and recurrent. Then $\gamma_k^k = 1$. Now this is very easy to see because what is γ_k^k? This is the expected time spent in k between visits to k. Now you see why it is equal to 1? Now remember T_K is the first time after 1 it visits k. So, initially that means that $n = 0, X_n = k$, why? Because you are starting from k and now in no other step it can be at k because this T_K is the first time after 0 it visits k so and since this sum is up to $T_K - 1$, so the number of visits to state k is just 1, which is at the zero-th step and in between the Markov Chain does not visit the state k that is because T_K is the first passage time. So, this $\gamma_k^k = 1$ is trivial. Anyway we will not see a complete proof of this theorem.

Now the second one says that this, now if I look at this, row vector which is γ_i^k. So, I can look at this row vector γ_i^k where $i \in S$. Now it says that this row vector is a stationary, it satisfies the stationary condition. Anyway see this is all these gamma is are non-negative because this number of visits that is obviously is a non-negative thing. You are taking expectation of that so anyway $\gamma_i^k \geq 0$ that is not an issue. What this point 2 is saying is that it satisfies this stationarity condition and also the third condition says it is not just greater than or equal to 0 but each $\gamma_i^k > 0$ and also it is finite for all $i \in S$. So, what this theorem is telling you, if you define γ_i^k in this way then this is a stationary measure or an invariant measure where each component is strictly positive. So, γ^k defined in this way is a stationary measure or an invariant measure and also one more property. So, you have this thing, that this $\gamma_k^k = 1$, so it is a stationary measure such that the k-th component is equal to 1. That is the content of this theorem.

So, now we move. So, why did we define this spatial stationary measure? Now the next theorem says that, let $\{X_n\}_{n \geq 0}$ be an irreducible Markov chain. So, it has only a single communicating class. Now fix a state k and let π be an invariant measure with $\pi_k = 1$. So, we saw in the previous theorem that this γ^k was an invariant measure with $\gamma_k^k = 1$ or the k-th component is equal to 1. Now suppose we look at some invariant measure, we do not know whether it is exactly that γ_k^k. It is some invariant measure with this property that $\pi_k = 1$. Then $\pi \geq \gamma^k$, what does that mean? So, what is this meaning of greater than or equal to? That means $\pi_i \geq \gamma_i^k$ for all $i \in S$. That is the meaning of entry wise. So, what this says is that, if you take any other invariant measure with the condition, so this condition is important that $\pi_k = 1$, then $\pi \geq \gamma^k$. So, in that way this γ^k is what is called a minimal stationary measure. Again it is just a terminology, you do not need to bother about that but the important thing is this inequality that if π is any invariant measure with this constraint that $\pi_k = 1$, then that π has to be greater than or equal to γ^k entry wise or in other words this should be true. If in addition, the chain is also recurrent. So, if in addition the chain is also recurrent then $\pi_k = \gamma^k$.

So, if you are working with if $\{X_n\}_{n\geq 0}$ is irreducible and recurrent then what this theorem tells you that this γ^k is the only invariant measure with $\gamma^k = 1$, with the k-th component equal to 1. So, if it is just irreducible then you take any invariant measure with the property that the k-th component in one then π should be that invariant measure $\pi \geq \gamma^k$, if in addition the chain is also recurrent then $\pi = \gamma^k$. Again, this equality means component wise that means $\pi_i = \gamma_i^k$ for all $i \in S$. That means if $\{X_n\}_{n\geq 0}$ is an irreducible and recurrent Markov chain, then this γ^k which we defined here in this way is the only invariant or is the only stationary measure with the property that the k-th component is equal to 1. So, if I give you a stationary or invariant measure with and I tell you that its k-th component is 1 and also if you know that it is an irreducible and recurrent Markov chain, then straight away you can say, this stationary measure is actually this gamma k which we defined in the previous slide. That is what is the content of this theorem. Now, this theorem gives us a kind of corollary which we write here as a remark. So, let $\{X_n\}_{n\geq 0}$ be an irreducible and recurrent Markov chain. Let π and μ be two non-zero invariant measures, now what is the meaning of non-zero invariant measures, that means each component is strictly greater than 0. See, just non-zero means it tells you that there will exist at least one, so strictly speaking non-zero invariant measure means there will exist at least one component which is strictly greater than 0, but since you have irreducibility now from the first theorem what we proved today we know that if it is irreducible Markov chain then if you take a stationary measure either it is the 0 measure or each component is strictly greater than 0. So, the setup that we are looking here since it is irreducible, when we say π and μ be two non-zero invariant measures that means each component is strictly greater than 0 but in general if I just say π is a non-zero invariant measure that means there is at least one component which is strictly greater than 0 but irreducibility forces that each component has to be strictly greater than 0. So, π and μ are two such non-zero invariant. So, suppose π and μ are two such are two non-zero invariant measures then there exists c such that $\mu = c\pi$.

Now, let us see the proof of that. Before going to the proof of that, so what is it saying that if you are having an irreducible and recurrent Markov chain then and if π and μ be two non-zero invariant measures, then one is a scalar multiple of the other. So, if you have a reducible and recurrent Markov chain, then you take any two non-zero invariant measures one is a scalar multiple of the other. So, this c actually will be not equal to 0. Now what is the proof of that, so for that again fix a state k. Now by this above theorem remember it says that if you have an irreducible and recurrent Markov chain and you are given an invariant measure whose k-th component is equal to 1, then that is equal to γ^k. Now you are given these two invariant measures π and μ but you do not know whether their k-th component is equal to 1 but there is a way to make their k-th component equal to 1. You divide by π_k. So, you look at the measure so each component you divide by π_k. Then since each component you are dividing by π_k so the k-th component now will become 1. Because it will be π_k over π_k and hence the k-th component will become 1. So, π over π_k now initially we did not know whether π for π the k-th component is equal to 1, but now if we look at this modified slightly modified invariant measure pi over π over π_k, now remember if something is an invariant measure so if π is an invariant measure then $c\pi$ is also an invariant measure. This you can easily check, if you take an invariant measure then any scalar multiple of that is also an invariant measure. So, if I divide by 1 over π over π_k, it still remains an invariant measure but now the advantage is this modified measure for this the k-th component is equal to 1.

Now, here $\{X_n\}_{n\geq 0}$ is an irreducible and recurrent Markov chain and this pi over π over π_k is a stationary measure with k-th component equal to 1. So, by this theorem this has to be equal to γ_k. Now similarly if I look at μ, then again if I look at μ over μ_k, that is again an invariant measure whose kth component is equal to 1 but again by this theorem this μ over μ_k should be equal to γ_k and remember like we can easily divide by this π_k or μ_k because these are two non-zero invariant measures so each component is strictly greater than 0. Since we also have irreducibility. So, π over π_k is an invariant measure whose k-th component is 1,μ over μ_k is an invariant measure whose k-th component is 1. So, the chain $\{X_n\}_{n\geq 0}$ is irreducible and recurrent, so by this theorem above we get that both of them should be equal to γ_k but now from this, you just take this π_k to that side what we get is $\mu = \frac{\mu_k}{\pi_k}\pi = c\pi$ where c is equal to this and since this μ_k and π_k are both non-zero, so this $c \neq 0$. So, what it basically says that $\mu = c\pi$ at the same time you can write it as $\pi = \frac{1}{c}\mu$. So, if you are having an irreducible and recurrent Markov chain and if you have two non-zero stationary measures then they are actually scalar multiple of one another. So, in this way you have kind of uniqueness. See, if you have an invariant measure its scalar multiple is again an invariant measure. That is in general true. But what this theorem is telling is that if you have an irreducible and recurrent Markov chain, then any two invariant measures and you take any two non-zero invariant measures then one is a scalar multiple of other. So, in that way again, there does not exist a unique invariant measure but if you take any two invariant measures one is a scalar multiple of the other. So, in that sense, so you have this kind of weak uniqueness that one is a scalar multiple of the other provided the chain is irreducible and recurrent. So, you need these two additional conditions on the Markov chain. If it is any Markov chain then such a thing is not true but see what is always true is that, if you have an invariant measure any scalar multiple is again an invariant measure. But what this theorem is telling is that, if you have an irreducible and recurrent Markov chain then any two non-zero invariant measures are scalar multiple of one another. But remember the two important assumptions are required irreducibility and recurrent.

So, moving on, now so till now we have classified states as recurrent and transient. For recurrent starting from i the probability that it will come back to i in a finite time is 1. So, with probability 1 starting from i it will come back to i in a finite time and if it is transient then there is a positive probability that starting from i it will never come back to i. So, for recurrent you have $P_i(T_i < \infty) = 1$ and for transient $P_i(T_i < \infty) = 1$ for recurrent and for transient $P_i(T_i < \infty) < 1$. So, there is a positive probability that $P_i(T_i) = \infty$. So, just let me write it here, $P_i(T_i < \infty) = 1$ and $P_i(T_i < \infty) < 1$. So, this is recurrent and this is transient. Now, we will see that we will further classify recurrent states into two more sub-categories what are those? So, a recurrent state i is said to be positive recurrent, if $m_i = E_i(T_i) < \infty$ that means that the expected time in which it will come back that is also finite and if i is not positive recurrent then it is called null recurrent. So, see if I give you a non-negative random variable which is finite valued it is not guaranteed that its expectation will be finite. Its expectation can be both finite as well as infinite. So, now if I look at this T_i which is the first passage time to i or the first return time to i, now recurrent means it is a finite random variable with probability 1 but just finite random variable with probability 1 does not guarantee that its expectation is also finite. So, if its expectation is also finite then you say it is positive recurrent and if it is not then you say it is null recurrent. And how you should think about it or what is the intuition? So, positive recurrent means, that it will come back, so recurrent means it will come back starting from i it will come back to i in finite time with probability one. But null recurrent means it might take actually a very long time with high probability. So, that is why when you look at this weighted, what is expectation? Expectation is basically weighted average so null recurrent means, it is finite but its expectation is not finite. So, the intuition is that it will come back in finite time but it will, see finite does not mean how big or small it is just finite means, it is not infinity but null recurrent means, that again that this is the way or this is the way you should think about it, this is the intuition that it will come back in finite time but with a substantial probability it will come back, it will take very long to come back. That is why the expectation is not finite. So, if $E_i(T_i)$ is finite, we say that it is positive recurrent, if $E_i(T_i)$ is not finite or it is equal to ∞ we say it is null recurrent. So, recurrent states we have further classified into two subcategories namely positive recurrent and null recurrent. So, the initial division was recurrent and transient and now we further divide recurrent into positive and null. So, initially we divided it into recurrent and transient and now we further subdivide it into positive and null. So, that is the definition of recurrent state but now why suddenly we have brought in this positive recurrence and null recurrence, because it has very close connection to this stationary distribution. Now that brings us to the last theorem of today's lecture which says that let $\{X_n\}_{n\geq 0}$ be an irreducible Markov chain, then the following are equivalent. What is the meaning of the following equivalent? That means, all these things so 1 implies 2, 2 implies 3. So, all these are equivalent if and only if one implies another. So, again so it says that 1 implies 2, 2 implies 3, 3 implies 1. So, all these like, if you assume 1 then the other is true. That is the meaning of that the following are equivalent. So, let $\{X_n\}_{n\geq 0}$ be an irreducible Markov chain then the following are equivalent. What are those statements which are equivalent? Every state i is positive recurrent? Some state i is positive recurrent, the Markov chain has a stationary distribution π. So, that means if every remember we are in an irreducible, we are working with an irreducible Markov chain, it says that if every state i is positive recurrent then some state i is positive recurrent. Now this is kind of trivial if every state is positive recurrent but some state i is positive recurrent but it is also saying the other way is true, that if some state i is positive recurrent then every state is positive recurrent not only that these two statements are also equivalent to the statement that the Markov chain has a stationary distribution. So, for example, if you know that every state in an irreducible Markov chain is positive recurrent then you know that the Markov chain has a stationary distribution. So, this gives you a sufficient condition. Not only sufficient because this is an if and only if condition. So, if you are in an irreducible Markov chain then if you can show that each state is positive recurrent then you know that it has a stationary distribution but not only that it is not that it just has a stationary distribution, it says that moreover when 3 holds $\pi_i = \frac{1}{m_i}$, for all, where $m_i = E_i(T_i)$. So, again let me try to explain this because this is a very important theorem. So, it says that if $\{X_n\}_{n\geq 0}$ is an irreducible Markov chain, then all these statements are equivalent. So, if you assume one then the other is true so all these are if and only. So, it says that every state i is positive recurrent, some state i is positive recurrent, the Markov chain has a stationary distribution pi. Say for example, if you know every state i is positive recurrent and it is an irreducible Markov chain then the Markov chain has a stationary distribution. Similarly, you do not actually need that every state is positive recurrent. If you know that just some state i is positive recurrent and it is an irreducible Markov chain then also you know that the Markov chain has a stationary distribution. But not only that under reducibility it also tells you that this stationary distribution is, so it tells you what that stationary distribution is or in other words if it is an irreducible Markov chain then there can exist at most one stationary distribution. So, it says that when three holes that means when the Markov chain has a stationary distribution, say suppose you are having an irreducible Markov chain, now then just look at this statement 3, it says that the Markov chain has a station, if the Markov chain has a stationary distribution then it is necessarily $\frac{1}{m_i}$ where $m_i = E_i(T_i)$. So, it tells you that, if it is an irreducible Markov chain then there can exist at most one stationary distribution. So, remember for any Markov chain, I said three possibilities are true. Either it is 1 or either it is 0, 1 or ∞ but if it is an irreducible Markov chain, the only possibilities are 0 and 1, infinity is not possible. If it is an irreducible Markov chain, there will exist at most one stationary distribution. It also tells you, if it exists what that stationary distribution is, so π_i is 1 over m i where m i is nothing but E i of T $\frac{1}{m_i}$ where $m_i = E_i(T_i)$. So, you see how the stationary distribution and positive recurrence is connected. So, all these statements are if and only if. So, another implication of this is, so if I tell you $\{X_n\}_{n\geq 0}$ is an irreducible Markov chain and it has a stationary distribution i then from that you can straight away say that all states are positive recurrent, that is like 3 implies 2, 3 implies 1 sorry. So, because all these are if and only if, so if it is an irreducible Markov chain and I say that the Markov chain has a stationary distribution π that straight away tells you that every state is positive recurrent. So, if it is an irreducible Markov chain then there can exist at most one stationary distribution. But remember if it is a finite state Markov chain, we have already proved that there will exist at least one stationary distribution. So, if it is an irreducible finite state Markov chain, then there exists a unique stationary distribution. So, if it is finite state Markov chain, that implies a plus irreducible sorry, so finite state Markov chain plus irreducible implies existence of unique stationary distribution. So, if it is an infinite state but is irreducible there can exist either 0 stationary distribution or 1 stationary distribution. If it is a finite state Markov chain plus irreducible then there exists exactly one stationary distribution. So, you now know a case where you have, you know both existence and uniqueness. So, if you take a finite state irreducible Markov chain, then it has a unique stationary distribution but if it is not finite, if the state space is not finite but infinite but still irreducible then you know that if there exists a stationary distribution then there will that will be unique. I am not, for infinite state Markov chain, existence of stationary distribution is not guaranteed. But if it exists, there will be unique. For example, you saw this. So, we will come to one example slightly later, but if it is a reducible infinite state Markov chain, then existence is not guaranteed but uniqueness is guaranteed. So, if there is one, there will be exactly one, but if it is a finite state Markov chain then there can be, then there will be exist exactly one stationary distribution. So, finite state Markov chain plus irreducible implies existence of a unique stationary distribution. That is one implication of this theorem, another implication is this remark. What it says? That this positive recurrence and null recurrence are class properties because why? Again, so if it is an irreducible Markov chain then that means, so it because that is coming from 2 implies 1. So, it says that if some state i is positive recurrent then every state i is positive recurrent. So, it is not possible that, again, why I am saying class property, irreducibility here it says about irreducibility but again if I can say, for example I start with a positive recurrent state. Now I look at the class of that, now since it is positive recurrent so it is recurrent. So, if I just look at the class of that, that is a closed class so I can think so, I can restrict my Markov chain on that so it will be just an irreducible Markov chain on that class. So, now it tells you that in that class there cannot be any null recurrence state why because of this theorem. It says that if some state i is positive recurrent then every state i is positive recurrent. So, till now we knew that this transience and recurrence are class properties but what we get from this theorem is that both positive recurrence as well as null recurrence is also class property. So, if you have a communicating class of recurrent states then either all the states will be positive recurrent or all the states will be null recurrent.

Now we will finish today with two examples, first recall the simple random walk. Now we have proved that asymmetric simple random walk is transient, we have proved that. Now thus, by previous theorem it cannot have a stationary distribution. Why? Because the previous theorem said that if you have an irreducible Markov chain, remember asymmetric simple random walk is also irreducible. So, if you have a reducible Markov chain then if it has a stationary distribution then all states has to be positive recurrent but for an asymmetric simple random walk, we also irreducible, forget positive recurrent we know that all states are transient. So, it cannot have a stationary distribution. So, you see for an infinite state Markov chain, even if it is irreducible it is possible that it may not have a stationary distribution. But if it has, then it has to be unique and it is given by $\frac{1}{m_i}$. So, here is an example of an infinite state Markov chain which does not have a stationary distribution which is this asymmetric, simple, random walk and why is that because each state is transient and the previous theorem already told us that, if it has a stationary distribution then each state has to be positive recurrent.

So, that is example, the first example which was of asymmetric simple random walk. Now the second example, here we look at the simple symmetric random walk. Now for simple symmetric random walk, we know that all states are recurrent but now the question are whether they are positive recurrent or null recurrent because now we have also know that positive recurrence and null recurrence are also class properties so either all states will be positive recurrent or all states will be null recurrent because a simple symmetric random walk is irreducible. Now, let us try to examine what those states are positive recurrent or null recurrent. Now for simple symmetric random walk, we have shown that all states are recurrent. Now if I look at this π_i, so I am looking at this vector of all 1's. So, it will be an infinite vector. So, it will go to both sides because in this case for a simple symmetric random walk, the state space is the set of all integers which is an infinite set. So, if $\pi_i = 1$ or in other words if I look at this infinite vector of all 1's then it easily, it satisfies this $\pi_i = \frac{1}{2}\pi_{i-1} + \frac{1}{2}\pi_{i+1}$ and why this? Remember, so this is precisely the stationarity condition for simple symmetric random walk because again what is, so π_i what it should be so I said. So, in the summation is over only those things like from where you can come to i. So, now in a simple symmetric random walk, you can come to either from $i-1$ or from $i+1$. So, you come from $i-1$ with probability $\frac{1}{2}$ and you come from $i+1$ with probability $\frac{1}{2}$. So, the stationarity equations become this $\pi_i = \frac{1}{2}\pi_{i-1} + \frac{1}{2}\pi_{i+1}$ and if I take this $\pi_i = 1$ for all $i \in S$, then these equations are satisfied. Or in other words, this is a stationary measure for simple symmetric random walk. Also, this is a non-zero stationary measure. Now again either you, so, 0 is always an invariant measure. Now remember we are trying to, so what we are initially trying to see that whether a simple symmetric random walk has a stationary distribution or not? Now what is a stationary distribution? A stationary distribution is also a stationary measure with the added condition that the sum of the entries has to be 1 but now if the sum of the entries has to be 1, it cannot be the 0 stationary measure. It has to be a non-zero stationary measure. Now non-zero stationary measure, the simple symmetric random walk is irreducible so non-zero means each component has to be positive. So, now we already have a stationary measure which is the vector of all 1's. So, if there exists a stationary distribution or an invariant distribution then that also has to be a stationary measure which where each component is strictly greater than 0. But now this is an irreducible and recurrent Markov chain. Now we have seen this that if it is an reducible and recurrent Markov chain, then if we have two invariant measures or two stationary measures then one is a scalar multiple of other. So, if you have a stationary distribution that should be a scalar multiple of this. So, again it should be of the form say $[c, c, c, \ldots]$ it should be a vector of all c's for some $c > 0$. Obviously, c cannot be 0 because then the sum will not be 1. But again, if every component is same then the sum will not be equal to 1. So, and then another way of saying this that since any invariant measure must be a scalar multiple of π but now π is all 1, so this sum is ∞. So, if you and now there has to exist a stationary distribution, it should be some $c\pi_i$. So, the components should be of the form $c\pi_i$ but then the sum that simply c will come out but since this series or this sum is infinite by multiplying by any non-zero c you cannot make it equal to 1. So, the thing is that if there has to be a stationary distribution in this case, all these components has to be equal but if all its components are equal then that sum cannot be equal to 1. In fact that sum will be equal to ∞. So there cannot exist a stationary distribution. So, the main thing that we are using here is that since this is an irreducible and recurrent Markov chain and we have already found out a stationary measure which is the vector of all 1's. So, the stationary distribution has to be a scalar multiple of that so it has to be of the form of all, so it has to be a vector where all components are equal, if all components are equal, then the sum of the components cannot be equal to 1, it will be equal to ∞. We have seen this thing before as well, so that if all components are same the sum cannot be finite. Or other words sum cannot be equal to 1, so forget 1, it cannot be even finite. So, in this case also, there cannot exist a stationary distribution. So, for a asymmetric simple random walk, it is very simple, we can say very easily that there does not exist a stationary distribution because all states are transient but for simple random walk we need to do a little more how we are claiming, but we finally see that for even for a simple symmetric random walk there does not exist a stationary distribution but since there does not exist a stationary distribution, all states are null recurrent. Why? Because, if the states are positive recurrent, then again if I go back to this theorem it says that if every state i is positive recurrent then the Markov chain has a stationary distribution. But we have shown that the, for a simple symmetric random walk there does not exist a stationary distribution. So, the states all has to be null recurrent because if there exists. If both the states are positive recurrent then the theorem that we just saw in previous slide tells you there has to exist a stationary distribution. In fact, there will exist, because we are in this irreducible setup, there will exist exactly one stationary distribution but we know that there exists. But we have shown that there cannot exist a stationary distribution, so all states of a simple symmetric random walk are null recurrent. So, for both simple symmetric random walk as well as asymmetric simple random walk, there does not exist a stationary distribution. So, both these are examples of infinite state irreducible Markov chains where stationary distribution does not exist. So, when it is infinite state then either, so it is possible that stationary distribution does not exist but if it exists and we also know that it is irreducible there will be exactly one. So, existence is not guaranteed but uniqueness is guaranteed, but for finite state Markov Chain, both existence as well as uniqueness is guaranteed, if it is irreducible. So, that is all. So, we will stop here today. Thank you all.