Discrete-Time Markov Chains and Poisson Processes
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Lecture 4
Stochastic Processes

Welcome to the fourth lecture of the course, Discrete-Time Markov Chain and Poisson Processes. In the last three lectures, we have gone through some of the basic concepts of probability, and then random variables, and then jointly distributed random variables, and finally conditional distribution and conditional expectations.

Today, we will start with the main part of the course which is the stochastic processes, more discrete-time Markov chain, that is what basically we are going to start. And before that, I will give you a brief introduction of general stochastic processes. Markov chain is basically a special case of stochastic process. So, what is a stochastic process? Stochastic process is basically nothing but a collection of random variables. So, the definition goes like that, let T be a set, which is either the set of all-natural numbers along with 0, or the interval 0 to infinity, 0 is included here. So, it is either the set 0, 1, 2 up to 0, 1, 2, 3, 4 and so on so forth, or T is a set which is the interval $[0, \infty)$, here 0 is inclusive. Then a stochastic process is a collection of real valued random variable. For a stochastic process the notation we are going to use is this one, and this is a collection of real valued random variable. What is the role of T? T is works as a index set, T is basically works as a index set. And when I have T is equals to 0, 1, 2 so on so forth. Of course, this collection X_t such that, $t \in T$ can be also written as $X_0, X_1, ...$. But, when T is $[0, \infty)$, I am not able to write in this manner, because in this case T is a uncountable set. But, when T is a at most countable set, I can write in this manner. So, the stochastic process is nothing but I have a index set and then I am talking about a collection of random variables X_t decides that $t \in T$. That is basically the stochastic process, the collection of random variable is a stochastic process. Here what happens is that, the first point I have already discussed that each X_t is a random variable, for each $t \in T$. Each X_t is a random variable, then a very very important intuition of the stochastic process is the second part, which tells that this T is our time. I will come to some example then it will be more clearer. So, we will think that t is some time. So, when capital T is $\{0, 1, 2, ...\}$, then this case I am talking about the time 0, time 1, time 2 and so on so forth. Whereas, when t is a uncountable set $[0, \infty)$, the interval 0 to infinity, then I am talking about a continuous time. And we will think that t as a time that is an important notion in case of stochastic process. And X_t is the value of something at that time, which we generally call state of the process at the time t. So, t is time and X_t is the value of the process or state of the process at some time t. Then, when this T is 0 to 0, 1, 2, 3 and so on so forth, we call that X_t is said to be a discrete time stochastic process. Why it is discrete time stochastic process? Because the times are discrete now. It is 0, then 1 then 2, then 3, then 4. So, we have a discrete integer, we have integer values, so, they are basically discrete. So, this is called discrete time stochastic process. When T is 0, infinity the interval $[0, \infty)$, so time is continuous in this case, and in this case, we call the corresponding stochastic process as continuous time stochastic process. So, in this slide we have learned what is basically stochastic processes, which is nothing but a collection of random variable, and the collection is made based on some index. i index set will be, either of the form $\{0, 1, 2, ...\}$. Or the T will be $[0, \infty)$, the interval 0 to infinity. And keep in mind one point I should mention here that see 0 is included in both cases, and that is a convention that to start the stochastic process from the time 0. That is a convention. Of course, we can change that, but this is the convention and this can help us to write many things. So, this convention we will follow that the first time the time start from 0, whether it is a at most countable, it is a countable time or whether it is a continuous time, whether it is discrete time or continuous time does not matter, the time start from time 0. And then we learn what is a discrete time stochastic process, if the time is discrete, then we call the corresponding stochastic process as a discrete time stochastic process. If the time is continuous, the corresponding stochastic process we will call continuous time stochastic process. In the first part of this course, we are going to talk about the discrete time stochastic process. And in the end of the course, we will talk about continuous time random stochastic process. So, let us move.

So, let us now talk about some examples. So, first example I have written is nothing but price of a gold on each day. So, suppose I start taking reading from maybe 31st January of the year 2006. So, if first reading is taken on this day, then the next reading on the 1st February on 2006, then 2nd February 2006 and so on it is going on. So, what we are going to do is that the first time point will map it to 0, second time point will be mapped to 1, third time point will be mapped to 3 and so on so forth. Now, if the gold price is denoted by P, then P0 is the price at the time 0, that means the price of the gold on the 31st January 2006. P1 is the price of the gold at on 1st February 2006, this will be mapped to 2. And then P2 is the price of gold on that second February 2006 and so on and so forth. Finally, when we take all this collection that $P_0, P_1, P_2, ...$ everything together, then this is a markup. This is a stochastic process, and then P_n is nothing but a price of the of the gold, price of gold on nth day. And when I am writing nth day, keep in mind that it is basically starting from the first day. First is 0, then we are we are proceeding, so n can take value $\{0, 1, 2, ...\}$. So, the idea is that first the real date that 31st January 2006 or maybe 31st, June 2019, whatever it is does not matter. We have to map the time real time to either to T and T can be either 0, 1, 2 so on and so forth, or T can be the interval $[0, \infty)$. And then we are going to talk about what is the state of the stochastic process at that time t. So, in this case, of course T has the form 0, 1, 2, and then the stochastic process can be given in this form. Now, let us go to the example-2. Example-2 is again of the same nature that it is the minimum temperature of each day, so I can again do in this way. So, for example, maybe if I try to talk about citing the reading. I have first reading has been taken on 15th of maybe March in 1970. And then 16th March 1970, then 17th March 1970, I will map this time to 0. The first time the first reading has been taken on 15 March. So, the first point first-time point has been mapped to 0, second time point has been mapped to 1, third time point has been mapped to 2. Then, maybe temperature maybe I denote it by $X_0, X_1, X_2, ...$ So, we have now again a stochastic process, $X_0, X_1, X_2, ...$ So, in this case, so what is X_n? X_n is basically the minimum temperature of nth day. Again, notice that when I am talking about this nth day, this is with the reference from the day of first reading. So, in this way I can have stochastic processes in different real-life examples. So, in both the cases both the example, example-1 and example-2 we have discussed, we have seen that the time index is nothing but this set. Time index is nothing but the set, it is the $\{0, 1, 2, ...\}$. Now, let us talk about example-3. Basically, time index is a little bit different than what we have discussed till now.

So, in the third example, we are going to talk about something where the thing is like that total number of visitors in a park up to time t of a day. So, for example, suppose this park opens at the at 9 am, so let the park opens at 9am. So, that 9 am time is mapped to 0. It opens at 9 am, so we map the time 9 am to 0, then the time is a continuous one, so I cannot write it exactly. But in this case, you can see that T is basically nothing but 0 to infinity now because time is now continuous one. Time is now continuous one, and then we have maybe second and the milliseconds everything I can talk about, and we have a continuous time in this case. And people can come any time, it is not that people can come at 9am, then 10 am, then 11 am, times are not discrete here. The visitors are not coming at some discrete times, but they can come any time. So, in this case, more appropriate thing is to take T is a continuous set, which is 0 to infinity. And in this case with this if I say that Xt is the is the number of total number of visitors total number of visitors up to time t, then of course this Xt is such that t belongs to a capital T, which is in this case 0 to infinity. This is a stochastic process. And this is of course a different example, then a little bit different example then the two first thing. The first two our t was countable say 0, 1, 2 and so on and so forth that most countable set. But, in that last example, our t is a continuous time which is the interval 0 to infinity. But, keep this thing in mind whether it is example-1, example-2 or example-3, whatever, we basically need to map the real time to either t this t or to this t. We have to do this one and the starting time we will always keep it to be 0. With that, let us proceed.

And the next one basically we will see some terminology here. So, first term we are going to talk here about is the state space. So, what is state space? State space is nothing but the collection or the set of all possible values that can be taken by a stochastic process is called the state space of that stochastic process. So, if I again go back to the previous example, see the first example, the price can be anything, any positive value can be our price. So, that means in this case, the first example the state space could be anything positive, the price of gold can be anything. But, it has to be a positive one, if I talk about the price in rupees, or if I talk about price in any currency does not matter, it has to be a positive one. Similarly, the minimum temperature that should be a real number, because the minimum temperature that of course, depends, on which area we are talking about. Of course, if I talk about the temperature in Chennai, of course, the minimum temperature in Chennai is always positive. So, in that particular case, the state space will be the positive, positive real numbers. But, if I talk about the temperature at Siachen, the minimum temperature at Siachen, of course, that go below 0. And so, in that case, the temperature will be any real number, whether it can be positive it can be negative, it can be 0 anything is included. So, in the case of example-2, when I am talking about minimum temperature, it depends on which region I am talking about minimum temperature. So, depending on that, the state space can be only positive real numbers, or it can be any real number positive, negative, and 0 are included. In the case of the third example, the state space is exact integers, because the number of visitors is always some integer values. So, that means, in the third case, the state space is an integer. So, that means the state space is all possible values or values that can be taken by a stochastic process is called state space. The collection of all possible values that can be taken by a stochastic process is called the state space. Then the next point is a convention that we are going to follow in this course, that we will take the state space is always almost countable. We will take the state space is almost countable, and most of the cases not always, we will take that state space, which we denote by s. The state-space which we will denote by $\{0, 1, 2, ...\}$. That is not always but in most cases, we will take this one. In most cases, we will state the state space to be 0, 1, 2. So, this is a convention for this particular course, because this is the first course, so, we are only going to talk about the atmost countable state space. The other state, the general state space that contiguous state space, and all those things, will not discuss in this course. But of course, they can be part of some advanced courses in the stochastic process. So, then the next one is that when state space is something atmost countable. What does this mean? This basically means that X_t is a discrete random variable for all t. So, X_t is a discrete random variable for all $t \in T$, which is basically we are going to use in this course. And if X_t is a discrete random variable, that means it has PMF. Each X_t has PMF and the PMF of the first time X_0, the PMF of X_0 that is the state at the point t equals to 0 at the first time, that is called initial distribution. So, initial distribution is nothing but the distribution or PMF of the random variable X_0. Notice that all because we have assumed that the state space is atmost countable, that implies that X_t is a discrete random variable for all t. So, all X_t has PMF, and the PMF of X_0, the initial time the state at the initial time that is X_0, and the PMF of X_0 is called initial distribution. So, finally, any realization of X_0 is called a sample path. So, the sample path is basically when we observe something for example when we observed the price of gold, we will see a price of today may be some amount of money maybe, maybe 600 rupees tomorrow is 601 rupees, day after tomorrow is 599 rupees. So, these values are observed values and these observed values are called the realization, and any realization of a stochastic process is called the sample path. (Refer Slide Time: 20:40)

So, let us now talk about some more examples. So, this example, again we go back to one of example, which we have talked about earlier, that is the example-3, where people are coming to the park. So, in this case, as I told that maybe Park is open at 9 o'clock 9 am in the morning. So, the time 9 am is our time 0, and at the time 0, there is no individual, they are in the park, so, I have this is the time 0. So, I have that point the stochastic process, the value of the stochastic process or the sample realized value of the stochastic process is 0. Then, at the time 9:30 am, maybe the first visitors come, so 9:30 am time is mapped to t equals half, because half an hour has been elapsed. So, maybe I can map it to $t = \frac{1}{2}$ and that then $t = \frac{1}{2}$, one person comes, so the realization jumps. So, the sample path jumps to this one. And then at the time 10:15 am, then the second visitor comes, so, that means 10:15 am will be mapped to 1.25, and then another visitor comes, so I have another jump here. The temple path has another jump here and it is coming here. Then, maybe at 10:30 two more visitors come now, two more visitors come together, maybe a couple has come to the park. So, both of them come together and this 10:30 will be mapped to 1.5. And at the time 1.5, I have a jump, and the height of the jump now is 2. Here the height of the jump is 1, here also the height of the jump is 1. So, in this way, it is going on throughout the day, and I can accumulate this data. I can write down these data to have a realization for a stochastic process. And of course, these jumps where we have the jumps that may change every day, maybe the next day, maybe this is the graph for Monday. But when we will go for Tuesday, the graph may change completely, the sample part may change completely. Maybe the first customer has come at the time 9:15, and maybe at the time, 9:15 two customers come together. So, if I go for Tuesday, Tuesday case, and on Tuesday if at 9:15 am two visitors come, I have a jump here. And the height of the jump will be 2 in this case. Then, it again continues, maybe then it has a jump somewhere, and somewhere it goes like that. So, of course, the sample path will change, based on which day I am talking about. Monday it is different, maybe Tuesday it is different, or maybe it is different day to day, better two way in this way, it is different from day to day. Today is something, tomorrow it will be something, yesterday it was something, and so on so forth.

So, let us now talk about the next example example-5, where we are talking about throwing off a fair die. So, the fair die has 6 faces, and any die has 6 faces. And then fair die means that the probability of occurring any face is $\frac{1}{6}$ or equal probability of occurring any 6 any face is equal. And because there are 6 faces, the probabilities are $t = \frac{1}{6}$. So, now suppose the Xn denote the outcome of the n throw. So, if I throw for the first time, whatever the outcome may be, the first time I throw, I got a 5. So, that X_5, the realized value of X_1 is equal to 5. Then, the second time I throw I got a point 1, so that means the realization at the time 2 is basically 1 and so on so forth, it is going on. Now, notice that if I define it in this way, I have not defined X_0 here. So, I have defined X_0 separately in this case, and by convention I take that X_0 equals 0, that before throwing I have nothing, so X_0 equals 0 that is the convention generally we take. But, of course, you can define X_0 to be any other value, that is not a problem. Now, I throw if I do this way, then it continues, and finally I have a stochastic process $\{X_0, X_1, X_2, ...\}$. And for that stochastic process, we have this is the state space. Again, keep in mind that X_n belongs to the set $\{1, 2, ..., 6\}$, for all $n \geq 1$, because it is the outcome of a throw of a die. But, why does this 0 come here? 0 comes here because the X_0 has been defined to be 0. So, this example actually emphasizes that state space is the set of all the values taken by the stochastic process. So, all the values taken by the $\{X_0, X_1, X_2, ...\}$ X_t, all the values taken by the X_t. So, 0 we need to incorporate here because X_0 can take value 0 with probability 1, and then 1 to 6, this value taken by any of the rest of the X_n's. So, in this case, of course, X_n is our independent because the outcome of the first toss first throw does not have any effect on the second throw or third throw, or fourth throw. Similarly, the outcome of the second throw does not have any effect on the rest of the throws. So, in this case, X_n's are independent, there is it is a very very intuitive one. The initial distribution of X of the stochastic process just recalls that, that PMF of X_0 is called initial distribution, the probability that the PMF of X_0 is called the initial distribution. So, in this case, what is the distribution of X_0? The distribution of X_0 is that it is a probability that X_0 equals 0 is 0. So, the PMF of X_0 if I use this notation, the PMF of X_0 is nothing but 1, and this is 1 if x equals 0, and it is 0 otherwise. Just recall that PMF at the point small x is nothing but the probability that x naught equals to small x. So, when x is small x is 0, then the probability of X_0 equals 0 is 1. And for any other value of x, that probability has to be 0. So, the initial distribution is this, the PMF of X_0 is this, and just recall that we have defined this δ in one of the previous lectures. And the δ_c is defined by 1, if x equals to c delta c at the point x maybe at the 0, otherwise. So, in this case, if I take c equals 0 I get exactly this function, I get exactly this f function. That is why we have written that the initial distribution is delta naught, in this case, the initial distribution is δ_0 in this case. So, the last part again, of course, is very very simple to see, it is nothing but because it is a fair die, the probability of any face coming up is $\frac{1}{6}$. And note that in this case, $n \geq 1$. So, we have that probability X_n equals to k equals to $\frac{1}{6}$, for all $n \geq 1$, and all $k = 1, 2, ..., 6$. So, in this slide, we talked about a stochastic process, which comes from a very simple experiment that throwing a die again and again. And if we do this way, then basically X_n is defined to be the outcome of the n throw, and if the through this manner, if I define X_0 is not defined there. So, we define X_0 separately, and then we have seen some of the properties of this stochastic process. What is the state space? What is the initial distribution? We will talk about this concept in this particular case.

Let us proceed to the next example. So, the next example is again the same experiment, this throwing of die, but that definition of the stochastic process, the definition of X_n changes. So, what is X_n now? X_n denotes the number of sixes in the first n throws. So, if the first throw is 1, then X_1 is basically 0, the second throw is 3. The outcome of the second throw is 3, X_2 is 0, the outcome of the third throw is 6, then X_3 equals 1. Then, the outcome of the fourth throw may be 5, then X_4 remains 1, and so on it is going. So, I hope the hat definition of X_n is fine with everyone, that it is the number of sixes in the first n throws. For example, in this case, this is the first throw, the outcome is 1. So, in the first up to the first throw, there is no 6, so X_1 is 0, then the outcome of the second throw is 3. So, up to the second throw, there is no head, sorry, no 6, so that means the X_2 is 0. The outcome of the sixth throw is 6, the outcome of the third throw is 6. So, up to the third throw, there is only one 6, so X_3 is 1, the outcome of the fourth throw is 5. So, up to the five, fifth, sorry, up to the fourth throw, there is only one 6, so X_4 is 1. So, it goes in that manner. So, then finally I have a collection of this form, where X_0 again defined with the convention that it is X_0 equals to 0. Again, the idea is that if we do not throw anything, there is no success. So, X_0 equals 0 is the convention people take, so with that, I have a stochastic process. And in this stochastic process, we can see that the state space is nothing but $\{0, 1, 2, ...\}$, which is coming from here or 0 other values also can take value zeros, then 1, 2, 3, 4 in this way it is going on. So, if you compare it with the previous example, you see in the previous example, the state space was a finite set. There were seven points in the state space. But, in this current example, the state space is not a finite one, but a countable one, so, it is a countably infinite set. In this case, X_n is not independent now, and this is very easy to see because once you reach one, the next will be not less than one. So, once you reach a point, the next level will be higher than t. So, clearly, there has to be dependent, see somewhere. What is the initial distribution? Initial distribution is the same as before it is δ_0, because, again, X_0 takes value identically 0. And so, the probability that X_0 equals 0 is 1. What about the other values? Now, in this case, because they are not independent, constitutive throws X_n's are not independent. So, it is easier to write this conditional probability. What is this conditional probability? So, first, let us talk about what is the probability of X_0? $(X_1 = j|X_0 = 0) = 0$. Because X_0 can take value 0, I will start with 0. So, what does this mean? This means that what is the probability that in the first throw, maybe I talk about first, first 0. First 0, what is $(X_1 = j|X_0 = 0) = 0$? So, what does this mean? This means that, I start from 0. This I always start from 0, and what is the probability that in the first throw, I do not get a 6. So, this probability will be $\frac{5}{6}$. Then, if I try to find out the probability that X_1 equals 1, given X_0 equals 0, that basically means that the next throw, the first throw give me a 1, sorry, give me a 6, the first throw give me a 6. I start from 0, and after the first throw, I have X_1 equal to 1, which means I have 1, 6. And that tells me that the first throw has to be 6. So, I have this probability is $\frac{1}{6}$, because it is a fair die. Then, if I try to find out what is the probability that X_1 equals two, given X_0 equals 0. Now, after the first throw, the number of 6 cannot be 2, the number of 6 can be atmost 1. So, this probability has to be 0. Not only this probability, even if I take probability X_1 equals to j, given X_0 equals to 0, but this probability will also be 0 for all $\{j = 2, 3, ..., 6\}$. None of them are possible. So, in this manner, I can go for the general one here. And the general one is the previous one that $X_n - 1$ if is I, and I am talking about what is the probability, next one will be j. The X_n will be j, that is the next one will be j. So, after $n - 1$th throw, I have i number of sixes. And then I do one more throw and I try to find out what is the probability that after the nth throw, there are j sixes. So, again in the same manner, if you see that $j = i+1$, what does mean? That means basically, nth throw is a 6. If $j = i+1$, that means I have increased by 1. So, that means that the nth throw is a 6, so the probability is $\frac{1}{6}$. If $j = i$, whatever i value I have, the next stage after the next throw I have the same number of sixes out of n throws. That means the nth throw is not a 6. And this shows that this is, the probability is $\frac{5}{6}$, it is not said it can be anything between 1, 2, 3, 4, 5 up to 5. So, that means there is five possibilities out of 6 possibilities and the probability is $\frac{5}{6}$. And you take any other value of j that is not possible from i, at after $n - 1$th throw. I cannot go to move to any value of j, where $j \neq i$ and $j \neq i + 1$, so, that probability is 0. So, this is about this example. So, we have in this lecture we have seen what is a stochastic process. Then, we have learned some terminology like what is the state space? What is the initial distribution? Those things we talk about, then we have seen some examples. And through the example, I try to try to illustrate that in what kind of scenarios we can have stochastic processes in practice? And how the state space are localized in different stochastic processes? And then, of course, we talked about initial distribution. We talk about some probabilities like in the last example, we talk about this probability. In the previous example, throws are independent. We talk about these kinds of probability and so on so forth. With that, I stop. Thank you for listening.