Discrete-time Markov Chains and Poission Processes
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Lecture 26
Some Problems

Hello everyone, welcome to the 26th lecture of the course Discrete Time Markov Chains and Poisson Processes. So, in the last lecture, we saw several properties of the exponential distribution. So, the first property was what we call as memoryless property, which said that, if X has exponential distribution then the probability that X greater than t plus s given that X is greater than t is simply just the probability X greater than s. So, the distribution does not have any memory, that was the first important property. The second property was if you have i.i.d. exponentials that means independent and identically distributed exponential random variables $X_1, X_2, ..., X_n$. And if you look at their sum, then it has what is called gamma n lambda distribution. And we also define what is a $\Gamma(n, \lambda)$ distribution is, although you should be familiar with what $\Gamma(n, \lambda)$ is, but we also for completeness we gave the density of $\Gamma(n, \lambda)$ distribution, which was the second property.

The third property was again if you are given some i.i.d. exponentials with say parameters $\lambda_1, \lambda_2, ..., \lambda_n$ and if you are looking at the minimum. So, $\min_i X_i$ is again an exponential distribution with parameter $\sum_i \lambda_i$. So, the minimum of exponential is again exponential, but now, the parameters get added up. And the fourth property was like again you are given $X_1, X_2, ..., X_n$ independent exponentials then, like the probability that the i'th random variable will be the minimum among the n random variables that probability is given by $\frac{\lambda_i}{\sum_i \lambda_i}$. So, these are the four properties that we saw in the last lecture. Today, we will work out a few problems based on those properties. So, let us start.

So, the first problem. Consider a post office that is run by two clerks. Suppose that Mr. Amar enters the system, like suppose that when Mr. Amar enters the system, he discovers that Akbar is being served by one of the clerks and Mr. Antony by the other clerk. So, in a post office, there are these two clerks. When Amar enters, he sees that one of the clerk is serving Mr. Akbar and the other one is serving Antony. Now, suppose also that Mr. Amar is told that his service will begin as soon as either Mr. Akbar or Mr. Antony leaves. So as soon as one of the clerks becomes free, Mr. Amars service will start until then he will have to wait. Now, if the amount of time that a clerk spends with a customer is exponential with mean $\frac{1}{\lambda}$. What is the probability that of the three customers Mr. Amar is the last to leave the post office? So, that is the problem.

So, when Mr. Amar enters, he sees that Mr. Akbar and Mr. Antony are being served by the two clerks. Now, he is told that as soon as one of them finishes their service, Mr. Amars service will start. Now, you are being asked what is the probability that Mr. Amar will leave the post office lost? That is the question. And the information that is given to you is that the service times or the amount of time a clerk needs to serve a customer is exponential with mean $\frac{1}{\lambda}$. Now, when we define the exponential distribution, we said that exponential with parameter λ, but if you recall, I also said that if X has expertise, so if x has distribution $Exp(\lambda)$ then the expectation of X is $\frac{1}{\lambda}$. So, if you are told what is the mean of the exponential, then also you know what the parameter is, it is just the inverse of that. So, here you are told that is exponential with mean λ which means basically it is exponential with mean $\frac{1}{\lambda}$.

So, the distribution is exponential with parameter λ. So, for an exponential distribution if you are told it is mean, then the parameter is $\frac{1}{mean}$. So, you will see, in many questions you may find it like it is written that a certain random variable has an exponential distribution with mean something. Instead of saying that it has an exponential distribution with parameter something, it is it may be told that it has an exponential distribution with mean something. So, if you are given the mean, then the parameter is just $\frac{1}{mean}$.

So, here the mean is $\frac{1}{\lambda}$. So, the parameter is λ. That is one point. Now, you are being asked what is the probability that of the three customers Mr. Amar is the last to leave the post office. Now, how will that happen? That will happen say among Akbar, Antony say, Akbar leaves first, then Mr. Amar will go to this clerk and Mr. Antony is still in service. Now, Mr. Amar will leave last provided.

Now, this remaining service time of Mr. Antony is less than the total service time of Mr. Amar. Let me repeat this again. So, again this can, so suppose Mr. Akbar leaves first, then Mr. Amar will go to this clerk. Now, who are in service, Mr. Amar and Mr. Antony. Now, Mr. Amar will be the last person to leave the post office if Mr. Antony finishes before Mr. Amar, that means what if the remaining, because Mr. Antony was already in service, so, if the remaining service time of Mr. Antony is less than the service time or total service time of Mr. Amar. So, you have to find out what is the probability that such a thing happens.

So, what I said is written here. So, Amar will be the last person to leave the post office if his service time is longer than the remaining service time of the person who is still in service when he joins service. So, again, when I was explaining I said like suppose Mr. Akbar leaves. So, it is also possible that not Mr. Akbar but Mr. Antony leaves first, but then Amar will come to this clerk, and then again, the comparison will be between Amar and Akbar.

So, Amar will be the last person to leave the post office, if his service time is longer than the remaining service time of the person who is still in service when he joins service. So, that person can be either Amar or it can be either Akbar or Antony, it does not matter, but who is so still in service when he joins.

So, the person who is still in service his remaining service time should be lesser than the service time of Mr. Amar or in other words, Mr. Amar service time should be longer than the remaining service time of the person who is still in service when Mr. Amar join service. But now, since the service times are exponentially distributed, you know that exponential distribution has this memoryless property.

So, if you are asked okay, how, what is the remaining service time? So, how much long will it take to complete the service? If you ask that question, and if I have to answer that question, that question is independent of the fact that how long either Mr. Amar, Akbar or Mr. Antony is being served. So, how long he is being in, he has been in service does

not affect that how he is remaining service time, that is because we are assuming that the service time has exponential distribution and exponential distribution has this memoryless property.
So, since the service times are exponentially distributed by the memoryless property, the distribution of the remaining service time of the other person, wherever it is either Mr. Akab or Mr. Antony, depending on who leaves first, the other person will again be Exp(λ). So, that is because if X has exponential distribution, then $\mathbb{P}(X > t + s | X > t)$ is just simply $\mathbb{P}(X > s)$.
So, how long the person has been in service does, has no effect on the remaining service time. So, that is the meaning of memoryless property. So, since we are assuming that the service times are exponentially distributed, so, the remaining service time will again be exponential, again have exponential λ distribution. So, now, Mr. Amar has exponential λ distribution and the other person also has the remaining service time that also has exponential λ distribution.
Now, you want that Mr. Amars time should be higher than the service time of the other person. So, it is basically what you want that the other exponential should be minimum among the two exponentials. So, now, there are, these two exponential random variables one service time of Mr. Amar and the other the service time of the other person who is in service. So, what you want is that the other persons should be the minimum one.
Now, you already know. So, if you are given to exponential that the probability that one will be minimum is just parameter of that divided by the sum of the parameters, but here everything is λ. So, the required probabilities $\frac{\lambda}{\lambda+\lambda}$, which is equal to half. So, here you are using the fourth property of exponential that we learned in the previous lecture, that if you are given say n independent exponential random variables, again, everything here is independent and independent is very, very important.
So, if you are given n independent exponential random variables, the probability that the i'th random variable will be the minimum among them is $\frac{\lambda_i}{\sum \lambda_i}$. So here, there are two independent exponential random variables. So, the probability that one of them will be minimum is the parameter of that divided by the sum of the parameters, but here, everything is the same parameter.
So, it is $\frac{\lambda}{\lambda+\lambda}$, which gives you half. So, the final answer is half. So, the probability that of the three customers Mr. Amar is the last to leave the post office is half.
So now, continuing, we move on to the second problem. So again, you have post office and clerks. So, suppose you arrive at a post office having to clerks at a moment when both are busy, but no one else is waiting. So again, the situation is similar to the previous problem. So now you arrive at a post office, which again has two clerks. And when you arrive, you see that both the clerks are busy, so they are serving one, one customer each and no one is waiting.
Now, you will enter service when either clerk becomes free. Again, so the situation is similar to the previous problem. Now, if service times for clerk i is exponential with mean $\frac{1}{\lambda_i}$, $i = 1, 2$. So, here, so in the previous problem, the service times were all same, but here clerk

1 has service time exponential with mean $\frac{1}{\lambda_1}$ and clerk 2 has service time exponential with mean $\frac{1}{\lambda_2}$.

So basically, it is exponential with parameter λ_1 and exponential with parameter λ_2. So here again, it is the distribution, the exponential, it said exponential with mean something. So, if you have to find the parameter, just invert the mean. Now, what is the question? The question is to find $\mathbb{E}T$ where T is the amount of time that you spend at the post office. So, the total amount of time from the time you enter the post office until the time you leave the post office. So, you enter the post office, you see two clerks serving two people, one, one people each. So, you wait. Now, once one of the clerks become free, you go to that clerk, then you get your service and then finally you leave. So, what is the total amount of time that is, so if T is the total amount of time that you spend in the post office, then this question is asking you to find $\mathbb{E}$ of T or expectation of T or mean of T.

So, let R_i denote the remaining service time of the customer with clerk i. So, when you enter the post office, you see that two people are already in service. So, what is important is their remaining service time. But again, the service times are exponential. So, by the memoryless property, now just R_1, and so the remaining service time is again exponential with the same parameter.

So, R_i's independent again because these are two different clerks serving. So, the service times are independent, independent exponential random variables with parameters λ_1 and λ_2 respectively. So, again we use the memoryless property to claim that the remaining service time is same as the, as the same distribution as the original service time, that is memoryless property. So, now you are interested in $\mathbb{E}$ of T. So, again we do it via conditioning.

So, expectation of T we write it as expectation of T given this event $R_1 < R_2$, $R_1 < R_2$ means clerk 1 becomes free first, so, you will basically go to clerk 1. So, the remaining service time of the first customer is less than the remaining service time of the second customer that means, the first customer is the first to leave or again here first and second means I like the customer who is with clerk 1 and the customer who is with clerk 2.

So, R1 less than R2 means the customer with clerk 1 leaves first as compared to the customer with clerk 2. So, in this situation, you will join clerk 1. So, expectation of T given R1 less than R2 times probability of R1 less than R2. So, that is one situation. The other situation is when the customer with clerk 2 leaves first. So, that is R1 is greater than R2 or R2 is less than R1. So, expectation of T given that event times probability of that event.

So, in this case you join clerk 2. Now, this first quantity we will calculate in the next slide, but you know what probability $R_1 < R_2$. Again, it is basically there are two exponentials. What is the probability that one of them is the minimum? So, now R_1 is exponential with parameter λ_1. So, this probability is the parameter of R_1 divided by the sum of the parameters.

So, you get $\frac{\lambda_1}{\lambda_1+\lambda_2}$. Again this we will calculate later, but the other one that probability $R_1 > R_2$ that will be just $\frac{\lambda_2}{\lambda_1+\lambda_2}$. So, you get in the numerator the parameter of the second exponential. So, here again we are using the fourth property that we learned in the previous lecture. Now, we need to calculate these two things.

Now, if S denotes your service time, then $\mathbb{E}(T|R_1 < R_2)$. So, what will be, so remember T is the total time you spend in the post office. So, we can divide it in two parts the time you wait and then the time you get service. So, how much time you wait? You wait R_1 amount of time, why? Because the customer with the first clerk leaves first and then you just immediately join clerk 1.

So, in this situation when $R_1 < R_2$, so, first we write T as $R_1 + T$, why? Because $R_1 < R_2$. So, your waiting time is R1 plus your service time is S. Now, we know that conditional expectation is again linear. So, we break it into two parts. So, $\mathbb{E}(R_1|R_1 < R_2)+\mathbb{E}(S|R_1 < R_2)$. Now, if you are told that $R_1 < R_2$ that means, $R_1 = \min(R_1, R_2)$ and you know that the minimum of two independent exponentials is again an exponential, but what happens the parameter gets added up. So, now, R_1 has $\text{Exp}(\lambda_1+\lambda_2)$ distribution because, you are told that it is a $\min(R_1, R_2)$, that is where you are using the fact that $R_1 < R_2$, which means $R_1 = \min(R_1, R_2)$ in that situation.

Remember in general R_1, so without any information R_1 has exponential λ_1 distribution, but if you are told that $R_1 = \min(R_1, R_2)$, since minimum of two exponential is again exponential, but now the parameter changes, the parameter becomes sum of the parameters. So, under the information that $R_1 < R_2$, R1 has exponential distribution, but now with parameter $\lambda_1 + \lambda_2$.

So, the unconditional distribution is exponential λ_1, but conditioned on the event that $R_1 < R_2$, now it has distribution $\text{Exp}(\lambda_1 + \lambda_2)$. So, that is the effect of this extra information or the conditioning. So, if R_1 has exponential $\lambda_1 + \lambda_2$, then you know what is its mean, its mean is just 1 over the parameters. So, you get this. And in this case, now, see, now, this information that $R_1 < R_2$ tells you that you are being served.

So, now, how are we going to use this information that $R_1 < R_2$, this tells you that you are served by clerk 1. Now, for clerk 1 the service time is exponential with parameter λ_1. We know that the service time or the time clerk 1 takes to serve a customer has an exponential distribution with mean $\frac{1}{\lambda_1}$ or in other words, it has an exponential distribution with parameter λ_1.

So, since you are being sold by clerk 1, and why, how you know that you are being served by clerk 1, that is because you are given the information that the customer with clerk 1 left earlier or left first. So, you are being served by clerk 1. So your service time has exponential distribution with parameter λ_1, so the mean is $\frac{1}{\lambda_1}$.

By a similar argument, you will get that $\mathbb{E}(T|R_1 > R_2)$ will be $\frac{1}{\lambda_1+\lambda_2}$, in this case, R_2 will be the minimum. So, in this case, R_2 will be $\min(R_1, R_2)$. But again, R_2 will have the same distribution, $\text{Exp}(\lambda_1 + \lambda_2)$, because now R_2 is the minimum. But now, when R_1 is greater than R_2, or R_2 is less than R_1.

That means you are now served by clerk 2, because the customer with clerk 2 left first. So, you will go to clerk 2. And for clerk 2, the service time is exponential with mean 1 over lambda 2. So, this second thing will be $\frac{1}{\lambda_2}$. So, you do basically the same argument as you did for this case $\mathbb{E}(T|R_1 < R_2)$. So, you do for this case, similarly, $\mathbb{E}(T|R_1 > R_2)$.

So, the first term will remain unchanged. But the second term will be $\frac{1}{\lambda_2}$ because now you

are being served by clerk 2 and not clerk 1. So now, so you now know everything you know these two values, you already knew these two, these two. So now if you plug in everything in this formula, that $\mathbb{E}T = \mathbb{E}(T|R_1 < R_2)\frac{\lambda_1}{\lambda_1+\lambda_2} + \mathbb{E}(T|R_1 > R_2)\frac{\lambda_2}{\lambda_1+\lambda_2}$.
So, these two when you plugin, you get finally that expectation of T is $\frac{3}{\lambda_1+\lambda_2}$. So, that is your meme, the expectation of the time you spend in the post office, your waiting time plus your service time. So, the main trick here or the main step here is breaking this expectation of T in this way.
So, this conditioning argument is very, very important in many, many places in probability. So this, so we see yet another example where this conditioning argument is very helpful. So, we first condition on this event whether $R_1 < R_2$ or $R_1 > R_2$, and then we calculate each component and we finally get that expectation of T is $\frac{3}{\lambda_1+\lambda_2}$. So, we will stop here today. Thank you all.