Discrete-Time Markov Chains and Poisson Processes

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Lecture 7

Chapman-Kolmogorov Equations: Examples

Welcome to the 7^{th} lecture of the course Discrete-Time Markov Chains and Poisson Processes. In the last lecture, we have talked about Chapman-Kolmogorov equations and in this lecture we are going to see some of the uses of the Chapman-Kolmogorov equations through several examples. So, just recall what were Chapman-Kolmogorov equations. We have, the Chapman-Kolmogorov equations basically talk about m-step transition probabilities and or in m-step transition probability matrix.

Now, what is Chapman-Kolmogorov equation?. Let p_{ij}^{m+n}, this is basically the probability of going from i to j in $m+n$ steps. So, this is a $m+n$ step transition probability. And Chapman-Kolmogorov equation basically tells us that this quantity can be written as $P(X_{m+n} = j | X_0 = i) = \sum_{k \in S} p_{ik}^{(m)} p_{ki}^{(n)}$. And I have to take over all possible values of k in that state space. So, the idea was that this is the probability of going from i to j in $m+n$ steps. So, here I am writing that, well, I am going from i to j, but through a path which moves through k at the mth step and then that path can go through any k in the state space, so I am taking the sum over all possible values of k belongs to S. And then we have discussed that this can be written in matrix notation like that,

$$P^{(m+n)} = P^{(m)} P^{(n)}$$

i.e., it is basically $P^{(m)}$ into $P^{(n)}$ and where $P^{(m)}$ or $P^{(n)}$ is the m-step or n-step transition probability matrix for the Markov chain. Then finally, we talk about another thing that $P^{(n)}$ can be written as P^n. So, that basically tells us to find out n-step transition probabilities or and n-step transition probability matrix, I can start with one-step transition probability matrix or I can take the nth power of the one-step transition probability matrix. So, that is basically this. So, we will see this three points in the last lecture, in the lecture number 6. Now, we will move to see how we can use these ideas, these knowledges to solve or to find out probabilities in different cases. Of course, along with these things we will go to use some other ideas also. So, let us see how it proceeds.

Let us talk about the first example, example number 11. So, this example we have seen earlier. Example goes like this, suppose that the chance of rain tomorrow depends on previous weather conditions only through whether or not it is raining today and not on past weather condition. So, this says that I can make a Markov chain out of this weather condition. Suppose that if it is raining today, then it will rain tomorrow with probability $\alpha = 0.7$ and if it is not raining today, then it will rain tomorrow with probability $\beta = 0.4$. Calculate the probability that it will rain four days from today, given that it is raining today. So, of course, we can answer this particular problem using conditional probabilities, but the computation of that one is a little cumbersome. So, we can solve this problem with Markov chain also. As we seen in one of the previous example that this problem can be easily writen in terms of a Markov chain and to do this one what we did as follows, we take two states 0 and 1. 0 means raining, 1 means not raining and then we define X_n, which is the state of the nth day. So, $X_n = 0$ means that on the nth day it is raining and $X_n = 1$ means on the nth it is not raining. So, this one we have seen earlier also, one of the example and we have argued because of this given condition that this X_n is a Markov chain and clearly in this case state space will consist of two points, 0 and 1. And finally, what we try to find out?. We try to find out the probability that it will rain four days from today, given that it is raining today. From today, after fourth day it is raining and we try to find out that probability. It will rain on the fourth day from today, given that it is raining today. That probability we are trying to find out. So, this probability is basically nothing but on the fourth day in terms of the Markov chain. So, X_4 that is equals to raining, raining means 0, given it is raining today, and today is basically our X_0. This probability I need to find out and this probability is exactly this one, so $p_{00}^{(4)}$ that we basically try to find out. How we can proceed to this?. First what we will going to do is that, we will try to find out what is the one-step transition probability matrix corresponding to the Markov chain X_n. And this one is fairly simple, because we have again seen this one earlier also. So, this is p_{00} in one-step, moving from 0 to 0 that basically mean that given today it is raining, what is the probability that tomorrow it will be raining. And that probability is given here that if it is raining today, then it will rain tomorrow with probability $\alpha = 0.7$. So, I have written 0.7 here; then the row sum of a one-step transition probability matrix has to be 1. So, clearly this has to be 0.3 because the row sum has to be 1. Similarly, if I go for the second row now, the first quantity will be that from 1, I am moving to 0. That means, it is not raining today. What is the probability that it will be raining tomorrow?. So, this probability is given again here, if it is not raining today, then it will rain tomorrow with probability $\beta = 0.4$. So, you wrote 0.4 here, then row sum has to be 1. So, this quantity has to be 0.6. So, this way we can easily write that transition probability matrix in this particular example. Now, our aim is to find out this probability. To find out this proobabiility, what we will calculate?. We will calculate $P^{(4)}$ because the first entry of $P^{(4)}$ is nothing but $p_{00}^{(4)}$. So, we have calculated this and how we can calculate the standard process that we know that $P^{(4)}$ can be written as P^4; that we know that $P^{(n)} = P^n$. So, I have to find out the fourth power of this matrix and that I can easily do like that, I can first find out $P^2 = P \cdot P$, so this matrix needs to be multiplied by itself to find out P^2 and then $P^4 = P^2 \cdot P^2$. So, that means that first I will multiply this by P matrix with itself, I will get P^2, then I will multiply P^2 with itself and I will get P^4. And this calculation I am not going into in this case because it is a simple matrix calculation that you have to do the row multiplied by column and write it into a proper place and you have to do this thing repeatedly. So, this is a simple one or alternatively what you can do is that you can put this P matrix in any mathematical software and then mathematical software can calculate P^4 for you. So, both the way you can do?. You can do it by hand, it will be a little bit cumbersome or alternatively you can take help of a mathematical software like MATLAB, R or anything like that. You can can proceed, so just take it. Believe me that this is basically P^4 matrix and once we get this P^4 matrix, this entry is $p_{00}^{(4)}$. So, this is the probability. So, the conditional probability that today it is raining, what is the probability that on the fourth day it will be raining from today, that conditional probabilities are given by 0.5749. So, probability is a moderate probability in this case that with almost more than half probability, there is more than 50 percent chance of having rain fourth day after if we have rain today. If the assumption of this particular example hold true, then there is about 57 percent chance of having rain fourth day down the line from today if it is raining today. So, this is one example or one case where we can use Chapman-Kolmogorov equation directly to find out required probability. Of course, there is one cumbersome step to calculate P^4 from the matrix P. That step of course having some numerical or some computational complexity, but we will accept that, this general framework is quite simple to handle with.

With that, let us proceed to next example, and this example is a little bit more complicated than the previous one. I would not say complicated, but actually the point is that if I can take the Markov chain properly in this case, you will see that computation is simpler compared to the standard Markov chain, most likely that which comes to our mind first if we use that, we will discuss this thing. Let us first go through the problem first. This says that suppose that the balls are successively distributed among 8 urns. So, I have 8 urns, urn 1, urn 2, 3, 4, 5, 6, 7, 8. So, I have 8 urns numbers from 1 to 8. And what I am doing is that I have plenty of balls, maybe infinite balls and I am just placing the ball randomly in one of the urns. And then, so with each ball being equally likely to be put in any of these urns. So, first line says that I have 8 urns and I have enough number of balls, maybe infinite balls, I am just distributing balls in these 8 urns and these distributions are random. So, I will take a ball I put it in any of the 8 urns and the probability of putting any of the urns is $\frac{1}{8}$ because they are equally likely to put in any of them. Now, our question is that I try to find out whether is that, what is the probability that there will be exactly three occupied urns after 9 balls have been distributed. So, what is the probability that there is exactly 3 occupied urns after 9 balls have been distributed?. So, I distribute nine balls now, and I try to find out what is the probability after distribution of 9 balls into these 8 urns randomly equally likely. What is the probability that all these 9 balls goes to 3 of the urns?. So, 3 of the urns are occupied, rest of the urns that means 5 urns remained empty. I hope the statement of the problem is fine. So, let us now proceed to solve this problem. Because I try to find out the probability of exactly three occupied urns, that basically I try to find out. So, one thing which comes to mind first is that maybe I take this X_n, be the number of occupied urns after distribution of n balls. That first comes to our mind that I can take X_n to be the number of occupied urns after n balls are distributed. So, before any ball distribution, number of occupied balls is 0, so clearly $X_0 = 0$. When I distribute one ball, one ball has to go to any of the urn. So clearly after distribution of the one ball there will be exactly 1 occupied urn, so X_1 also has to be one. Now for X_2, I have two cases possible, one is that, after distribution of one ball, the first ball goes to any of the urn. I have 8 urns, 1 to up to 8 urns. So suppose first ball goes here. So, when I am distributing the second ball, the second ball can go to this urn or second ball can go to any of the rest of the urns. If the second ball comes here, then the number of occupied urn remains same. So, it can be 1 or it can be 2 also if the second ball goes to any of the urn, except the second urn in this case. So, X_2 can be either 1 or 2, then X_3 can be 1, the reason is that all 3 balls go to one urn, X_3 can be 2 that two of them go to one urn, one goes to a separate urn or it can be 3, like 3 of the balls can go to any of the different 3 urns. So, this way this particular X_n, this particular stochastic process proceeds and in this case it is very easy to see that X_n is a Markov chain, the reason being that, well, if I know the previous one, the next probability I can find out, that is basically we are now going to do.

So, let us argue in this manner that was that if I have this thing that suppose, now I try to find out what is the, what is this probability?. This probability we know that it is a probability from moving i to i in one-step or i to $i+1$ is one-step or basically I try to find out what is the probability that moving i to j in one-step, $i,\ j$ are some member from the state space. So, if I try to find out that because what we have discussed here, suppose now I am in the ith state, in the next time step, I can only move from i to i or I can move to $i+1$. I cannot move to any other states, why?. The reason is quite simple because at the current stage the i urns are occupied, out of 8. Now, when I am distributing one more ball, after distribution of one more ball, if that ball goes to any of these i urns, then the state will remain i because again after distribution of one extra ball, exactly i urns are occupied. On the other hand, if that new ball goes to any of the empty urns, then the state will increase to $i+1$ from i. So, there are two possibilities, one from i, I can remain in i or I can move to $i+1$, this is the two possibilities there. For rest of the states the like $i+2$, it is not possible, I can go from i to $i+2$ in one-step. So, I try to find out these probabilities what is the probability that moving from i to i?. Moving from i to i as I have already discussed that means that i urns are already been occupied. I am distributing one new ball. When I am distributing one new ball, if the state of the next step is i that means that particular new ball has to go to any of the i already occupied urns. And what is the probability of that?. The probability of that is $\frac{i}{8}$. The reason being that out of 8 urns the ball has to go to any of the i occupied urns. So, that is why this probability $p_{ii} = \frac{i}{8}$. And what is the probability that p_{ii+1}?. So, how many unoccupied urns are there?. There are $8-i$ unoccupied urns. So, that if I have to move from i to $i+1$, the new ball has to go to any of the empty urn. That basically means that I am having $8-i$ empty urns. So, the probability is $\frac{8-i}{8}$. And this is true for any $i \in \{0, 1, 2, \cdots\}$, just for discussion let us see what is p_{00}?. p_{00} has to be 0 because $i = 0$ that gives me that $\frac{i}{8}$ that is 0 that gives me 00 is 0. And what is p_{01}?. $p_{01} = 1$ because $i = 0$, it is $\frac{8}{8}$. And then $p_{0j} = 0$ for all $j = 2, 3, \cdots, 8$. And this is very very easy to see because, when all the urns are empty, I start from 0 that means, all the urns are empty, there is 0 occupied urns, then when I am basically distributing the new ball, it has to go to any one of the urn. So, clearly the next step from 0 has to be 1, there is no other possibilities. So, the probability from moving from 0 to 1 in one-step has to be 1 that is written here and when it is 1, then because we know that probability p_{0j} has to be 1 if I take the sum of over all possible states in S that has to be 1, i.e., $\sum_{j \in S} p_{0j} = 1$, so that says that all other probabilities has to be 0. There is no other option. I can move 0 to 1 only, there is no other option. Then, let us discuss p_{10} that is not possible, because once I have, 1 urn occupied, I cannot actually move to 0 now. So, this has to be 0 that from 1 urn occupied if I distribute one more ball, the state will remain same or it will increase but it cannot decrease. So, the probability has to be 0. If I talk about p_{11} that is basically one urn is already been occupied. I am distributing a new ball, so the state will remain same, number of occupied urn will remain the same if that new ball goes to the urn which is already been occupied. So, that means this probability has to be $\frac{1}{8}$ that there are 8 total possible cases out of that only 1 is there on which if I put it, I will have the same state, so that the probability is $\frac{1}{8}$. Then I can find out 1 to 2, it will be that already one urn is occupied, when I am distributing one extra ball, one new ball, after distribution of the new ball, state will be 2, if the new ball goes to an empty urn. And how many empty urn are there?. Because I have only one urn was occupied. So, just before distribution of the new ball there are exactly 7 empty urns. So, the probability is $\frac{7}{8}$. Now, from the fact that summation p_{1j} has to be 1 for $j \in S$ i.e., $\sum_{j \in S} p_{1j} = 1$, you see that if you sum that $p_{11} + p_{12}$, that sum is 1. So, rest of the p_{1j} has to be 0 for all $j = 3, 4, \cdots, 8$. So, that is basically the idea to find out the transition probabilities in this case. Now, you see that in the state space I have 9 states. So, clearly I can also write that one-step transition probability matrix, but that one-step transition probability matrix will be of ordered 9 by 9, 9 cross 9 transition probability matrix I can write in this particular case. So, that is why I have not written that complete matrix here because it is a very very big looking matrix, but writing this one is very very simple as long as we have this particular thing. Now, what we want?. We try to find out this one. Because I try to find out after 9 ball being distributed, what is the probability that exactly 3 occupied urn. So, initially the state is 0, there is no other option that before distribution of any ball, the state is 0, number of occupied urn is 0. So, from that particular fact, I am trying to move to 3 after 9 balls are distributed. So, I try to find out what is the probability that $X_9 = 3$ after 9 balls being distributed, the state is 3, number of occupied urn is 3 given $X_0 = 0$. Note that $X_0 = 0$ with probability 1. The reason being that before distribution of any urn, the number of occupied urn is 0. So, this one we basically try to find out. Let us see how we can find it out?.

Now what we can do is that first point we can do is as follows notice that because we have already discussed $p_{01} = 1$ that basically means that moving from 0 to 3 in 9 steps is same as moving from 1 to 3 in 8 steps because when we first distribute one ball, the next step that number of occupied urn is 1. So, after 9 balls being distributed, I move exactly 3 occupied urns that probability is same as finding out the probability that there is exactly 1 urn is occupied. So, basically I have just distributed one ball, exactly one urn is occupied and from that, I will distribute 8 more balls and after distribution of 8 more balls exactly 3 occupied urn will be there, that two probability are same. Alternatively, you can see it, of course, this intuitively makes sense very easily that this probability is same as that because of this or you can see it from the Chapman-Kolmogorov equation. Because using Chapman-Kolmogorov equation, I can write this $9 = 1 + 8$, then I can write this way that p_{0k} and then p_{k3} in 8 steps. So, one step it is and then 8 step it is, and one basically there is no point of writing 1 here, and you see that $p_{10} = 1$, rest of the $p_{0j} = 0$ for $j = 0, 2, 3, \cdots, 8$. So, that means that only one term will remain here, and that term is nothing but when I take $k = 1$ and that is what these becomes p^8_{13}. So, that means, finding out the probability that exactly 3 occupied urns after 9 balls have been distributed is same as finding out that starting with one occupied urn, we need to find the probability of exactly 3 occupied urns after distribution of 8 balls, this probability I need to find out. This I can do in several ways; one of them, I can write that P matrix, P matrix notice that it is a 9 by 9 matrix. I can take this to the power 8 and then I can take the one-third element of this P^8 matrix or I can also do that P which is again in same matrix to the power 9 and then I take that 03th element of the matrix, that is what we can take. But the problem here is that because I have the matrix P is a 9 by 9 matrix, taking it to the power 8 or taking it to the power 9 may be very very complicated issue. By hand it is very very difficult to do, even if you take the help of some software, of course, you can do it, but before that you have to write this 9 by 9 matrix. So, basically you have to write this 81 entries of the matrix. So, this is a quite complicated job to do. So, what we can do alternatively?. Alternatively we can make this problem a little bit simple. And how can we make it?. We can make it simple by defining the stochastic process differently. In the previous example, previous case, we have defined the stochastic process like this; it is the number of occupied urns after distribution of 9 balls. So, we will basically change this definition.

So, what I am going to do is that I am going to use this two things here. One is that we have already discussed that the state of X_n does not decrease. So, once I reach i, I cannot reach $i-1, i-2, \cdots, 0$. I can only remain in i or I can move forward, I can go to $i+1, i+2, \cdots$. One step of course, I can go from i to i or $i+1$ not others, but in many steps I can move from i to i, maybe $i+1, i+2$ anyone in the higher value. Now, this particular thing I can use and what I can do is that I can collapse the state 4 to 8 in the previous case. So, what I have states that $0, 1, 2, 3, 4, 5, 6, 7, 8$. These are the states I have. What I can do is that these states I can take as it is $0, 1, 2, 3$ and I collapse all of them and maybe I call it 4 prime or rather I just call it 4 just. So, I will take all those things together and I call this is one-state. Why we are doing this?. Because finally I have to find out the probability of exactly 1 occupied urn if I try to find this out, once my chain reaches any of these states, it will not come back to 3 because the state cannot decrease. So, if X_n goes any of these states, 4, 5, 6, 7, 8, it will not come back to 3 that is why I can take all the states together and I can write it as 1 of the state which may be I call as 4 now. So, we try to define a new Markov chain, Y_n having this states. What are the states?. As I mentioned it is 1, 2, 3 because 0 again I am not taking because finally, as we pointed out that we try to find out this probability that starting from 1 after distribution of 8 balls there will be exactly 3 occupied urns. So, 0 is I can just forget about. So, here I am not taking 0. So, I start at 1, 2, 3, so these three states as it is, so that i means if I use any of 1, 2, 3, i means that exactly i urns are occupied and 4 means that at least 4 urns are occupied. So, at 4, 5, 6, 7, 8 at least 4 urns are occupied that is basically now 4 and i means that exactly i urns are occupied, where i can take value 1, 2, 3. Now, we can write, how many state in this case, the state of Y, let me use this notation to distinguish between state of X and the state of Y. So, that is basically nothing but 1, 2, 3 and 4. This is the state, so it has 4 elements here, clearly when I write the transition probability matrix of Y, this is basically transition probability matrix of Y_n. This is the transition probability of matrix Y_n, because I have 4 states here clearly the Q is a 4 cross 4 matrix. Now, I we can work with this 4 cross 4 matrix much easier way than to work with a 9 cross 9 matrix what we have in this case. So, what we now try to find out?. We try to find out starting from one occupied urn there is exactly 3 occupied urns after distribution of 8 balls. So, I tried to find out this probability in this case, and in this case instead of P notation I write Q notation to distinguish between P is for the matrix the chain X and Q is for the chain Y.

So, let us try to find out this probability and to do this one what I have to do, I have to first find out the matrix Q. And once I get Q, what I will do?, I will take the eighth power of Q and then I will take the appropriate entry of the Q^8 and that entry is nothing but I have to take the first third entry of that matrix Q and that is basically my final answer. First third entry, first third element is basically the final answer of the problem. So, let us try to see how this Q can be calculated. As we have discussed earlier that when I am in state 1, one urn is occupied, what is the probability that after distribution of one more ball there will be exactly one urn is occupied, that is $\frac{1}{8}$. This is coming from the fact that, the new ball has to go to the same urn which is already been occupied. So, that why is the probability is $\frac{1}{8}$. From 1, I can move to 2 and I can move only to 2 if the new urn go to one empty urn and they are exactly 7 empty urns, when just one urn is occupied, so from moving from 1 to 2 that probability is $\frac{7}{8}$. From 1, I cannot go to 3, I cannot go to 4, that is why these two entries are 0. Similarly, from 2, the state cannot decrease. So, I cannot come back to 1. So, 2 to 1 is 0, 2 to 2 is $\frac{2}{8}$ because in this case the new ball has to go to any of that two occupied urns, and 2 to 3 is $\frac{6}{8}$, because in this case, the new ball has to go to any of the empty urn and there are 6 empty urns when exactly two occupied urns are there. And then from 2, I cannot move to 4 so that is why that probability is 0. Similarly, the third row can be filled up from 3, I cannot come to 1 and 2, because state cannot decrease. From 3, I can go to 3 and in this case there are 3 occupied urns. So, the probability is $\frac{3}{8}$ and 3, I can move to 4 and when there are 3 occupied urns there are 5 empty urns so moving from 3 to 4 in one step that probability is $\frac{5}{8}$. And finally come to the fourth row, this is something interesting, see from 4, I cannot come to 0, I cannot come to 1, I cannot come back to 2, I cannot come back to 3 in one-step in any step because the state cannot decrease in this particular example, but from 4, I can go to 4 and once I enter into 4, because the state do not decrease and actually 4 includes 4, 5, 6, 7, 8 all of them. So, once Y in this 4, that Y will remain in 4 forever. And that is why in this case, I can move from 4 to 4 only and that probability is exactly 1. Once I have this matrix, then I have to do some cumbersome calculation, again we can do it using any software. So, first I will find out Q^2 that I can find out multiply Q with itself, then I will take square of it that will give me Q^4. And then again I can take the square of it, which will give me Q^8, that way I can calculate or alternatively once I get Q^4, in this case Q^4 is given by this. I can calculate the required probability very easily because I need to find out one-third element of Q^8 which is basically nothing but, if you take the product between first row multiplied by third column of Q^4, I am done and that is what we have done here and finally, if you calculate the probability is this. So, this shows that the chance of occupying of exactly 3 urns after distribution of 9 balls randomly is very very low it is just 0.00756. So, probability is quite small in this case. So, it says that, well, when I will equally likely, when I will distribute balls randomly in the urns and I have 8 urns, it is very very likely that after distribution of 9 urns more than 3 urns will be occupied or there is the probability is very very thin, if we try to find out the probability of exactly 3 urns being occupied after distribution of 9 balls in this particular scheme. So, recap this example, this example of course, we have used the Chapman-Kolmogorov equations, but top of that, to make the problem simple, we need to define the Markov chain, the computation little bit simple I should say, to computation make it little bit simple, I can also do one thing, we have done also. We have discussed also one thing that we define our Markov chain a little bit cleverly, little bit appropriately so that we can reduce the computation of it. So, instead 81 entries in a 9 cross 9 matrix, we can reduce it to a 4 cross 4 matrix which has 16 entries. So, this is basically something to take away from this example. One of this, Chapman-Kolmogorov equation can be used to find out this kind of probabilities. And second takeaway from this example is that in many many examples, if we define our Markov chain cleverly, we can reduce the amount of calculation, we can reduce the computation bargain or the complexity of the problem. With that takeaway, let us proceed to the next example.

This example again goes like this, in a sequence of independent flip for a fair coin. So, I have a fair coin, fair coin means that probability of head and probability of tail are same. So, probability of head is half and probability of tail is also half. So, I have a fair coin, I am tossing the coin again and again, again and again, repeatedly I am tossing the coin. So, now, suppose N denote the number of flips, until there is a run of 3 consecutive heads. So, if I get 3 consecutive heads, up to that point, how many tosses I have done that is basically the random variable N. So, just start with if the first is head, then I get head, then I get again head. So, in this case, after 3 tosses, I got a chain of 3 heads. So that is why $N = 3$. If the scenario is like that tail, head, head, head, in this case, after fourth toss I got a chain of three heads. So, $N = 4$. Similarly, I can have something like that head, head, tail, head tail, tail, head, head head. So, in this case, you see that this is the first chain of consecutive 3 heads in this particular sequence. So, in this case, N will be 1, 2, 3, 4, 5, 6, 7, 8, 9. So, in this case, $N = 9$ so on so forth it proceeds. So, I hope that definition of N is fine with everyone that this is basically N denote the number of flips until there is a run of 3 consecutive heads. So, I keep on tossing it and N is basically number of tosses till I get a run of 3 consecutive heads, clear?. So, one interesting thing is that see in this case I have two consecutive heads, then again tail come, so it basically means that I can go back to 0, again I look for next step. Next state comes here again I get a tail. So, I have not chain yet, chain of 3 consecutive, run of 3 consecutive it comes here and there are 1, 2, 3, 4, 5, 6, 7, 8, 9 tosses needed for that. So $N = 9$. So, that is the definition of N. Now, what we are trying to do in this, we try to find out what is the probability that $N \leq 8$?. So that basically means I try to find out that what is the probability out of 8 tosses there is a run of 3 consecutive heads, within 8 tosses there are run of 3 consecutive heads that basically we try to find out here. The next one is very simple once I can find out this next one, this probability we will discuss when we come to that point. So, how can we do that again we do using a Markov chain. So, we define the Markov chain in this case in this manner: X_n is a Markov chain which having the following states. What are the state?. State i if i is 0, 1 or 2, that basically mean that currently on a run of i constitutive heads. So, that means that when i equals to 0, currently on a ran of i constitutive heads. So, that basically means that if you see this one, when I have the first entry only tail, that means that chain of head has not started. So, in this case, after this result of the state X_1 is basically 0, that I am currently in a run of 0 heads in the sequence of tosses. Then I have a head, so now state will be 1 so that the current length of the consecutive head is 1. Then next head comes in this case, so X_2, X_3 is 2. And then the next state comes, so X_4 is 3, then if tail comes, then X_5 will be 0. So, this way basically we defined our Markov chain that, if i is any of 0, 1 and 2, then it means that I am in a currently in run of i consecutive heads and 0 means basically I am currently getting a tail. So, I have not a run of any consecutive heads in the just current toss, so that means, if I have a tail in the current toss it means that state is 0. If I have one head in the current toss that the previous toss is a tail, then I am in state 2. If the current toss is head, the previous toss is head and the previous to previous toss is tail, that means, I am in state 2. So, for state 2, the thing has to be something than tail, head, head, for state 1 it will be tail head, something than tail, head and state 0 means that tail is here something than tails. And here in something head, head, head, should not be there. And what is this 3 means?. 3 means that a run of 3 constitutive heads have been occurred. So, 3 means many thing. 3 means that just now I get the head, head, head, that may happen or the head, head, head, 3 constitutive heads has occurred sometimes in the previous. So, if the chain is like that tail, head, head, tail, head, head, head, then of course, I have listed 3 or if I next head is there tail, already I have these, so even in this case also in the state 3. So, in this case basically $X_1 = 0$, $X_2 = 1$, $X_3 = 2$, $X_4 = 0$, $X_5 = 1$, $X_6 = 2$,

$X_7 = 3$, $X_8 = 3$, $X_9 = 3$ whatever comes here does not matter, tail, head, $X_9 = 3$ and so on so forth, rest once I reach 3, I will remain in 3 forever that basically means. And I hope that this writing is clear to everyone that $X_0 = 0$ because tail is there, there is a run of 0 head. $X_1 = 1$ because now currently I have a run of one head and head, head, head has not occurred before. $X_3 = 2$ because currently I am a run of 2 heads and head, head, head has not occurred yet. $X_4 = 0$ because currently I am in a run of 0 head, and head, head, head has not occurred. Then $X_5 = 1$ because currently I am a run of one head. $X_6 = 2$ because currently I am a run of two heads and $X_7 = 3$, because currently I have a run of 3 heads. So, consecutive 3 heads have already been occurred. Then X_8 is basically, already this 3 head, head, head occurred, so $X_8 = 3$. X_9 already this head, head, head has occurred. So, $X_9 = 3$ and so on so forth, it will go on. So, from this understanding, now, let us write what is transition probability matrix in this case. The transition probability matrix P can be given this form and writing this one is very simple. See from run of 0 head, I will go to run of 0 head. So, currently I am a run of 0 head that means currently have a tail. I will go to 0 head again if I have a tail again. So, in the next toss I am getting it tail that probability is half because I have a fair coin and I am doing the tosses independently. Now from, currently I am in 0 run so I am in tail. So, from that I will go to 1 if I have head here and what is the probability of getting head, that is half. Now from 0, I cannot go to 2 because after doing just one toss, the state cannot go from 0 consecutive head to 2 consecutive heads. So that probability is 0, similarly from 0, I cannot move to 3, so that probability is also 0.

Now, what is the case in case of 1?, for row 1, I basically start with the probability that head, I am currently something and then just 1 head. I am in a run of exactly one head. Just currently I have just one head before that I have to have a tail. So, now from this one I can go to 0 if a tail comes. From this one I can go to 0 if it a tail comes. So, basically, what is the probability of coming this tail?. This is basically head. So, I am in 1 that means this a tail, head this is there something and then tail has to be there. And from that I will go to 0 if it tail comes, that probability of getting this tail is half. So, this probability is half. Then, again I have tail, head, because I am in 1, then what is the probability of going to 1?. I cannot go from 1 to 1 because if tail comes I will go to 0, if head comes here, I will go move 2. So that means from 1 to 1, I cannot go and that probability is 0. Now, 1 to 2, I can move, and that probability is again half because the probability of getting this head is basically half. So, that is why this is half. And similarly, I cannot move from 1 to 3, because after just tossing one coin from a run of one constitutive head, I cannot move to a run of 3 consecutive heads. So, that is why these are the cases. Now in 2, the scenario is like this something then tail, head, head, I am currently in this state, there are two possibilities for the next one, either a tail comes or something then tail, head, head, head, these two possibilities are there. If tail comes here, I will go back to 0, that is why 2 to 0 is half. And if head comes here, I will go to 2 to 3. So that is why that probability is half and rest of them has to be 0, the reason is that, I can argue it different way that from 1, 2 to 1, I cannot go from 2 to 2, I cannot go, this way I can argue or I can argue it like row sum has to be 1. So that is why basically these two probabilities has to be 0, because if I add these two, I have already reached 1. And once I am in 3, I will be in 3 forever because I have defined this third state like that. So that is why for 3, to 3 is 1, rest of them at 0. So, this way I can write this matrix. Now, what I am trying to find out that I am trying to find out the probability of this event and this event is basically nothing but finding out the probability that at the eighth step I am in the third stage, that $N \leq 8$ that basically means that within 8 tosses, I have run of three consecutive heads. That means at the eighth step I should be in the state 3. And that is why we have defined this particular state as an observing state as an state, if we reach the state, I will not come out of this state because once I get that head, head, head somewhere within this 8 tosses, I am done. That is why we have defined this third stage like this. So, I have to find out. To find out this probability, that means starting from 0, because doing any tosses, that state is 0. Starting from 0, I will go to state 3, in 8 step that probability I have to find out and the finding out again I have to find out P^8 then I have to take the entry corresponding to the first row and third column of P^8. I am done. I have to take the entry corresponding to the first row and third column of the P^8 then I am done. And then the computation again you can do it and you can check that this comes out to be this. So, see that if I try to solve this problem directly from the very crash of the very basic definition of the probabilities and using the conditional probabilities, this would be a very very complicated problem. But if we convert it to a Markov chain problem, we see that this problem can be solved very easily if I can define our Markov chain properly. If I can define our Markov chain cleverly, we can solve this problem very easily. And that is basically the takeaway of this particular example. Of course, we can solve this problem using our basic idea of the probability, the conditional probability, but that would have been a very very complicated problem. But using the Markov chain thing and Chapman-Kolmogorov equation, I can solve this problem quite easily. Let us just talk about the last one that probability that $N = 8$. See that probability $N = 8$, I can write as $P(N \leq 8) - P(N \leq 7)$. $P(N = 8) = P(N \leq 8) - P(N \leq 7)$. $P(N \leq 8)$ that we have already discussed how to find out, $P(N \leq 7)$ I can again find out from the same thing that in the 7th step I have to be in the state 3. So, it is p_{03}^7. This two things I can, we can find out similarly that. So for that I need P^7 and then I have to take the entry in the first row, third column of P^7. So, these two probabilities I can find out, I can plug in, I can subtract, I can have this probability. Of course, the exact calculation, exact value I am not giving here because this is just merely computation, nothing else. You can put it you are in the P matrix given here in any software and can check that what probability you get for P, $P(N = 8)$. With that, I stop. Thank you for listening.