Discrete - Time Markov Chains and Poisson Processes
Professor Subhamay Saha
Department of Mathematics
Indian Institute of Technology, Guwahati
Module: Poisson Processes
Lecture 31
Conditional Arrival Times

Hello, everyone, welcome to the 31st lecture of the course Discrete Time Markov Chains and Poisson Processes. So, in the last couple of lectures, we learned about the concept of Poisson thinning, and we solved a couple of problems using that concept. So, today, we will learn another new property of the Poisson process, which is about conditional arrival times. So, let us start. So, before starting with the concept, let us first look at an example or a problem.

Suppose that you are told that exactly one event of a Poisson process has occurred by time t. So, since this is time t, and you were told that in this interval 1 event has occurred. Now, you are given the information that there has been 1 event within the time interval 0 to t. Now, the question is that under this information, what is the distribution of the time at which the event occurred? So, you are told that there has been exactly 1 event in the interval 0 to t. And now under this information, you are asked about the conditional distribution of the time of occurrence of the event.

So, that is what this problem is asking you to do. So, let us see how we can solve this. So, basically, we are interested in this quantity. So, we are interested in the distribution of the time at which the event occurred, and this is 1 event. So, this is basically the first event. So, we are interested in the probability that the time of occurrence of the first event is less than or equal to s given that there has been exactly 1 arrival up to time t. So, obviously, this $s \leq t$, because you know that up to time t there has been an arrival to the time of arrival or the time of occurrence of the first event should be less than or equal to t.

So, if it is $s > t$ then probability T_1. So, if say, if $s > t$, then the $\mathbb{P}(T_1 \leq s|N(t) = 1)$ is just 1 because you are already told that there has been an arrival by time t. So, obviously, the time of that arrival should be less than or equal to s if $s > t$. So, the interesting part is when $0 < s < t$, that is the interesting part. So, $\mathbb{P}(T_1 \leq s|N(t) = 1)$ equal to 1. Now, what does that mean? So, this is again a conditional probability. So, we write down the formula of conditional probability.

So, this is just $\frac{\mathbb{P}(T_1 \leq s \cap N(t)=1)}{\mathbb{P}(N(t)=1)}$. Now, you are saying that the first arrival. So, if this s the first arrival that has happened within this time s. And you also know that up to time t, there has been exactly one arrival. So, combining these two what we get is that one event in 0 to s and 0 events in this part, open interval s close t because, you are saying that there is the first arrival happened within time s and up to time t there has been exactly one arrival, that means, there is 1 event in 0 close to 0 s and 0 events in open s close t divided by, sorry, not $N(t) = s$, but rather it should be $N(t) = 1$.

So, that is a type I. So, it should be probability $N(t) = 1$. But now, when I write it in this way, this 2 intervals, $[0, s]$ and this interval opening close to these are disjoint intervals, Poisson process has this property of independent increments. So, events in these joint intervals are independent. So, you get probability. So, this probability just basically breaks into probability of 1 event in 0 to s times probability 0 event in s to t. So, this is equal to this by independent increments, independent increments property.

Now, 1 event in 0 to s that is just $\lambda s e^{-\lambda s}$ that is the probability of 1 event in the interval 0 to s and 0 events in s to t again, it has this property of stationary increments. So, the number of events in an interval just depends on the length of the interval. So, that probability is $\lambda s e^{-\lambda(t-s)}$ and the denominator is just probability $N(t) = 1$, which is just $\lambda t e^{-\lambda t}$.

And now, if you do the calculations, do the cancellations you will see you will get $\frac{s}{t}$, but this is what is $\frac{s}{t}$ remember, we are looking at $0 < s < t$ and we have already seen that for $s > t$ it is 1. So, this is equal to $\frac{s}{t}$ in this regime. So, but that is nothing but the distribution function or the cumulative distribution function of a uniform distribution on the interval $[0, t]$.

So, given that there has been one arrival up to time t, the time of that arrival, sorry given that there has been 1 event up to time t the time of occurrence of that event will be uniformly distributed on the interval $[0, t]$. So, if you are told that there has been exactly 1 event or exactly say one arrival in the interval $[0, t]$, then the under this information, the conditional distribution of that occurrence or time of occurrence of that event or time of arrival is uniformly distributed on the interval $[0, t]$ that is what we have solved or found out via this problem. Now, motivated by this.

We come to this theorem, which basically generalizes what we saw in the previous slide. So, given $N(t) = n$, so, now, you are told that within time 0 to t there has been n arrivals. Now, under this information the n arrival times remember in the previous problem, this is there was just one arrival, we were trying to find the distribution of that arrival time, but now, there has been n arrivals. So, we are now trying to find out the joint distribution of these n arrival times. So, the n arrival times $S_1, ..., S_n$ have the same distribution as the order statistics corresponding to n independent random variables uniformly distributed on the interval $[0, t]$.

So, you take n independent uniform $[0, t]$ random variables, so, $U_1, U_2, ..., U_n$ then what this theorem is telling you if you are looking at this conditional distribution of $S_1, ..., S_n$ given that $N(t) = n$ that has, so, this has this. So, this has the same distribution as this quantity has the same distribution as $U_1, U_2, ..., U_n$ where each U_i is uniformly distributed on the interval 0 to t. So, each Ui has a uniform $[0, t]$ distribution.

So, given that there has been in arrivals up to time t, the conditional distribution of $S_1, ..., S_n$, the n arrival times is the same as the distribution of order statistics of n uniform $[0, t]$ random variable. So, in the previous problem, we saw that if it is a single arrival then the time of arrival is uniformly distributed on $[0, t]$.

And now if you are looking at $S_1, ..., S_n$, now, obviously, you can understand that why that this order statistics is coming into the picture because the S_1 has to be less than S_2 the time of arrival of the first customer has to be less than the time of arrival of the second customer or the time of occurrence of the first event has to be less than the time of occurrence of the second event, so which has to be further less than the time of occurrence of the third event and so on.

So, you see, this now, order statistics thing comes in very naturally. So, what this theorem is telling you is that given that there has been n arrivals up to time t, the conditional distribution of $S_1, ..., S_n$ is the distribution of order statistics of NIID, uniform $[0, t]$ random variables, that is what this theorem is telling you. So, again you can easily see that this generalizes what we saw in the previous slide or in the previous problem. So, now, let us see the proof of it.

So, remember, we are interested in the conditional density or the conditional joint density of $f(s_1, ..., s_n|N(t) = n)$. Now, in order to find that conditional density observe that if you take $s_1 < ... < s_n$ obviously, all the n arrivals should happen before time t because you are told that up to time t there has been n arrivals like the previous problem the time of occurrence of the first event must be less than or equal to t because you are told that up to time t there has been one event.

Similarly, here because we were told that up to time t there has been n arrivals, so the time of or the time of up to time t there has been n events. So, the time of occurrence of these events should be less than t all these n events should be less than t and obviously, the time of occurrence of the first one should be less than the time of occurrence of the second event and so on. So, you can see why we are looking at such a condition. Now, for this for given $S_1, ..., S_n$ which satisfies this constraint the event that the time of first arrival S_1 the time of second arrival S_2 and so on.

The time of nth arrival is S_n and up to time t there has been exactly n arrivals that is equivalent to the event that the first $n + 1$ inter-arrival times satisfy this. Now, if the first arrival is at time S_1 that means the first inter-arrival time is S_1. Now, if the first second arrival time is S_2 that means the second inter-arrival time is $S_2 - S_1$. So, going like this way, since its capital the time of Nth arrival is s_n. So, the inter-arrival time the nth entered inter-arrival time should be $S_n - S_{n-1}$.

But now, you are also told that up to time n there has been exactly n arrivals. So, between S_n and t, there should not be any arrival. So, that so, let me draw a picture that should make things clearer. So, this is 0 say this is s_1, this is s_2, so on. This is s_n and this is t. Now, so, this is the first inter-arrival time, this is the second interval arrival time and so on. Now, also says up to time t there has been n arrivals. So, you know that in this interval there has been 0 events or in other words, the n plus first inter-arrival time should be greater than the length of this interval which is $t - s_n$.

So, why this last thing is the first n terms you can easily understand because once you are told the arrival times you can easily find out what are the inter-arrival times are, now you are also told that up to time t there has been exactly n arrivals and the nth arrival happened at time S_n. So, in this interval, which is from $[s_n, t]$ there has been no arrival which means the $n + 1$ if inter-arrival time is greater than $t + s_n$ because in this interval there has been no arrival or no event. So, the $(n + 1)$th inter-arrival time should be greater than $t + s_n$ because

remember what was $(n+1)$th inter-arrival time that is basically the time between the nth event and the n plus first event, you know the nth event happened at time s_n, and up to time t the n plus first event has not happened. So, the nth inter n plus first inter-arrival time should be greater than $t - s_n$.
So, you see that these now, this event that the first arrival time first arrive or the first event happened that time S_1 the second event happened at time S_2 and so on, nth happened at time S_n and in $t = n$ that is equivalent to these conditions on the first $n+1$ inter-arrival times and once we get conditioned like why are we trying to get conditions equivalent condition in terms of inter-arrival times, that is because, it is very easy to deal with interarrival times because they are independent and identically distributed.
The arrival times are not do not have such an easy structure that of independence and identically distributed but if you look at the inter-arrival times, they have a much simpler or easy to deal with structure namely i.i.d structure. So, that is why we converted the event in terms of arrival times to an either we converted it into an equivalent event in terms of inter-arrival times. Now, once you observe this thing, now, it becomes very easy to find the conditional density that we are trying to find. So, since the inter-arrival times are i.i.d with exponential lambda, so, we are assuming that it is a Poisson process with rate λ the conditional density of $f(s_1, ..., s_n | N(t) = n)$ is this quantity.
Clean it up a bit. So, remember, so, this we are trying to find out the conditional densities. So, this you can think of as this exactly that $f(S_1 = s_1, ..., S_n = s_n | N(t) = n)$, that is the conditional density. Now, again, if you use the formula of conditional probability, so, this conditional density should be this event that $\frac{event(S_1=s_1,...,S_n=s_n \cap N(t)=n)}{\mathbb{P}(N(t)=n)}$. So, that should be the conditional density.
But now, once we have observed the fact that this event is equivalent to this event. So, now see, this is basically the probability that $T_1 = s_1$, this is the probability that $T_2 = s_2 - s_1$. Again so, this is slightly heuristic because these are all continuous random variable. So, I am writing just the density, but the density is not actually probability, but again this heuristic very much works in every in most of the situations. So, here this is basically you can think of it as $T_2 = s_2 - s_1$ and so on.
This is $T_2 = s_n - s_{n-1}$ and this is the last thing that $T_{n+1} > t - s_n$ that probability is just $e^{-\lambda(t)}$, so, it is just an exponential λ and a variable taking value greater than $t - s_n$ that is just equal to this and this is the probability that $N(t) = n$. Now, if you do all the cancellations, you just end up with this and we have already seen this is true, if this is true otherwise, this is equal to 0.
And now, you can easily see that this is the density or the joint density of order statistic corresponding to like NIID uniform random variables having NIID uniform $[0, t]$ random variables. Recall the density what was so if say X_1,X_2,...,X_n were IID continuous random variables having the common density small f then the joint density of the order statistic was given by $n!$ times product of the densities. So, the provided that in a region where like in this kind of a region, so, remember what is the density of uniform $[0, t]$ is just $\frac{1}{length\ of\ the\ interval}$. So, which is t, so, n factorial times 1 over t to the power n. So, you precisely get n factorial

over t to the n. So, just recall what was the formula for joint density of order statistic corresponding to NIID continuous random variables having common density f. So, here the density is 1 over t and hence you get that this is the density in this region that is because here like for order statistic, this thing has to be true, the first order statistic should be less than second order statistic and so on. And also you get this less than t that is because here you are told that there has been n arrivals up to time t.

So, in this region, you get this density and 0 elsewhere and this is precisely the density of order statistic corresponding to NIID uniform 0, t random variables. So, we have proved the theorem. So, what we have shown is that if you are told or if you are given the information that there has been n arrivals up to time t, then the conditional distribution of the n arrival times S1, S2, Sn has the same distribution as the order statistic of NIID, uniform 0, t random variables. That is the main theorem and this theorem generalizes the problem which we solved in the first slide, which was where there was just one arrival.

So, if it is one arrival there is nothing like uniform there is nothing like order statistic. So, it is just uniform in the first slide, it was just uniform on 0, t, but now there are n arrivals or n events. So, it is order statistic corresponding to NIID uniform 0, t random variables and why this order statistic is coming into the picture that is because of the simple reason that the first arrival time should be less than the second arrival time and so on, which should be fine a less than the nth arrival time and all these should be within the interval 0 to be.

And what was the main thing that we did in the prove the main thing was to convert the event in terms of arrival times to event in terms of inter-arrival times and the advantages inter-arrival times are IID having distribution exponential lambda. So, once we write the event in terms of arrival times as even in terms of inter-arrival times now, we can easily write down that probability. So, here basically we are writing down the density, but heuristically that is what we did. So, fine that way we got the conditional density.

So, now, let us see few problems based on the concept that we just learned namely the conditional distribution of arrival times. So, let us start with the first problem. Customers arrive at a bank according to a Poisson process with rate lambda per hour. So, here the unit is per hour. Suppose, 2 customers have arrived during the first hour. So, this is the information that you are given that within the first hour that means, within one unit of time, there has been 2 arrivals or there has been 2 events that you are told.

Now, the question is what is the probability that both arrived during the first 20 minutes and the second problem is at least one arrived during the first 20 minute. So, this is t, now, remember, so, your rate is given in terms of hour, so, 20 minutes is just 1 by 3. So, here this is t equal to one and this is 1 by 3. So, you are told that in this there has been 2 events. Now, the first there has been 2 events on this interval of length 1, now you are being asked what is the, so, under this information, what is the conditional probability that both the events happen within this time interval of 1 by 3.

And the second one says what is the probability that there has been at least 1 arrival in this time interval of length 1 by 3, let us see how to solve this again, we will use the fact that given that there has been n arrivals up to time t, the conditional arrival times are unique are had the conditional distribution of the arrival times is same as the distribution of order statistic corresponding to NIID, uniform 0, t random variable. So, here n is 2, t is equal to 1, 2 arrivals on a time interval of length 1.

So, the probability that both arrivals will happen in the first 20 minutes. Now, both arrivals will happen in the first 20 minutes. That means, if so, the arrivals, the conditional distribution of the arrivals is same as the distribution of order statistic. Now, both arrivals are happening within first 20 minutes means, if I look at the maximum that is less than 20, if both of them are less than 1 by 3, that means the maximum is less than 1 by 3. So, the probability that both the arrivals will happen in the first 20 minutes is same as the probability that the maximum of 2 independent random variables having uniform 0, 1 distribution is less than 1 by 3.

Because you are saying that both the arrivals happened by time 1, so you are asking about the probability that both arrivals happen by time 1 by 3, but that is same as the probability that the maximum is less than 1 by 3. Now, so that is same as the maximum of 2 independent random variables having uniform 0, 1, 0, 1, because here t is 1, distribution is less than 1 by 3, but you know how to find the distribution of maximum, so, the probability that 2 max of say U1 and U2 is less than or equal to 1 by 3.

But since, if the maximum has to be less than 1 by 3, that is same as both has to be less than 1 by 3, these are independent. So, this is just simply U1 less than or equal to 1 by 3 times, U2 less than or equal to 1 by 3. So, the first one is 1 by 3 because it is uniform 0, 1. Second one is also 1 by 3, so, you get 1 by 9. So, the probability that both the arrivals will happen in the first 20 minutes is 1 by 9. Now, let us look at the second problem. So, here we are asked that at least one arrived during the first 20 minutes.

Now, if we can find the probability that no one arrived within the first 20 minutes, then if I just do 1 minus that probability that will give me my required probability. So, this is 1 standard trick. So, in many places in probability, or in many problems in probability, if you are asked to find out at least something, you find the probability of the complement of that and then do 1 minus that probability.

Again, this is a very standard trick in probability. So, here we are being asked the probability that at least 1 arrival happened within time 1 by 3. So, first let us try to find out the probability that no arrival happened within time 1 by 3. But what is the meaning of that that no arrival happened within time t that means both arrivals happen after time 1 by 3, that means the minimum of them is greater than 1 by 3. So, the probability, so the required probability is 1 minus here. The term probability is missing. So, it is 1 minus the probability that minimum of 2 independent random variables having uniform 0, 1 distribution is greater than 1 by 3.

So, if both the arrivals has to happen after time 1 by 3, that means the minimum is greater than 1 by 3, that means the minimum of 2 uniform 0, 1 random variables is greater than 1 by 3. If we can find that probability then the probability that at least 1 arrival happens within time, 1 by 3 is just 1 minus that probability. That is precisely what we have done here. Now, again, since we have 2 independent random variables, the probability that the

minimum of them is greater than 1 by 3 is just that each of them is greater than 1 by 3, these are independent. So, it is just probability that the first random variable is greater than 1 by 3.
And again, probability, both of them are greater than 1 by 3. So, that is just 1 minus 1 by 3 whole square. That is because the distribution is uniform 0, 1. And they are independent, the whole square is coming because of independence, and 1 minus 1 by 3 is coming, because both of them has uniform 0, 1 distribution. So, the probability that a uniform distribution or a random variable having uniform distribution takes value, any form 0, 1 distribution takes value greater than 1 by 3 is 1 minus 1 by 3.
Now, this has to be true for both of both the uniform random variable, so you get 1 minus 1 by 3, the whole square. And now you have to do 1 minus because you are actually asked about at least 1 arrived during the first 20 minutes, but what you calculate by 1 minus 1 by 3 whole square is none of them arrives within the first 20 minutes. So, if you do 1 minus that, you get the required probability. So, finally, you end up with the answer 5 by 9. So, the probability that at least one of them arrived within time within the first 20 minutes, you get it to be 5 by 9 and how do you calculate?
You calculate it as 1 minus the probability that none of them arrived within the first 20 minutes, and that probability is just simply the minimum of 2 independent uniform 0, 1 random variables is greater than 1 by 3 and that is just 1 minus 1 by 3 whole square. So, that is the very simple idea. So, you see, this problem was very simple in one case, you needed to find out what is the probability that maximum of 2 independent uniform random variables is less than or equal to 1 by 3 and in the other problem, you needed to find out that the minimum of 2 independent uniform 0, 1 random variables is greater than 1 by 3. So, you see, once you have the theorem, that the conditional distribution has the same distribution as the order statistic of uniform distribution, then solving this problem becomes very, very simple.
So, the previous problem was a very simple problem. Now, we look at a slightly more difficult problem. So, suppose insurance claims arrive at an insurance form according to a Poisson process with rate lambda. So, you know about insurance claims. So, suppose there is an insurance problem, which sells insurance and claims come according to a Poisson process with rate lambda. Now, the successive claim amounts are independent random variables having mean meal, now each claim that will have an amount what is the claim amount?
Now, you are told that each claim is independent, and the mean is are the average size of the claim his mu. And also the size of the claim is independent of the time of arrival of the claim. If I tell you when the claim has arrived, that does not tell you anything about the size of the claim, the amount of the claim. So, the claims have IID. And again, we the problem does not specify what the actual distribution is, but it only just specifies what the mean is, but the for the problem that will be enough.
Now, let Si and Ci denote respectively the time and the amount of the ith claim. So, Si is the time of arrival of the ith claim and Ci is the amount of size of the ith claim. Now, I define Dt to be equal to i running from 1 to Nt e to the power minus alpha Si times Ci. So,

this is basically the discounted or other total discounted claim up to time t. Now, why do you do this discounting that is because of the fact that there is some time value of money, if something is worth 100 rupees today, that is not worth 100 rupees say 1 year from now. Because if you invest this 100 rupees in a bank or a post office that will grow to something more so, there is what is called time value of money or something that is worth 100 rupees in future is not worth 100 rupees today it is less. So, there is what is called this time value of money. So, if you are going to get some payment or some amount in the future and you are trying to analyze it today, so, you will have to discount it or you will have to keep into you will have to bring in the fact that there is this concept of time value of money because if you invest money today, it will grow in future like it will increase in a future date because of that, we are doing this discounting this is just for time value of money.

So, this is the total because you are doing the analysis today, but the claims will come in future. So, basically, say from the insurance firms point of view, this is very important to know say up to a time t what will be the total amount of claim that will arrive but again because you are doing the analysis today, so the future claims you will have to just discount it because of the fact that there is something called time value of money. So, this is the total discounted claim that arrives up to time t. So, again this is a very important quantity from the insurance forms point of view.

Now, the question is fine expectation of Dt or the average of Dt, average total discounted claim which will arrive up to time t. So, you can easily understand that such a thing is very important from an insurance forms point of view. So, again this is a very real life problem. Now, how to do that, now, we are interested in we want to find out the expectation of Dt again we will use the trick that we have used so many times in this course before and which is used many many times everywhere in probability which is this trick of conditioning. So, the expectation of Dt is the expectation of Dt given Nt equal to n times probability Nt equal to n.

So, this is just probability Nt equal to n and then you sum over n running from sorry, it is not sum from n it should be sum from 0, n running from 0 to infinity because Nt is a Poisson random variable. So, it takes values from 0 to infinity. So, we find the expectation by conditioning on the number of arrivals up to time t. So, first so, in order to find this now, we already know what probability Nt equal to n is. So, we need to find out what this quantity is?

For that let us denote by U1, U2 up to Un NIID random variables with uniform 0, t distribution again from where this is coming that is because you are conditioning on the fact that there has been n arrivals up to time t or n events up to time t. Now, we want to find the conditional expectation of Dt given the event that Nt is equal to n or given that there has been n arrivals, n events up to time t. Now, recall the expression for Dt. So, we have just written down that here.

Now, since Nt equal to n this quantity is basically equal to the expectation of summation i running from 1 to n because now, you are given that Nt equal to n, e to the minus alpha Si times Ci given Nt equal to n. But now, given that Nt equal to n, you know what is the distribution of a site that has the same distribution as the ith order statistic of n independent and identically uniform distributed uniform 0, t distribution. So, again here we are using the theorem that we learned in the second slide.

So, now and Ci is our independent of all this because Cis are independent of the arrival times. So, this Si I can replace by Ui and also now, so, in this case, I could not push the expectation inside the sum because this Nt was also a random variable but now, Nt is equal to n so, I have replaced Nt by n. So, this just simply becomes equal to this. So, there are many things that I am doing here. So, first I am plugging in Nt equal to n that is because it is given that Nt equal to n and also the condition distribution of S1 up to Sn given that Nt equal to n is same as the distribution of order statistic of n uniform 0, t random variables.

So, that is why it says we are interested in expectation replaced Si by Ui. And then I have just pushed the expectation inside the sum because now the sum is a deterministic sum. And now this condition will vanish because now I have already used the fact that under this conditioning Si has same distribution as Ui. Now, you know that this Cis are independent of this Ui is because that is because of the fact that Ci is independent of the arrival times. So, this expectation will just become the expectation of e to the power minus alpha Ui, times expectation of Ci, that is, because if 2 random variables, X and Y are independent, then the expectation of X, Y is expectation X times expectation Y.

So, that we are using here, now the expectation of Ci is equal to mu that is given. So, that just comes out because it is constant. So, I take that out of the summation. So, you get this is equal to mu times i running from 1 to n, the expectation of e to the power minus alpha Ui. So, here, I am using the fact that the size of the claims are independent of the time of arrival of the claims. So, the expectation of this just becomes expectation of e to the power minus alpha Ui times expectation of Ci and that expectation of Ci is equal to mu for each i. So, it comes out of the summation as a constant. But now see, whether, so when you are doing sum of some numbers, the order does not matter. See, 1 plus 2 plus 3 is same as 1 plus 3 plus 2 is same as 1, say 2 plus 1 plus 3. So, the order in which you are doing the summation does not change the actual sum. So, this sum, so this is basically you are looking at the sum of the order statistics. But this sum is same as this sum, because these terms are just a rearrangement of these terms. This each, so, if I look at so here, this sum is of e to the power minus alpha Ui.

But here this e to the power minus alpha Ui without the bracket is just a rearrangement of these terms. And since summation does not depend on the order, like 1 plus 2 plus 3, same as 1 plus 3 plus 2, which is same as 2 plus 1 plus 3, and so on, so this sum is actually equal to this sum. So, the difference here is now I have removed the bracket, because both these sums have the same terms. Same number of terms and exactly the same terms, only the ordering is different, but the ordering does not change the sum.

That is what we have used to come from this step to this step. But now, now again, this each Ui is identically distributed uniform 0, t random variables. So, the sum this sum will be n mu over t times integration 0 to t, e to the power, sorry, not lambda, but alpha. So, e to the power minus alpha x dx. So, mu is there, since each of these expectations is same, you get a n and then you just find the expectation of 1 term. So, remember, Ui has uniform 0, t distribution, so it has density 1 over t on the interval 0 to t.

So, and you are trying to find out this expectation, so, this t comes out and it just becomes integration 0 to t, e to the power minus alpha x dx. So, this lambda is a typo, it should be e to the power minus alpha x dx. And now if you just do the integration, you get in n mu over alpha t, 1 minus e to the power minus alpha t. So, we have calculated this term. So, this is where we have used all you we have used the fact that the conditional arrival, the conditional distribution of the arrival times given of the n arrival times given that there has been n arrivals up to time t has the same distribution as the order statistic of NIID, uniform 0, t random variables.

And in other steps, we have used the fact that this size of the claim is independent of the time of arrival of the claim. So, that is another thing we used. And one last thing that we used is that if when you add summing up some numbers, the order does not change, if you reorder those numbers, the sum does not change, or the sum is independent of the ordering of numbers. So, using all these, we get that this expectation of Dt given Nt equal to n is this.

So, finally, we are interested in the expectation of Dt, which is equal to n running from 0 to infinity n mu over alpha t, 1 minus e to the power minus alpha t times this probability that Nt equal to n. Now, this whole thing, this thing, this thing and this thing is does not depend on n, so, it comes out of the summation. So, what you get is just mu over alpha t summation n running from 0 to infinity n e to the power minus lambda t, lambda t to the n over n factorial, but that is just simply the expectation of a Poisson random variable with parameter lambda t.

So, this is nothing but we know that the expectation of a Poisson random variable with parameter lambda is just lambda. So, here the parameter is lambda t, so, the expectation is lambda t so, you just this is equal to lambda t. So, again this term is also there that 1 minus e to the power minus alpha t. So, now this lambda so, this t and this t gets cancelled and finally, you get this answer. So, things that are not dependent on n comes out of the summation and then you just notice that the summation is nothing but the expectation of a Poisson lambda t distribution.

So, which is equal to lambda t and you just then do the (())(47:07) cancellations. And you end up with this answer. So, ethe xpectation of Dt is lambda mu over alpha times 1 minus e to the power minus alpha t. So, you see, we have calculated so initially, it looked this kind of a complicated expression. But you see, the first thing we did do is we condition on the number of arrivals up to time t. And once we do that conditioning, then we use the fact that given that there has been n arrivals up to time t, the conditional distribution of the arrival times is equal to the distribution of order statistic of NIID, uniform 0, t distribution, so that we used use in order to calculate the conditional expectation of Dt, given Nt equal to n.

And then we use some small other elementary facts like that sum of numbers does not depend on the order of the numbers. And also we use the fact that the size of the claim is independent of the time of arrival of the claim using all these, we calculate the expectation of

Dt given Nt equal to n and then we again, plug that in the original formula, that expectation of Dt is sum over n running from 0 to infinity expectation of Dt given Nt equal to n times probability of Nt equal to n. And then we just do the algebraic manipulation. And finally, we get the answer.

So, again, the main idea behind this problem is this conditional arrival times, which we learned in todays lecture. So, we have solved 2 problems using this concept. The first one was relatively simple. The second one was slightly longer not difficult, but I will say slightly, the solution was slightly longer. But again, the central idea is this condition, distribution of conditional arrival times, given that there has been no arrivals up to time. So, we will stop here today. Thank you all.