Discrete – Time Markov Chain and Poisson Processes
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Lecture 22
Lecture: Some Problems I

Hello everyone, welcome to the 22nd lecture of the course Discrete-time Markov Chains and Poisson Processes. So, in the last lecture we saw a very important result, which said that if the Markov chain is irreducible, positive recurrent and aperiodic then no matter from what initial distribution μ you start with, if you look at probability under the initial distribution μ that $X_n = i$ that as $n \to \infty$ converges to π_i where pi π_i is the unique stationary distribution, that means π_i is equal to $\frac{1}{E_i(T_i)}$ for all i. So, no matter what initial distribution you start with, if you look at the distribution of X_n or the distribution of the Markov chain at time n, then as $n \to \infty$, it converges towards the stationary distribution. So, an irreducible, positive recurrent, aperiodic Markov chain converges towards stationarity. So, one result we saw about stationary distribution is that, if you start from a stationary distribution, the distribution of all X_ns is also equal to the stationary distribution. But the last result which we saw, it said that, okay, you start from any initial distribution, but as $n \to \infty$ the distribution will converge towards the stationary, unique stationary distribution, because we are working with an irreducible, positive recurrent and aperiodic Markov chain. Such a Markov chain is also called an ergodic Markov chain. So, that is the result which we saw in the last lecture. So, in today's lecture we will work out some problems, fine. So, let us start.

So, the first problem is consider the Markov chain with the following transition probability matrix, which is given by this matrix, and the question is find all stationary distributions. First of all, you can easily check that this matrix is not reducible, is not irreducible. In fact, it has three classes, one class is 0 and 4, so one class is 0 and 4, another class is just this single class 2. So, 2 is actually an absorbing state and the other class is the remaining 2, so which is 1 and 3,. So, this Markov chain is not irreducible, we know that for an irreducible Markov chain, if it is a finite state irreducible Markov chain, then there exists a unique stationary distribution, but here it is not irreducible, it is finite state Markov chain, so there will exist at least one stationary distribution. But since it is not irreducible, we do not know whether it will be unique or not let us check. So now also if you just investigate a bit, you will see, if you just look at the transition matrix closely, you will see that these two classes that this one and this one, these two are closed and hence recurrent. We know that if it is a finite state Markov chain, then if you have a closed class, it is recurrent and this 3 is actually transient, you can easily check that. Because like 1 easy way to see that from 3 you can actually go to 1 with probability one by 4. So from 3 you can go to 1 with probability 1 by 4 or, sorry, from 3 you can go to 2 with probability 1 by 4 and when you are in 2, you remain at 2, because it is an absorbing state. So, that is one way that you see that starting from 3, you will never come back to 3. Because from 3 you go to 2 with probability 1 by 4 and then you stay in 2 forever because 2 is an absorbing state, so, and since 3 and 1, they form a single communicating class, so 1 is also transient. So, we have 3 communicating classes here, 0, 4, 2, and 1, 3. So, 0, 4 and 2 are recurrent classes and 1, 3 are transient, but our main aim is to find all stationary distributions. So, for that we write these equations that pi p basically, we write down the equations $\pi P = \pi$, fine, and we get these equations.

So, let me call this as equation 1, let me call this as equation 1, this as equation 2, this as equation 3, this as equation 4 and this as equation 5, this as equation 6. Now, from the first equation, you can easily check that you get pi 0 equal to pi 4. If you bring this to this side, you will get half pi or rather if you bring this one to this side, then you get $\pi_0 = \frac{1}{2}\pi_0 + \frac{1}{2}\pi_4$ and from there you get $\pi_0 = \pi_4$. Now, from the third equation, you can easily check that you get $\pi_3 = 0$ because this pi_2 and pi_2 will cancel and you will get $\pi_3 = 0$. Now, using that and equation 4, so $\pi_3 = 0$ so from equation 4, so these 2 things are not there you will get that $\pi_1 = 0$. So, you get these constraints, so you do not get any constrain on π_2, but you get that $\pi_0 = \pi_4$, $\pi_3 = \pi_1 = 0$. So, thus if you take any vector of length 5 of this form $(\alpha, 0, \beta, 0, \alpha)$, why I am taking like this, because hese constraints, so you do not get any constrain on π_2, but you get that $\pi_0 = \pi_4$, $\pi_3 = \pi_1 = 0$ and you have no restriction on π_2. So, this is a stationary distribution, but again you also need this condition 6 to be true. So, the sum of all the components should be equal to 1 and they should be non-negative. So, what you want is basically this and this to hold, so any such vector is a stationary distribution if these two are satisfied. So, for example, one particular thing is, if you, so say $(\frac{1}{2}, 0, 0, 0, \frac{1}{2})$ is a stationary distribution, another stationary distribution is $(0, 0, 1, 0, 0)$. In this way just so any such vector with these two constraints that α and β should be between 0 and 1 and $2\alpha + \beta = 1$. Then this way, you can find infinitely many stationary distributions. So again, this is an example of a finite state irreducible Markov chain, which has infinitely many stationary distributions. And you see like how, so basically when you analyze the stationarity equations, you will find out that whether it has a unique stationary distribution or infinitely many stationary distributions, because if we are in a finite, if we are working with the finite state Markov chain, then either there will be a unique mark stationary distribution or infinitely many stationary distributions. If it is irreducible, then it will be unique, if it is not then you do not know you have to check, but anyway even if you just analyze the stationarity equations, then also you will see whether the, it is a unique stationary distribution or infinitely many stationary distributions. So, for this particular example, we see that there are infinitely many stationary distributions and we have calculated it, how the stationary distribution will look like.

So, moving on, a particle performs random walk on the vertices of a tetrahedron. At each step it remains where it is with probability 0.25 or $\frac{1}{4}$ or moves to 1 of its neighbouring vertices each with probability 0.25, that means what say, suppose the vertical is at state A, fine, or is at the vertex A, then it will, the next step, it will stay at vertex A. So, in the next step, it will stay at vertex A with probability $\frac{1}{4}$, sorry, so suppose the particle is at vertex A, then it will, in the next step, it will stay at stage A with probability $\frac{1}{4}$, it will go to D again with probability $\frac{1}{4}$, it will go to B with probability $\frac{1}{4}$ and to C with probability $\frac{1}{4}$. So, if the current state is one particular vertex A, then in the next state or in the next time period, it will be at A with probability $\frac{1}{4}$ and it will move to the other three neighbouring vertices with probability $\frac{1}{4}$ each. That is why you have seen a random walk on the set of all integers. So, this is a random walk on a tetrahedron, so that is the dynamics.

So, suppose the particle starts at A, so suppose the particle starts at A, find the mean number of steps until it is first returned to A. Now, how do we model this movement of a particle, if X_n be the position of the particle at time n, then X_n is a Markov chain that is very easy to see because of the way that the particle is moving. If you know the current state, then in order to know the future like say where it will be in the next step, nothing in the past is important, only you need to know where it is in the current state. So, it is a Markov chain and what is its transition probability matrix? Its transition probability matrix is this because if it is at A in the next state or in the next stage or in the next time period it will be at A with probability $\frac{1}{4}$ with at B with probability $\frac{1}{4}$ at C with probability $\frac{1}{4}$ and at D with probability $\frac{1}{4}$. Similarly, for other states, so the transition probability matrix is this simple matrix. Now, so we already know that if it is a transition probability matrix, the row sum is equal to 1, because what it gives? So, what is a particular row the one particular row is, say for example, what is the ith row? The ith row each element is starting from i, what is the probability of going to state j in the next step. So, if you sum over all j that is basically doing the row sum, you get 1. But if you look at this matrix closely here, even the column sums if you look at this column sum that is also equal to 1. So, column sum is column sum, so column sum is equal to 1, so you see a spatial property of this matrix is that every transition probability matrix its row sum is equal to 1, that is just because the weight is defined, but for this particular transition probability matrix. You see that the column sum is equal to 1, indeed this is a special matrix.

So, now we see a very useful fact, so an $n \times n$ transition probability matrix is said to be doubly stochastic, if its each column sum is equal to 1. So, just like the example we saw in the previous slide. So, in a transition probability matrix, we know already the row sums are 1, so if the each column sum is also equal to 1, it is called doubly stochastic. So, stochastic means just row sum is 1, but because column sum is also equal to 1, we are calling it doubly stochastic. Now, if X_n is an irreducible Markov chain with state space S where the cardinality of S is n, so it is a finite state Markov chain with n states. And if the transition probability matrix is doubly stochastic, then the unique stationary distribution of the Markov chain is given by this provided also the Markov chain is irreducible. So, if you have an irreducible Markov chain with a finite state space, and suppose the cardinality of the state space is n or the number of states is n and if the transition probability matrix is doubly stochastic, that means both row sum as well as column sum is equal to 1, then the unique stationary distribution is given by this$\pi_i = 1$ by n. So it gives equal probability to the stationary distribution gives equal probability to each state or the probability. So, basically the stationary distribution is the probability mass function of a discrete uniform distribution.

Again, if you do not know the terminology discrete uniform distribution, never mind, but what we mean is that the stationary distribution is $\pi_i = 1$ by n for all $i \in S$, where n is basically the cardinality of the state space. So since it is an, we are in an irreducible setup as well as a finite Markov chain, so we know that there exists a unique stationary distribution that we know. But if you see that the transition probability matrix is doubly stochastic, then you do not have to do all those calculations of stationary equations, right away you can say that it is the stationary distribution is 1 over the cardinality of the state space. If you do it you will end up with the same result. But what I am saying here is that, this fact tells you, okay, you do not have to do all those calculation, you can save that work because the stationary distribution will be $\frac{1}{n}$, if n is the number of states. Provided it is an irreducible Markov chain, finite state irreducible Markov chain with state space cardinality n and the transition probability matrix is doubly stochastic. So, we see that doubly stochastic matrices are special in the sense like for them the stationary distribution has a very special structure. But remember what are we trying to find, we are trying to find the mean number of steps, so you are starting from a you are trying to find the mean number of steps until it is first returned to A. Now, we know that this mean of the first passage time or the first return time and stationary distribution are connected. If it, if you are in an irreducible finite state Markov chain, then the stationary distribution is given by $\frac{1}{E_i(T_i)}$. So, if you are, if we are interested in finding the expectation of the mean return time, so if we can just find the stationary distribution that will be it, but now for this particular example, this transition probability matrix is doubly stochastic.

So, now using this useful fact, we get that the stationary distribution is this because the, here the state space is the cardinality is 4 because the states are A, B, C, and D, so the cardinality of the state space is 4. So, the stationary distribution is $\frac{1}{4}$, $\frac{1}{4}$, $\frac{1}{4}$, and $\frac{1}{4}$. So, now if we are interested in $E_A(T_A)$ that should be 1 over, so basically this is, this thing is actually 1 over E_A, so this thing is actually $\frac{1}{E_A(T_A)}$, so you get this thing because 1 over, if $\frac{1}{E_A(T_A)}$ is equal to $\frac{1}{4}$, then you get that $E_A(T_A) = 4$. So, you see just by finding the stationary distribution, we can say what this expected or mean return time to starting from A, what is the mean return time to A, and we get it as 4. So, this example has two important things, one this useful fact that if you have a finite state irreducible Markov chain whose transition probability matrix is doubly stochastic, then the stationary distribution has a very special form which is each component is $\frac{1}{n}$, where n is the size of the state space. So this one, this example is, that is one important part of this example. Another important part of this example is like, if you are asked to find the mean return starting from a particular state, if you are asked to find the mean return time to that state, then you also, you just find out the stationary distribution. And if you are being asked to find the mean return time to state i, you look at the ith component of the stationary distribution, just invert that and you get the mean return time. So here, we were asked about $E_A(T_A)$, so we looked at the first component which is $\frac{1}{4}$. So here, all components are same, so we looked at $\frac{1}{\frac{1}{4}}$, which is 4 and hence, that is the mean return time. So, we will stop here today. Thank you all.