Discrete-Time Markov Chains and Poisson Process
Professor Subhamay Saha
Department pf Mathematics
Indian Institute of Technology, Guwahati
Module Poisson Process
Lecture 34
Compound Poisson Processes

Hello everyone. Welcome to the 34th lecture of the course Discrete Time Markov Chains and Poisson Processes. So, today is going to be the last lecture of this course and we will end with another concept which is compound Poisson processes. So, let us see what it is.

So, a stochastic process Xt is said to be a compound Poisson process if it can be represented as this kind of a random sum. So, what is $X(t)$? $X(t)$ is $\sum_{i=1}^{N(t)} Y_i$, where what is $N(t)$? $N(t)$ is a Poisson process with some rate and Y_i are i.i.d random variables with some distribution which is also independent of $N(t)$. So, $X(t)$ is this kind of a random sum where this upper limit of summation $N(t)$ is a Poisson process with some rate λ and these summands are i.i.d random variables but also important thing is these Y_i's are also independent of $N(t)$.

So, if Xt can be written in this way, then we say that the process or the stochastic process $X(t)$ is a compound Poisson process. Now, remark, so, if $N(t)$ is Poisson with rate λ, so here if $N(t)$ is Poisson with rate λ then this expectation of $X(t)$ where $X(t)$ is of this form or in other words $X(t)$ is a compound Poisson process is given by $\lambda t \times \mathbb{E}Y_1$ and variance of $X(t)$ is given by $\lambda t \times \mathbb{E}Y_1^2$. So, here we will not prove this formula but I strongly encourage you to try proving these two formulas formula expectation of $X(t)$ and variance of $X(t)$ by conditioning on $N(t)$.

So, again, I strongly encourage you to do that exercise of finding the expectation of $X(t)$ and variance of $X(t)$ if $X(t)$ is of this form, where $N(t)$ is a Poisson process Y_i is our i.i.d random variables. Again, we are assuming here that Y_i's have mean and variance then only we can talk about mean and variance of $X(t)$ so, we are assuming that Y_i's have mean and variance or the first and I should rather say Y_i's have first and second moments.

So, we are assuming this then only these formulas are valid that expectation of $X(t)$ is λt which is basically expectation of $N(t) \times \mathbb{E}Y_1$ and variance of $X(t)$ is again λt which is the expectation or you can think of it as the variance of $N(t)$ because $N(t)$ is a Poisson Nt is Poisson λt. So, its mean and variance is the same, it is λt but one thing you should be careful about.

So, here it is not the variance of Y_1^2, the variance of Y_1^2 but rather the second moment of Y_1^2. So, one thing why I am Y_1, remember these are all i.i.d. So, the expectation of Y_i is same as Y_1 is same as the expectation of Y_i for all i and similarly, this is equal to expectation of Y_i^2 squared for all i.

So, since this is i.i.d, so this is just stated in terms of Y_1. So, the expectation of $X(t)$ is $\lambda t \mathbb{E}Y_1$ and the variance of $X(t)$ is $\lambda t \times \mathbb{E}Y_1^2$. So, you should be careful here. It is not the variance of Y_1^2, but the expected variance of Y_1 but the expectation of Y_1^2, the second moment of Y_1. So, again I will not work out this.

So, I have just given it as a formula. I will not prove these two formulas namely how do we get the expectation of $X(t)$ is $\lambda t \mathbb{E} Y_1$ and the variance of $X(t)$ is $\lambda t \times \mathbb{E} Y_1^2$. But I strongly encourage you all to try and prove these two formulas by conditioning this like a standard conditioning argument but here conditioning on Nt.
So, you will see that is where you will use all those facts that $N(t)$ is independent of Y_i and so on. But I am not going to do all those things in this course but I strongly encourage you to do those two exercises. But you should keep in mind these two formulas, the expectation and variance of a compound Poisson process if $N(t)$ is Poisson with rate λ.
Now, so, we have defined this new class of stochastic processes namely the compound Poisson process. Now, where does such thing such processes arise? So, let us see a couple of examples. So, suppose that buses arrive at a sporting event in accordance with a Poisson process and suppose that the number of fans in each bus is independent and identically distributed. Then, if I call then $X(t)$ is a compound Poisson process where $X(t)$ denotes the number of fans who have arrived by time t.
So, buses are arriving. So, you have a sporting event at some sporting venue say some stadium where some match is going to happen say buses are arriving according to a Poisson process with some rate λ say some rate and further the number of people that each bus brings that is i.i.d with some distribution, then if you are counting the number of fans who have arrived at the stadium up to time t and call it $X(t)$, then that will be a Poisson that will be a compound Poisson process. Why? because you can write it in this form that Xt so what will be the total number of people who have arrived up to time t or the total number of fans who has arrived up to time t? It should be just i running from 1 to $N(t)$, not 1 but sorry, 1 only.
So, yeah, that is how we wrote it. So, $X(t)$ is $\sum_{i=1}^{N(t)} Y_i$. And so, we will basically follow the convention. So, if $N(t)$ is 0 then the sum is 0. So, here you can say so, Nt can actually be equal to 0 in that case, what is this sum? So, if $N(t)$ is equal to 0, then $X(t)$ is equal or this sum we will consider say that this is sum is equal to 0 that convention we are following.
So, in that case, so, Y_i is the number of fans who come in the ith of the bus then up to time t there will be $N(t)$ arrivals of the bus. So, the total number of fans who will arrive up to time t will be given by this formula. So, you see again compound Poisson process arrives very naturally. Another example, is suppose customers leave a supermarket in accordance with a Poisson process.
If Y_i the amount spent by the ith customer for $i = 1, 2, ..$ are i.i.d then $X(t)$ is a compound Poisson process where $X(t)$ denotes the amount of money spent by the customers again not who arrived who left up to time t. So, basically, let me keep it as arrived only. For this one let me say that customers arrive. Again, it does not matter.
So, suppose customers arrive at a supermarket in accordance with a Poisson process if Y_i is the amount spent by the ith customer for i equal to 1, 2 and so on and suppose if that amount is i.i.d then if you denote by $X(t)$, the amount of money spent by the customers who spend up to time who arrived up to time t then again $X(t)$ will be compound Poisson because it will have this kind of explanation because here now, Y_i is the amount spent by

the ith customer, $N(t)$ is the total number of customers who arrived up to time t. So, the amount spent by the customers who arrived upto time t is given by this.
So, again you see this is a very real-life situation amount spent by customers in a supermarket. So, again that can be modeled by a compound Poisson process provided like the arrival process is Poisson and the amount spent is i.i.d. So, again, all these assumptions are needed only then you can say that it is a compound Poisson process but you can see, like, again, you can use compound Poisson processes to model very real-life scenarios fine. So, compound Poisson processes are not like just we have not defined such a process just like that. But those are important processes in the sense it can be used to model many real-life scenarios fine.
So, we will end with one example or one problem on the compound Poisson process. Suppose that families migrate into a territory according to a Poisson process with rate λ equal to 2 per week. Now, if the number of people in each family is independent and takes the values 1, 2, 3, 4 with these probabilities, then what is the expected value and variance of the number of individuals migrating into the territory during a fixed 5-week period?
So, again, here the unit is per week. So, per week, so the families migrate into a territory according to a Poisson process with rate 2 per week, fine. Now, each family that migrates will have 1, 2, 3 or 4 members with certain probabilities. Now, you are asked about the mean and variance of the number of individuals who migrate into the territory in a given period of 4 weeks. Now, let us see how we can be solved this using compound Poisson processes.
Now, let $N(t)$ denotes the number of families who migrate into the territory up to time t. So, before starting also remember, so here, the number of people in each family is independent. And again, this number is also independent of the arrival process. So, whether so a family, independent of everything else, will have 1, 2, or 3, 1, 2, 3, or 4 members, with probabilities, $\frac{1}{6}$, $\frac{1}{3}$, $\frac{1}{3}$ and $\frac{1}{6}$, respectively and you are asked what is the mean and variance of the number of individuals who migrate into the territory in a given period of 5 weeks.
So, let $N(t)$ denote the number of families who migrate into the territory up to time t. Then $N(.)$ is a Poisson process with rate 2 per week. That is because you were told that families migrate into a territory according to a Poisson process with rate λ equal to 2 per week. Let Y_i denote the size of the ith family that arrived into the territory.
Then so then you are given that these Y_i's are i.i.d and takes values 1, 2, 3 and 4 with certain probabilities. And it is also independent of the arrival process, then total number of individuals migrated into the territory during a fixed 5 weeks period is if you have to write it, then it is X equal to $\sum_{i=1}^{N(5)} Y_i$ Y_i because $N(5)$ will be the total number of families who will migrate in a given period of 5 weeks and Y_i is the size of the family. So, this will be the total number of individuals who migrate into the territory during a fixed period of 5 weeks.
So, now you have to find expectation and variance. Now, you already know how to findthe expectation and variance of compound Poisson processes. So, the expectation of $X(t)$ is $\lambda t \mathbb{E} Y_1$ and the variance of $X(t)$ is $\lambda t \times \mathbb{E} Y_1^2$. So, here $t = 5$ because the rate of arrival is 2 per week and you are asked for a 5-week period. So, here $t = 5$. So, the expectation of X is λt, so λ is equal to 2, t is 5 and now you have to find the expectation of Y_1. So, it takes

value 1 with this probability.
So, it is $1\frac{1}{6}$. It is 2 with probability $\frac{1}{3}$, so you get $\frac{2}{3}$. 3 with probability $\frac{1}{3}$ so you get $\frac{3}{3}$, and 4 with probability $\frac{1}{6}$. So, you get $\frac{4}{6}$. Now, if you do this calculation, you will end up with 25. And now for variance, what is the formula variance, that is $\lambda t \times \mathbb{E}Y_1^2$.
Now, again λ is 2, t is 5 now, you have to find expectation of Y_1^2, so, it is $1^2\frac{1}{6}$. It is 2 with probability $2^2\frac{1}{3}$, so you get $\frac{4}{3}$. 3 with probability $\frac{1}{3}$ so you get $\frac{9}{3}$, and 4 with probability $\frac{1}{6}$, So, you get $\frac{16}{6}$. So, that is just the expectation of Y_1^2, where Y_1 or Y_i denote the size of the ith family and if you do the calculation, you get this answer.
So, again, you see the main thing is we just wrote down the required thing as a compound Poisson process and then just we used the formula for mean and variance of compound Poisson process. So, that ends the course. So, just a very brief recap of what we have done in this course. So, we started with some preliminaries, and then we studied discrete time Markov chains in detail. So, we studied we started with the definition then the first important theorem was that the evolution of a Markov chain is completely specified by the initial distribution and the transition probability matrix.
Then, we saw this N step transition probability matrix and so on, then we talked about this classification of states recurrence, and transients, then we further subdivided recurrence into null recurrence and positive recurrence. Then, we looked at stationary distributions or invariant distributions, when it exists or about its unique existence, we analyzed all those things for finite-state Markov irreducible Markov chains for finite-state Markov chains in finite-state Markov chains, irreducible non-irreducible. So, we made a detailed study of existence and uniqueness of stationary distributions.
And then finally, we looked at some limiting theorems. So, that was Discrete Time Markov chains. Then, we learned about various properties of exponential distributions. So, all those properties are of independent interest, but in the context of this course, those were kind of prerequisite for studying Poisson processes. And then, after doing all those prerequisites, we started with Poisson processes.
We saw many properties of Poisson processes like it has in the stationary independent increments, the number of arrivals on a given time interval of length t is Poisson with parameter λt, if it is a Poisson process with rate λ, then we saw this Poisson thinning which basically says if you split a poison process then the splitted processes are again Poisson processes with appropriate rates and they are independent of one another. We also saw a kind of converse to this that if you have two independent Poisson processes and you merge them, then that is again Poisson process.
And then, we saw about this conditional. So, before that, we also saw that what are the inteinter-arrivales of a Poisson process, so that turned out to be exponential distributions and that is basically the connection between Poisson proceeds and exponential distributions. And then using that we also got that the nth arrival time has gamma distribution fine.
So, all these things we saw, then we saw this Poisson thinning, superposition of two independent Poisson processes, then we saw this conditional arrival times, so that if you are given that up to a time t there has been n arrivals, then what is the conditional distribution of

these n arrivals are these occurrence of these n events. So, we saw that is basically that basically has the same distribution as the order statistic corresponding to n i.i.d uniform 0 t random variables.

And then finally, we ended today's class with this concept of compound Poisson processes which again, we saw is helpful in modeling real-life scenarios and then we ended with a simple problem using the concept of compound Poisson process. So, during the course, we have solved plenty of problems so that the concepts gets clearer.

And also, in mathematics, solving problems is very, very useful. Unless you solve many, many problems, you will not be able to grasp the concepts properly. So, that is why we gave special emphasis on solving problems during the course we have solved many problems. So, that is all. So, hopefully, the course was enjoyable and hopefully, this will be useful to you in the future. So, with that, let me stop. Thank you all for attending the course. Thank you again. That is all.