Discrete - Time Markov Chains and Poisson Processes

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Lecture 24

Time Reversibility

Hello everyone, welcome to the 24th lecture of the course Discrete Time Markov Chains and Poisson Processes. So, in the last two lectures, we solved a few problems. In todays lecture, we will start with a new topic, which is time reversibility.

So, let X_n be an irreducible Markov chain. Before starting with the theorem, let me say a few words. So, what is a Markov chain? We say that a process X_n is a Markov chain if given the present state the evolution or the future evolution depends, does not depend on the past, that is what a Markov chain is. Now, so this is something talking about the forward evolution. Now, what if we look at the backward evolution.

So, what do I mean by that? So, suppose say you are told that, you are given the states X_{100}, X_{99}, X_{98} and you are asked about X_{97}. So, these states you are given X_{98}, X_{99}, X_{100} you are given and you are asked that what is the probability that X_{97} was a certain state. So, basically, you are being asked the question about backward in time. So, now the question is, can you answer this question?

And another question is, in order to answer that question, how much of this information is relevant? Do you need all these states X_{98}, X_{99}, X_{100}? Or do you just need X_{98}? Let us see. So, that is the motivation or that is the starting point. So, the Markov chain definition talks about future evolution, what if we look back in time, or we reverse time. So, we will start today with a theorem.

So, let Xn be an irreducible Markov chain with stationary distribution, one small thing. So, you see here we are looking at $0 \leq n \leq N$. Till now, we were just looking at for all $n \geq 0$. So, we are looking on an infinite time horizon, but here we are looking on a finite time horizon. Very soon you will see why we need that, but again nothing changes you can look at a Markov chain on a finite time horizon all definitions and everything remains intact. Here, we are looking at the Markov chain on the time horizon 0 to capital N. Now, define $Y_n = X_{N-n}$. So, you see that is why because, so, we cannot talk about some X infinity minus something. So, that is why we need to look at on a finite time horizon.

So, we are defining why $Y_n = X_{N-n}$. Then Y_n is again a Markov chain with initial distribution p_i and transition probability matrix $\hat{P}$, which is given by this equation. What is it? That $\pi_j \hat{P}_{ji} = \pi_i P_{ij}$. So, basically, it is saying that the entries of this matrix the transition probability matrix satisfies this relation that $\pi_j \hat{P}_{ji} = \pi_i P_{ij}$, for all i, j.

So, what it says is that, like now, if we go back to the question which we started with it said that, it says, so what this theorem tells you is that like okay if you are looking back in time that is again a Markov chain. That means, if you are given information about X_{100}, X_{99}, X_{98} and you are asked something about X_{97} a probabilistic like what is the probability that $X_{97} = i$ for some i in the state space.

Then you only need that what the 98th state is. So, again this you can think of as the current state, so you do not need too many things in the future. So, you only need X_{98}. That is the meaning of the statement that Yn is again a Markov chain. Now, remember, here we start, not, the initial distribution is also important, the initial distribution is, so if you start with π, so that is also important.

So, be an irreducible Markov chain with stationary distribution π and you also start the Markov chain at π. So, this is also important that probability, so $\mathbb{P}(X_0 = i) = \pi_i$, that is also important that the initial distribution is the stationary distribution. So, you start from the stationary distribution, so that is also there. So, if X_n is an irreducible Markov chain with initial distribution π where π is also the stationary distribution.

Then if you define Y_n in this way, where $Y_n = X_{N-n}$ then Y_n is again a Markov chain with initial distribution π and transition probability matrix $\hat{P}$ which is given by this relation. Moreover, Y_n is also irreducible and has an invariant distribution π. So, remember, so this point is important that you are starting from the stationary distribution.

That $\mathbb{P}(X_0 = i) = \pi_i$. The initial distribution is not π, this theorem is not true and again this π is not any initial distribution, but π is, it is also told that π is the stationary distribution of the Markov chain. So, here that means the stationary distribution. So, it is like, since it is irreducible the Markov chain will have, if there exists a stationary distribution then there will be a unique stationary distribution.

So, here basically you are told that okay the unique stationary distribution exists and you take that as the initial distribution. If that is the setup then if you define your Y_n in this way, then Yn is again a Markov chain with initial distribution π and transition probability matrix this $\hat{P}$ where $\hat{P}$ is given by this relation. Further, Y_n is also irreducible and has an initial distribution π.

So, what this is telling you is that if you, so for a Markov chain, we know that if you are trying to predict the future evolution, you only need the current state and nothing from the past. Similarly, if you start a Markov chain from the stationary distribution and an irreducible Markov chain from the stationary distribution at look backward in time, then that again a Markov chain. So, more specifically we have to give an example.

So, if you are given the states X_{100}, X_{99}, X_{98} and you are asked about X_{97} then only you need to know what X_{98} is, from that you can make a probabilistic statement. So, what is the probability that say suppose, so if I ask you what is the $\mathbb{P}(X_{97} = j | X_{98} = i)$, that is now equal to $\hat{P}_j$. That is the meaning of that this is the transition probability matrix.

And the Markov property means like in apart from this if you are also given a X_{99}. Anyway, let me now write this again. So, again if you are given some extra information, say X_{99},X_{100}still this probability is just this $\hat{P}$. So, only you need to know X_{98} in order to say something about X_{97}. You do not need anything more in the future like X_{99}, X_{100},... That is the content of this theorem.

But remember one important thing is you start the Markov chain from the stationary distribution then if you reverse it, then it is again a Markov chain with the same initial distribution and the reversed Markov chain is also irreducible. So, remark that this chain is called the time-reversal of X_n and you can easily see that from the definition, because you are looking backward in time or you are reversing time.

So, this chain Y_n is called the time-reversal of X_n. Now, one more definition, let X_n be an irreducible Markov chain with transition probability matrix P and initial distribution π, where π is also stationary distribution. Now, you know in this case if you reverse the Markov chain then it is again a Markov chain with initial distribution π and transition probability matrix $\hat{P}$.

Also, this time reverse chain is irreducible and has π as invariant distribution. Now, if for all $n \geq 1$ the time reverse chain. Now see, now, we are looking at X_n the Markov chain of infinite time horizon, but if you have to, so what is, but for the definition what we are saying is that if for all $n \geq 1$ this time reverse chain also has transition probability matrix P then X_n is said to be reversible.

Now, in order to define this time reverse chain, you need to just look at the Markov chain on a finite time horizon that is why what this definition is telling you, you say a Markov chain. Now, when X_n for $n \geq 0$ on the entire or the infinite time horizon it is said to be reversible if for all $N \geq 1$, if you look at this reverse chain, then that is.

So, this first theorem tells you that well that will be again a Markov chain with initial distribution, that will be again an irreducible Markov chain with initial distribution π, which will also be the stationary distribution and the transition probability matrix will be $\hat{P}$. Now, the, so that is for any irreducible Markov chain starting from the stationary distribution.

But further, if you know that this $\hat{P} = P$ that means, this transition probability matrix of the time reverse chain is same as the transition probability matrix of the original chain then you say that the Markov chain is reversible. So, again what is the meaning of also as transition probability metrics P?

That means, you know that the reverse chain transition probability matrix is $\hat{P}$. What this definition is telling is if this $\hat{P} = P$ or in other words what it means is that if $\hat{P}_{ij} = P_{ij}$ for all i, j then you say that the original Markov chain is reversible. So, if the transition probability matrix of the reversed chain is same as the transition probability matrix of the original chain, then you say that the Markov chain is reversed.

Now, we will see a couple of examples. So, first, consider the Markov chain with the following transition probability matrix. Now, again it is easy to see that this Markov chain is irreducible. Any two states communicate and have unique stationary distribution this now, why? Because, you can again check that this is a doubly stochastic matrix and we have already seen this result.

So, if the transition probability matrix is doubly stochastic and the chain is irreducible, then the unique stationary distribution is given by $\frac{1}{N}$ where N is the number of states. So, here the number of states is 3. So, the unique stationary distribution is given by $\frac{1}{3}$. Now, if P had satisfies this, remember, we are trying to check whether this Markov chain is reversible or not. Now, what will be the transition probability matrix of the reverse chain it will be P hat which will satisfy this.

But now, you know that, so here what is $p_{ij} = \frac{1}{3}$ for all j. So, this and this are both equal

to $\frac{1}{3}$ and hence, they get canceled. So, $\hat{P}$ will satisfy this. So, $\hat{P}_{ji} = P_{ij}$. Since, P_{ij} is same for all j, so this will get canceled from both sides. So, P hat will satisfy this relation or in other words, $\hat{P}_{ji} = P_{ij}$.
Now, if this Markov chain has to be irreducible that means what, that means, again see, so here it is, it says that$\hat{P}_{ji} = P_{ij}$ but I have written as $\hat{P}_{ij} = P_{ji}$. But since this is true for all i, j you can just interchange i and j. But then if this has to be time-reversible then $\hat{P}_{ij} = P_{ij}$. But what this relation is telling you this is equal to P_{ji}. So, this Markov chain will be reversible, if $P_{ji} = P_{ij}$ for all i, j.
But you can clearly see say for example, $P_{12} = P_{21}$. What is P_{12}? P_{12} is basically, so you are looking at P_{12}. So, the first row is 1, second column. So, basically this. So, you are looking at this, that is P_{12}, and this is P_{21}, you can clearly see, so $\frac{2}{3}$ is not equal to $\frac{1}{3}$.
So, this Markov chain is not reversible because you see like, if it has to be reversible, there at least for this particular case it has to satisfy this condition that $P_{ji} = P_{ij}$ for all i, j. But you can see $P_{12} \neq P_{21}$. Similarly, you can also check say for example, $P_{01} \neq P_{10}$, because this is P_{01} and this is P_{10}. So, this Markov chain is not reversible.
So, let us look at another example. Now, we consider another Markov chain with the following transition probability matrix. So, this matrix is very similar to this but with a slight modification, anyway. So, the Markov chain has this transition probability matrix. Again, it is easy to check that the Markov chain is irreducible and has unique stationary distribution because again you can check that this matrix the transition probability matrix is doubly stochastic.
Now, again, thus if $\hat{P}$ has to satisfy this by the same calculation as before, it should be $\hat{P}_{ij} = P_{ji}$. But now, you look at this matrix here $\hat{P}_{ij} = P_{ji}$ for all i, j. So, again, so if this chain has to be reversible you need $\hat{P}_{ij} = P_{ij}$. But since the stationary distribution has all components equal this is basically equal to P_{ji}. So, this chain will be reversible if $P_{ji} = P_{ij}$ for all i, j and that is precisely true for this given transition probability matrix.
So, you see this and this is same this and this is same this and this is same. And obviously, so you only need to check this for $i \neq j$ because for $i = j$ that is trivially true. So, here is an example of a transition probability matrix or here is an example of a Markov chain that is reverse.
So, you see, we have seen both the examples one of a Markov chain that is not reversible and another of a Markov chain that is reversible. Just the transition probability matrices are very similar, but with a slight difference, so in one case it is reversible in the other case, it is not reversible. So, moving forward.
So, that was about time reversibility. Now, we will see another definition. So, initially, you might find okay from where this definition is coming, but slightly later you will see that, okay, everything is connected. So, what is that definition? So, let X_n be a Markov chain with transition probability matrix P. A non-negative row vector λ, where the size of the row vector is equal to the size of the state space.
And P, P is the transition probability matrix, are said to be in detailed balance if this condition is true $\lambda_j p_{ji} = \lambda_i p_{ij}$ for all i, j. So, that is the definition of detailed balance. So, let

X_n be a Markov chain with transition probability matrix P. A non-negative row vector λ, non-negative means, each $\lambda_i \geq 0$. So, such a non-negative row vector.

And P this transition probability matrix P is said to be in detailed balance if this condition is true, $\lambda_j p_{ji} = \lambda_i p_{ij}$ for all i, j. So, the obvious question is why suddenly this condition, what is like, what this condition is going to give us? If we investigate a little more closely, we will see that. So, that is our next remark that if λ and P are in detailed balance that means, this condition is true, then λ is an invariant measure for the Markov chain.

And what is the proof? The proof is very simple. Now, $\sum_{i \in S} \lambda_i p_{ij}$, this is equal to this, why? Because $\lambda_j p_{ji} = \lambda_i p_{ij}$, that is the detailed balance condition. But now, so, again the summation is still over $i \in S$, but now when you, because now, look at this, this is . So, when you are doing P, $\sum_{i \in S}$ you are basically summing up a row of transition probability matrix and that is equal to 1, because you are summing it over i and this is p_{ji}.

So, you are actually summing up over the jth row, so that sum should be equal to 1. So, again λ, the sum does not depend on j. So, you can take out λ_j outside, so this becomes equal to λ_j. So, finally, you get $\lambda_j = \sum_{i \in S} \lambda_i p_{ij}$ and this is true for all $j \in S$. So, and this is precisely the condition for the stationary measure. So, recall the definition of a stationary measure it says that a row vector with non-negative entries is said to be a stationary measure if it satisfies this condition.

So, we see that if λ and P are in detailed balance, then lambda is an invariant measure for the Markov chain. So, you see. So, in general, in order to find a stationary measure, you need to solve this set of equations given by $\pi P = \pi$. Now, you see this. So, again this is also a set of equations, but this is a much easier looking or simpler looking set of equations.

So, in many cases you will see, you will see some examples later that calculating this detailed, finding solutions to detailed balance equations is easier than finding solutions to the stationarity equations. So, this is a special case. So, if it is, if the detailed balance equations is true or if λ and P satisfy the detailed balance condition, then λ is an invariant measure. So, if you can show that the detailed balance equation has a solution, then you can find an invariant measure.

Now, if that invariant measure, the sum of all the entries is finite, then you just divide by that sum and you can make it an invariant distribution. So, this is a way of finding invariant distribution. And why we are looking at this? Because this way is slightly easier than finding it from the original equation set of equations which is $\pi P = \pi$, which is slightly more complicated than this detailed balance equation.

So, detailed balance implies stationarity obviously, if it is a stationary measure it need not satisfy the detailed balance equation, but if it satisfies the detailed balance equation then it is a stationary measure. So, this detailed balance is a stronger notion. So, if it satisfies the detailed balance condition then it is a stationary measure or it is an invariant measure.

And further, if you can show that if that sum of the entries, so if the sum of the entries is infinite or if it is 0 then again it will be just an invariant measure you cannot do much with that, but if the sum of the entries is finite, then you just divide by the sum to make it an invariant distribution. Because, if something is an invariant measure then if you just multiply it with a constant it remains an invariant measure, but now, since we are dividing by the sum it will become an invariant distribution. So, this is a way of finding invariant distribution.

So, if you can show, if you can find a solution to the detailed balance equations that is a way of finding invariant or stationary distribution. But why we, the important thing is that you will see cases where finding a solution to the detailed balance equations is much simpler because this is a much simpler looking equation as compared to the stationarity equations.

So, now, the question is, in the first couple of slides, we looked at this time reversibility, and then suddenly we move to detailed balance equations. Detailed balance equations is useful, because if we can find a solution to the detailed balance equations, we can find invariant distribution or stationary distribution.

But is that the only reason why we brought this stationary distribution this detailed balance equations then we could have brought it when we dealt with just stationary distributions, why did we bring it in the lecture on time reversibility? That is because these two are connected, what is the connection is given by the next theorem. So, let X_n be an irreducible Markov chain with transition probability matrix P.

Let π be a positive row vector which means $\pi_i > 0$ for all i in S such that the sum is equal to 1. Or in other words, it is a distribution. Then, the following statements are equivalent. Again, what is the meaning of the following statements are equivalent. That means, the statements are if and only one implies the other and vice versa. So, π and P are in detailed balance, you know what that means, and the second statement is π is the stationary distribution for X_n and X_n is reversible.

So, if you can find, so if you are given an irreducible Markov chain, and if you can find a solution to the detailed balance equation, then it is not just that that solution is a stationary distribution. But further, the Markov chain is also reversible. Well, in this course, I am not able to give you the full justification of why reversible Markov chains are useful.

But if you do any further course on Markov chains or some advanced course in Markov chains, then you will find out why reversible Markov chains are useful. I will just make one statement. So, if you know what Markov Chain Monte Carlo is, then that is one place where this reversible Markov chain plays an important role.

So, if you do not know what Markov Chain Monte Carlo is, you do not need to bother, but just keep this thing in mind that why we are looking at reversible Markov chains, is because reversible Markov chains have some useful applications, but we are unable to keep such applications in this course, because we have some other things to do. But reversible Markov chains are important, keep this point in mind.

So, we are not just studying them just like that, but we are studying them because they are actually important and has many applications in various places. One such place is Markov Chain Monte Carlo if you know what it is, if you do not know do not bother. Anyway. So, again, what this theorem is basically telling you is that okay, if you are giving an irreducible Markov chain, and if there you can find a solution again a positive solution or to the detailed balance equation.

That means, if you have a π where each component or each component of that is positive, it sums up to 1 and π and P satisfy that detailed balance equation, then it is not just true that π is stationary, we have already seen that, that if the detailed balance equation has a solution then from that you can find stationary distribution, but further the chain is also reversible.
So, if the detailed balance equation has a solution such that each $\pi_i > 0$ and they sum up to 1. Then it is true that yes 1 that it is an invariant distribution or a stationary distribution. But moreover, it is the Markov chain is also reversible. And this statement is not just one way it is an if and only if statement that means, if you know that okay π is a stationary distribution such that and the chain is reversible then π also, π and P satisfies the detailed balance equation.
So, if you know that π is an invariant distribution, and again since we are in an irreducible Markov chain, so, if π is an invariant distribution, we know that all components must be positive. So, we saw independently that if you have an irreducible Markov chain, and you are given a stationary measure then either everything is 0 or everything is positive irreducibility is the important hypothesis.
So, since we are working with irreducible Markov chains, if you have a stationary distribution, then all, everything, each $\pi > 0$. So, if you have an irreducible Markov chain, which has a stationary distribution π and the chain is reversible, so suppose you can just show that the chain is reversible, then you can also say that this pi and the transition probability matrix P of the given Markov chain satisfy the detailed balance equation or π and P are in detailed balance.
So, now, we will see the proof of it the proof is very simple. So, first, we do 1 implies 2. So, by the previous remark π is a stationary distribution. We have already seen that if π and P are in detailed balance, then π is a stationary distribution. Because it is a stationary measure, but now, since it also sums up to 1, so it is a stationary distribution. And now, we need to, so this part is done that π is the stationary distribution for X_n.
Now, we need to show that X_n is reversible. So, for time reversibility by detailed balance condition, remember what was the detailed balance condition, the detailed balance condition was $\pi_i p_{ij} = \pi_j p_{ji}$. Now, we know what is $\hat{P}$, we know that this transition probability matrix of the reversed chain $Y_n \pi$ will be $\hat{P}$ where $\hat{P}_{ji}$. So, basically, you saw π_j, that it satisfies $\pi_j \hat{P}_{ji} = \pi_i p_{ij}$.
But now, since we know that each $\pi_i > 0$, so, we can bring it to the denominator. So, we have that $\hat{P}_{ji} = \pi \frac{P_{ij}}{\pi_j}$ but this quantity πp_{ij} because of this is equal to $\pi_j \frac{P_{ij}}{\pi_j}$ but then π_j and π_j cancels and you get equal to p_{ji}. So, $\hat{P}_{ji} = P_{ji}$ for all i, j that precisely means that $\hat{P} = P$ or the transition probability matrix of the reverse chain is equal to the transition probability matrix of the original chain and hence, the Markov chain is reversible.
So, that is 1 implies 2. Now, 2 implies 1. Now, that means, you are given that π is the stationary distribution for X_n and X_n is reversible, reversible means $\hat{P} = P$. Now, we know that $\hat{P}$ satisfies this condition that $\pi_j \hat{P}_{ji} = \pi_i p_{ij}$, that is how you define $\hat{P}$. But now, $\hat{P} = P$

which means $\hat{P}_{ji} = p_{ij}$. So, you get this is equal to this, now, for all i, j.
So, finally, what you get is $\pi_j p_{ji} = \pi_i p_{ij}$ for all i, j and that is precisely the detailed balance condition. So, hence π and P are in detailed balance. So, you see the proof is very, very simple. So, if it, if you have an irreducible Markov chain, the transition probability matrix P, π is a row vector such that of the size of the, whose size is the same as the size of the state space.
And such that each component is strictly greater than 0 and it sums up to 1, then the following are equivalent, the statements are π and P are in detailed balance. And the other statement is pi is stationary distribution and the chain is reversible. So, you see that detailed balance, not only the solution to the detailed balance equation, not only just gives you stationary distribution, but also gives you reversibility of the Markov chain.
So, we will finish today with an interesting example of time reversibility, or how time reversibility can be useful. Again, this is basically not about time reversibility, but about the detailed balance equation. Remember, I made this statement that you will see in most situations, finding solution to detailed balance equations is simpler than finding solution to the stationary equation provided there is a solution to the detailed balance equations, because the detailed balance equations are much simpler looking, I am not saying you will always find a solution to the detailed balance equation, but detailed balance equations are much simpler looking equations.
So, you first try with that set of equations. If not, then again, you will have to go to the original stationarity equation in order to find the existence of stationary distribution. So, what is the example or what is the problem? A random King. So, a chessboard has various pieces, one of them is the king the most important piece, so, King can move in any direction. So, if you have played chess, then you will know that a king can move in any direction, it can go sideways, it can go up down, it can go in all directions, provided those directions are available. So, a random King makes each, we are calling it a random King, because it is as if we are saying that like you know a random walk. It goes in all possible directions with certain probabilities. Now, suppose here, you have a random King who makes each permissible move with equal probability.
Now, if it starts in the square marked 1, which is this square, how long on average will it take to return to 1. So, from 1, it starts making this random walk, the king is making this random walk, what is a random walk, like, if you are in a particular square on the chessboard, you know in which possible directions the king can move, now the king moves in all those directions with equal probability.
So, that means what, so for example, if we just concentrate on this square 1, now from 1, it can go to 2, it can go to 3, or it can go in this direction, that is 5. Now, it will go to either of these squares with probability $\frac{1}{3}$. So, with probability $\frac{1}{3}$, it will go to 2, with probability $\frac{1}{3}$, it will go to 3 and with probability $\frac{1}{3}$, it will go to this 5. So, that is one situation.
Now say, for example, if we look at this yellow square. Now, you see what are the possibilities. So, for example, I am looking at square 7. Now, it can go this way, it can go this way, it can go this way, it can go this way, and it can go this way. These are all the possibilities.

So, there are five possibilities. So, each move it will make with probability $\frac{1}{5}$. Similarly, if you look at a square, so you see why, the reason why these squares are colored differently. So, you see this pink colored square, there are three possibilities.
In these yellow colored squares, there are five possibilities. Now if you look at a square here, now there are you will see there are eight possibilities, it can go here, here, here, here, here, here, so all possibilities are there. If it is a square in this green region, then each of, so the probability that it will go to any of these squares is $\frac{1}{8}$. That is the mechanism or that is how the king will make its move.
And you can clearly see that if let X_n denote the position of the king after the nth move, then X_n is a Markov chain, where squares of the chessboard are the states. So, again, because where it will be in the next step only depends on where it is currently, so it is a Markov chain. And what are the states of the Markov chain? Sorry, what are the states of the Markov then?
So, there are 64 states because there are 64 squares. Let me just clean up this square. So, then X_n is a Markov chain with squares of the chessboard, where squares of the chessboard are states. Now from the pink colored squares, the kink can move to three adjustment squares, each with probability $\frac{1}{3}$, this is precisely what I explained just now.
From here, it can go to three adjacent squares, which is this, this and this, and the probability of going to each square is $\frac{1}{3}$. From yellow-colored squares, the king can move to five adjacent squares, each with a probability $\frac{1}{5}$, again. So, these squares, these squares, and these squares, there are five possible directions to go, and the probability of going in each of these directions is $\frac{1}{5}$ each. So, that is for the yellow squares.
And for green colored squares, the king can move to 8 adjacent squares each with a probability $\frac{1}{8}$. So, again, if it is a square in this green region, then there are 8 possible directions to go and it goes in each of these directions with a probability $\frac{1}{8}$ each. So, thus, what is the transition probability matrix it is p_{ij} is equal to $\frac{1}{3}$ if i is pink and j is adjacent to i.
And I have explained what is the meaning of being adjacent to a square, p_{ij} is equal to $\frac{1}{5}$. If i is yellow and j is adjacent to i, it is equal to $\frac{1}{8}$ if i is green, and j is adjacent to i. Now, you can easily check from here that says the king can move in any direction, the Markov chain is irreducible. So, starting from any square, you can go to any other square. So, the Markov chain is irreducible.
Now, if I define λ_i in this way, so $\lambda_i = 3$ if i is pink. So, remember, so here, I am just grouping the states into three groups, pink-colored states, green colored states and yellow-colored states. So, for, if i is a pink-colored state, I am defining my $\lambda_i = 3$, I am defining it to be 5 if i is yellow, and it is equal to 8 if i is green, then it is easy to see that λ and P satisfy the detailed balance equations for all i, j.
Why is that? Because again, now, let us see for if say i is pink, so i is pink, then what, then what will be λ_i, it is 3, and what is p_{ij}, it is $\frac{1}{3}$. So, this $i = 1$. Now, j, j again, so j can be either green, yellow or pink, but say, for example, j is yellow fine, but then I have defined my λ_j. So, because j is yellow, so λ_j will be 5 because of the definition. So, it will be 5.
And again, what is p_{ji}, p_{ji}, so again, so here it is p_{ij}, but does not matter. So, if the starting

state is yellow, it is $\frac{1}{5}$ then it will be again $5\frac{1}{5}$, which will be again equal to 1. So, both sides will be equal to 1 if I define my λ_i'i in this way. So, the detailed balance equations are satisfied by this row vector. So for, where the pink states are 3, the yellow states are 5 and the green states are 8.

So, now, since this is, this satisfies the detailed balance equation. So, now if I just, so this is a stationary measure, now so if I just divide by the sum of all λ_j's, then this will be the unique stationary distribution because it is a Markov, irreducible Markov chain. So, if there is a stationary distribution that is the unique stationary distribution. And so, we have found out the solution to the detailed balance equation.

So, if I now just divide by the sum of all λ_j's, I will get the unique stationary distribution. Now, remember, what was the original question that starting from 1, how long on average will it take to return to 1. So, basically, you are interested in starting from 1, if T_1 is the first return time or the first passage time to 1. So, you are interested in $E_1(T_1)$, but we know this theorem that if it is, so again, this is a finite-state Markov chain, so it is irreducible, it has these stationary distributions.

And stationary distribution and this mean return times are connected in the sense that $\pi_i = \frac{1}{E_i(T_i)}$. So, $E_1(T_1) = \frac{1}{\pi_1}$, but here what is π_1, here it is basically, so it should be the $\sum_j \frac{\lambda_j}{\lambda_1}$. Now, there are 4 squares, 4 pink squares, 24 yellow squares, and 36 green squares. So, $4 \times 3 + 24 \times 5 + 36 \times 8$. If you do this calculation, you will see you will end up with 140. So, that is the answer. So, now, just a few remarks.

So, we started from here. Now say, suppose we started from say a yellow square, say 4, then the same calculation, so if we are interested in $E_4(T_4)$, then what it will be, it will be $\sum_j \frac{\lambda_j}{\lambda_4}$. But λ_4, what is it? Because it is a yellow square, it will be 5. So, you will see the answer will be different. Similarly, if you are looking at a green square, say, for example, if I look at $E_{25}(T_{25})$, in that case, you will divide by 8.

So, you see now, so starting from this pink square, the long, the average time it takes to come back to 1 is much, will be much longer, because here you are dividing it by 3, but if it is from a yellow square, you are dividing it, remember λ_4 is 5, so you are dividing it by 5, so it will be slightly less and if it is a central square, it will be even less, because you are dividing it by 8. So, these are some remarks.

But anyway, the important thing is, so here we have seen again, many things, we have seen that this particular Markov chain here the detailed balance equations has a solution using that we got the stationary distribution. And as soon as we have the stationary distribution, we can say things about mean return times. And that is what the question was about.

So, you see, again, initially this looks like a very difficult question to answer, but using the Markov chain, and these detailed balance equations we have answered this in a much simpler way. So, you see, it is a very, interesting example of Markov chains of detailed balance equations, and all those things. So, that is all. That is all. So, we will stop here today. Thank you all.