Discrete-Time Markov Chains and Poisson Processes

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Module: Hitting Times and Strong Markov Property

Lecture 9

Hitting Time I

Welcome to the 9[th] lecture of the course Discrete-Time Markov Chain and Poisson Processes. Recall that in the previous lecture we have talked about accessibility and communication and then we talked about closed class, communicating classes and closed communicating classes and then today we are going to discuss about something called hitting time.

What is hitting time?. Hitting time basically means that when I entered into a subset of other state space, so the definition of the hitting time goes as follows. Suppose $A \subseteq S$, S is our state space. If I have this is the state space and suppose this set is A, this consists of some states maybe it can consist multiple states, it may consist single state, I mean any anything is possible. Now, the hitting time T^A is nothing but defined by

$$T^A = \inf\{n \geq 0 : X_n \in A\}.$$

What is this equation?. Let us try to understand what this mean, first consider this set, what is this set?. I am taking the collection of all possible values of $n, n \geq 0$ of course, n are integers in our case, $n \geq 0$ such that $X_n \in A$. So, maybe X_0, then X_1 is also here and X_2 here X_3 is here then maybe X_4 comes out of the set, so in this particular example this set will look like something like $\{2, 3, 5, \cdots\}$ and maybe then X_4 is outside, X_5 may be inside something will come based on the chain whether the chain is inside the set A or outside the set A based on that some set will come. So, I am taking the collection of all possible values of n such that $X_n \in A$ that basically I am considering that particular set. Now, I am taking the infimum of that set, so when I take the infimum in this case, if I take the infimum here that turns out to be 2, so what is that, I mean that basically mean if the chain starts from outside this 2 is basically nothing but the time at which I enter the chain enters the set A for the first time. So, that means that the hitting time is nothing but for the first time when the chain enters to a particular set under consideration if the chain starts from the outside and of course, if the scenario is like that if this is my A set and if X_0 is inside then of course this set will consist 0, this set we will have 0 in this particular case and the infimum will also be 0, so T^A will be 0 in this case if I start from inside A

because in this case 0 will be there and I am taking the infimum, so infimum is basically in our case is nothing but the minimum element of the set, so that is why T^A will be 0 in this case but T^A will be 2 in this particular case. So, that two points are mentioned here that well if I start from A, T^A has to be 0, if I start outside A then T^A is the first time when the chain enters the set A after the time 0, it is the first time when the chain enters the set A after the time 0. So, these are basically the definition, now you see that I have written one convention here, that $\inf \emptyset = \infty$, why this convention we need to take?. First of all that we need to take because suppose something happened like that, this is my S and A is here and chain is starting from suppose this state is X_0 and this state is like that, from this state I cannot go to A, so any state in A is not accessible from this particular state suppose, any state in A is not accessible from this particular state then of course, if the chain starts from this particular state I will never visit the set A and that says that this set will be $\emptyset$ in this case because for any n, $X_n \notin A$ in this particular case. So, then the this set will be $\emptyset$ and that means that starting from this particular state I will never visit the states in A, the chain will never visit the states in A and that basically means that in any finite time I will not reach A and that is why the convention is taken that when this set is $\emptyset$, that means starting from something I cannot visit the state A, then the hitting time is taken to be infinity, that means that infinity time I will hit that, that means basically in practice I will never go into the set A. So, this is the definition of the hitting time, it is basically nothing but

$$\inf\{n \geq 0 : X_n \in A\}$$

and with the convention that $\inf \emptyset \infty$, then we have two points that if $X_0 \in A$ then $T^A = 0$ this is the scenario in this case and if $X_0 \notin A$ this is the scenario in this case then T^A is the first time after 0 when the chain enters the set A. So, now let us see that notation that well if $A = \{i\}$ then I can write T^i instead of T^A that is basically nothing but this, but for simplicity instead of writing this notation we will write this one, this curly bracket will not put here we will just use this notation. Now, one of the very interesting thing is that to see this probability, what is this probability?. T^A is finite so that means in the finite time I will be in the state A when I start from 0, so this I can write as $h_i^A = P_i(T^A < \infty) = P(T^A < \infty | X_0 = i)$ because here I am starting from i, so starting from the state i what is the probability that I will visit any state inside the set A in a finite time, this probability in many practical scenario is very very meaningful. So, in this particular lecture we are going to talk about this kind of probability and through some examples we will show that how we can calculate this probability for different kind of problems. So, the probability that starting from i the Markov chain will ever hit A is given by this probability and this is the notation we are going to use in this case, that it is nothing but starting from i in finite time the process will hit the state A and with our P_i notation we can write this in this form or in the standard notation we can write this in this particular way. And informally in many cases instead of writing this $T_A < \infty$ we will write that hit(A), that basically means that $T_A < \infty$, so hit$A \equiv T_A < \infty$. Now, i is there, so starting from i what is the probability that i will ever hit A that basically mean that starting from i what is the probability that the Markov chain will enter any of the state in the set A in finite time, and there is another terminology that absorption probability. If the A is a closed set, just recall what is the definition of close set, a communicating class is called closed if I cannot go out from that particular class. Once I am inside the class I will be there in the class forever, so that kind of class is called the closed class and if I try to find out what is the probability of hitting A closed class that is generally called the absorption probability because once I am inside this class I will not come out of the class. So, the thing is that once chain is entered into the class the chain will be absorbed into that particular A forever, it will never leave A in future, it will be inside in A forever, so it is as if the chain is absorbed into the set A, and that is why basically this probability if A is closed this probability h_i^A is called the absorption probability, where informally, we will write $h_i^A = P_i(\text{hit}A)$.

Let us proceed and let us try to see this absorption that hitting probability with help of an example. So, this example is taken from this book that book by Norris, the name of the book is Markov chain and this is published by Cambridge University Press in 1997 and this example you will find in the page number 12. The process goes like that, suppose we consider a Markov chain with one step transition probability specified by the following diagram. Earlier we have seen that if I am given with a one-step transition probability matrix I can draw a graph corresponding to that one step transition probability matrix and where basically the idea was that nodes are basically states and directed arrow, directed edge actually signifies that I can move from a particular state to another state in one step with some positive probability. Now, in this case you see that with that nodes and edges now I have another thing I have written some numbers here, what are these numbers, these numbers are nothing but what is the probability. So, in this case you see from 1, I can go to 1 only, so and that can happen with probability 1. So, from 1, I can go to 1 only, from 2, I can go to 1, I can go to 3 and from 2, I can go to 1 with probability $\frac{1}{2}$ and to 3 with probability $\frac{1}{2}$, so these are the probabilities, the directed edge actually tells us from 2, I can go to 1 and this quantity tells us that what is the probability of going from 2 to 1 in one step. Similarly, from 3, I can go to 2 with $\frac{1}{2}$ probability and from 3, I can go to 4 with probability $\frac{1}{2}$ and from 4, I can be in 4 only and that is with probability 1. Now, you can see that I can very easily write the probability matrix in this case from this diagram. So, I have the states 1, 2, 3, 4, so 1 to 1 that is 1 rest of them are 0, from 1, I cannot go to any other state in one step, from 2, I can go to 1 with probability $\frac{1}{2}$ and 3 with probability $\frac{1}{2}$ rest of the probabilities are 0, from 3, I can go to 2 and 4 with the equal probabilities, rest of them are 0 and from 4, I can go to 4 only in one step and rest of the states are not accessible in one step from 4. So, given this matrix, I can draw this diagram and given this diagram, I can write this matrix. So, this diagram and matrix has a 1 to 1 correspondence, given the diagram I can write the matrix, given the matrix I can draw the diagram. So, that means this diagram actually specifies the one step transition probability matrix of a Markov chain. Now, what we are interested in this case to find out starting from 2 what is the probability of absorption in 4, we are interested to do that. So, starting from 2, I start from here, now what is the probability of absorption in 4, note that from 2, I can go to 1 and once I reach 1, I will be here forever on the other end from 2, I can go to 3 and then maybe go to 4 in some steps and once I go to 4, I will be here, the chain will be here forever, it will not come out state 4. So, we try to find out the absorption probability in this case, absorption probability in 4 starting from 2 that is basically the idea here. And as we proceed we will see that I can write some linear equations and solving those linear equations I can able to find out the required probability. We can write a system of linear equations and by solving that system of linear equations we can actually find out the required probability in this particular case. How can we proceed, let us proceed. We denote this h_i is that probability, instead of writing 4 that I am not writing anymore, so just make the notation little bit simple I just use $h_i = P_i(\text{hit4}) = P(T^4 < \infty | X_0 = i)$, that basically mean that it is nothing but probability that I hit 4 that $T_4 < \infty$ when I start from the sum state i. And in particular we try to find out what is h_2, we want in this case what is h_2, because $i = 2$ is basically starting from 2 what is the probability that I will reach the state 4 in finite time and once I reach state 4, I will be absorbed here, I will not come out that particular state 4 for this particular Markov chain because from 4, I can only remain in 4 with probability 1. Now, the next point is very very simple from here, first point is that h_1, what is h_1?. Starting from 1 what is the probability of hitting 4 in finite time. Now, if I start from 1, I will be in 1 forever, I will not come out of 1 in this case. So, clearly h_1 the probability of hitting 4 starting from 1 will be 0, h_1 will be 0. Similarly, $h_4 = 1$ because if I am once in the 4, I am already in the 4, so that in finite time I will hit 4 with probability 1, so that was $h_4 = 1$. So, $h_1 = 0$, the reason is that once I am in 1, I cannot come out from 1, so in finite time I will not visit the state 4, so clearly that probability that visiting state 4 in finite time starting from 1 is 0. Similarly, if I am in 4, I start from 4, so in finite time I am already in 4, so that means basically $T_4 = 1$ because with 1 probability and finite time I will be in 4. So, these two are very easy to see from this diagram that $h_1 = 0$ and $h_4 = 1$. So, what we have done here basically we use a new notation h_i just to make it little simple, I do not want to write 4 in the superscript, in power here and then basically easy to see that $h_1 = 0$ because once I am 1, then I will not come out from 1. And so basically in this case I can write in this way also $T_4 = \infty$ if $X_0 = 1$, so in finite time I will not visit 4 so that means basically in this particular case $h_1 = 0$, then why $T_4 = \infty$?. Just recall the definition of T_4, $T_4 = \inf\{n \geq 0 : X_n = 4\}$. Now, in this case you see that if I start from $X = 1$ for any n, $X_n \neq 4$. So, that is why basically if I start from 1 this set is 5 and so $T_4 = \infty$. Similarly, $T_4 = 0$ if $X_0 = 4$ because for 0 that 0 element is in this set, now zero element is in this set, so when I take the infimum that will be 0. So, in this case in finite time I will not reach 4, so the probability of reaching 4 in finite time starting from 1 will be 0 and in this case the time is 0, $T_4 = 0$, so that is finite in fact, so starting from 4, I will reach 4 in finite time and in 0 time and that is why that probability is 1 when $i = 4$, so this is basically the idea behind these two.

Now, let us proceed and let us see how I can solve for other values of i, I got $h_1 = 0$, I got $h_4 = 1$, now in between I have h_2 and h_3, I have to find out what are these two values, two extremes we have find out in this case and in between I have h_2 and h_3 and I try to find out what are these values and in particular in this case I am interested in h_2 only, because I try to find out starting from 2 what is the probability of absorption in 4. So, that means I am basically interested in h_2. Now the thing is that you see that $h_2 = P(T^4 < \infty | X_0 = 2)$ which is this, this is basically probability that in finite time I will be in state 4 starting from 2. Now, using the conditional probability I can write it $T_4 < \infty$ given $X_0 = 2$, X_1 equals to from 2, I can go to 1, so 1 multiplied by probability that $X = 1$, given $X_0 = 2$ plus some quantity.

So, here is that plus $P(T_4 < \infty | X_0 = 2, X_1 = 3)$ multiplied by probability $P(X_1 = 3 | X_0 = 2)$ i.e., $h_2 = P(T_4 < \infty | X_0 = 2, X_1 = 1) P(X_1 = 1 | X_0 = 2) + P(T^4 < \infty | X_0 = 2, X_1 = 3) P(X_1 = 3 | X_0 = 2)$. So, what I have done, note that from 2, I can go to 1 or I can go to 3 in this case in 1 step. So, what I am doing is that I am doing the conditioning with respect to the next transition, I start from 0, from 0 the next transition can be 1 with probability $\frac{1}{2}$, it can be 3 with probability $\frac{1}{2}$, so I am conditioning with respect to the next transition and when I am doing that what I have written is that first is 2, then the next is 1 and then the probability that what is the probability of going from 2 to 1 in one step plus the first is 2, that 2 is kept here the second can be 3 and multiplied by the probability that what is the probability that I can go from 2 to 3 in one step. So, conditioning on the first transition I can write that h_2 in this particular form, so if I can go from 2 to another state that the contribution of that particular state I have to add, so if I can go from 2 to maybe n number of states, then there will be n such additions will be there, n such terms need to be added and then I can write h_2 equals to this plus, this plus, this so and so forth. In this case from 2, I can go to two states, so two contribution, two terms comes here. Now, look into this first probability, if I look into this first probability you see that we have discussed this one that Markov property tells that if I know the state of the Markov chain in some time, then the Markov chain starts from there afresh, if I know the Markov property, Markov property tells us that if I know the states of a Markov chain at some point of time, then the Markov chain starts from that time afresh. So, I can forget the previous part of the Markov chain, I can think that the Markov chain is at that particular state. So, suppose sometime the Markov chain is in the state 1, so in this case the first time point it is state 1, so I can now think that well X_0 I can remove and I can think that well X_1 is the starting point and the Markov chain now starts from 1 so the initial distribution is δ_1 and it proceeds from that particular initial distribution and so on so forth, I can forget the previous part completely. So, that means that in this case I can ignore this one, similarly, in this case I can ignore this one, these two terms I can ignore and now I can write this quantity same as $P(T_4 < \infty | X_0 = 1)$, again I can write $X_0 = 1$ because Markov chain starts from there itself. So, it is again starting point I can write X_0 or whatever you can write X_1, so it is basically again starting from that particular point that is basically important, so I can write X_0. And then multiplied by p_{21} in one step plus $P(T_4 < \infty)$ starting from 3 multiplied by P_{23} in one step, so we can write this way that it is basically probability that I will visit the state 4 in finite time given $h_0 = 1$ starting from 1 multiplied by P_{21} it is the one step transition probability from 2 to 1. Similarly, visiting the state 4 in finite time starting from 3 that probability of that event multiplied by one step transition probability from 2 to 3 and now the first quantity is h_1 because starting from 1, I am visiting the state 4 in finite time and this probability is basically h_3 and then $P_{21} = \frac{1}{2}$, so that $\frac{1}{2}$ comes here $P_{23} = \frac{1}{2}$ is also $\frac{1}{2}$ so $\frac{1}{2}$ comes here. So, conditioning on the first transition I can write h_2 is same as $\frac{1}{2}h_1 + \frac{1}{2}h_3$ in this particular case. So, we wrote this one similarly for h_3 also I can write, from h_3 I can go to 2 and I can go to 4. That means with $\frac{1}{2}$ probability h_2 and with $\frac{1}{2}$ probability h_4, so h_3 equal to $\frac{1}{2}h_2 + \frac{1}{2}h_4$. That is

$$\begin{aligned} h_2 &= P(T^4 < \infty | X_0 = 2) \\ &= P(T^4 < \infty | X_0 = 1)p_{21} + P(T^4 < \infty | X_0 = 3)p_{23} \\ &= \frac{1}{2}h_1 + \frac{1}{2}h_3 \end{aligned}$$

$$h_3 = \frac{1}{2}h_2 + \frac{1}{2}h_4.$$

So, now I have some system of equation along with I have $h_1 = 0$, $h_4 = 1$, now I have to solve this set of equation and then I can find out the value of h_2 as well as h_3 and once I have that I am done, and as our particular aim is to find out h_2 so let us start with the first equation. So, h_2 as we know that $h_1 = 0$, so that term does not contribute anything, I have $h_2 = \frac{1}{2}h_3$, and then h_3, I can write from here, so h_3 again this quantity is 1, so $h_3 = \frac{1}{2}h_2 + \frac{1}{2}$, from here I just plug in here now I have an equation in h_2, I can solve for h_2 and if you solve if you will find it is nothing but $h_2 = \frac{1}{3}$. So, we got that $h_2 = \frac{1}{3}$, we can also find out what is the probability of h_3 in this case that is not at all difficult because h_3 is basically nothing but $\frac{1}{2}h_2 + \frac{1}{2}$, so it is $\frac{1}{6} + \frac{1}{2}$ so that basically mean that $\frac{1}{6} + \frac{1}{2} = \frac{4}{6} = \frac{2}{3}$ in this case. So, $h_3 = \frac{2}{3}$, $h_2 = \frac{1}{4}$ and $h_1 = 0$, $h_4 = 1$. So, let us try to interpret this result, what does this basically tells us, that if I start from 2 there is a $\frac{1}{3}$ probability that I will visit the state 4 and in finite time and once I visit the state 4, I will be there forever, if I start first with 3 from 3 there is a chance of $\frac{2}{3}$ that I will be in state 4 and I will be there forever and this is very very intuitive because if I start from 2 then I have to basically first go to 3 and then go to 4 and on the other hand in one step I will go to 1. So, there is a less chance of absorption in 4, if I start from 2 compared to if I start from 3, there is the less chance of absorption in 4 if I start from 2 compared to if I start from 3, that is intuitive because from 2, 4 is far ahead whereas 1 is close enough, if I go close I will be I will stuck there, on the other hand from 3, 4 is closer compared to 2 and so that probability is higher. So, this is basically the way we can find out the hitting probability and you see that the basic idea is that we write the set of linear equations with some initial conditions may be and then we try to solve the set of linear equations to find out the required probability.

Let us proceed and let us now put that everything in terms of a theorem. So, the theorem goes like this, the vector of hitting probabilities h consists of h_i^A, so basically if I have a state space S, then basically like in this case I have four h values h_1, h_2, h_3, h_4. So, taking everything together I am calling it h, so how many states are there I have so many h_i^s there and taking everything together I am calling it h, so that is a vector. So, the vector of hitting probability is the minimal non-negative solution of the system of linear equation this, so few things I need to discuss here, one is what do you mean by minimal, what is non-negative and how we get this system of equations. So, the way we get this system of equation is exactly same what I have discussed here through this I got this one. So, if I am already in A, if I start from A, i is the starting point if I am already in A then I am already in A, so in finite time anyway I will reach A, so I am in A. So, basically if $i \in A$ then $T^A = 0$, so if I start from A then $T^A = 0$, so that says that probability that T^A is finite starting from $X_0 = i = 1$, and this is basically h_i^A. So, if I start from A, $h_i^A = 1$, so 1 comes from there, if I am outside A what happens, if $i \notin A$, then that first transition I will make and suppose from i in the first transition I can go to j, so it will be p_{ij} multiplied by h_j^A. Now, starting from j what is the probability that in finite time i will hit the set A, so that probability comes here, that probability comes here so it is p_j multiplied by h_j^A and the sum I need to take over all possible values of j in the state space i.e.,

$$h_i^A = 1 \text{ for } i \in A$$
$$h_i A = \sum_{j \in S} p_{ij} h_j^A \text{ for } i \notin A.$$

So, these two equations comes from this one as well as the understanding that once I am in A, I am in 4, that probably $h_0 = 1$, $h_4 = 1$, so from that two thing these two are coming, the only thing is that well in this case I am considering the singleton set but it may be either A can have multiple states, so I have written these two thing together this way. So, this way we got this system of linear equations. Now, I have to solve this linear equation, so when I solve the system of linear equation there are few possibilities are there, there could be unique solution, there could be infinite solution, not only that the solution can be positive, can be negative, the solution can be positive, can be negative means that in this case when I solve this h_i can take the solution of this system of equation can take some of them can take positive values, some of them take negative values, all can possibly positive values, all can take negative values, everything are possible, all can take 0, some 0, some non-zero all those possibilities are there. Now, because h_i^A are probabilities, so probability cannot go outside the range 0 and 1, so that means I have to take that solution which give me all the component of the h vector to be non-negative. Now, if there are multiple non-negative solutions which one I need to take, this theorem states that I should take the minimal non-negative solution among all the non-negative solution. Now, the question is that what is this minimal, this minimal basically means this, so if suppose I have another solution X consists of X_i, $i \in S$, and this is a non-negative solution, so that means that $X_i \geq 0\ \forall i$. Now, the minimality means that, that solution I will take for which any other solution is greater than or equals to that particular solution X_i, so this h will be minimal non negative solution, if all the $h_i \geq A$ and for any X which is a non-negative solution of this system of equation $X_i \geq h_i^A$. If this condition is satisfied then we say that, that h is minimal, h is non-negative because all the $h_i \geq 0$ and that particular solution of this system of equation I am going to take as the values of h_i which are given the probability that hitting the set A starting from the state i. So, with that let us move on and let us see, okay one thing I should discuss here, that in this case you see that I have not talked about the minimality here, so from where the minimality comes, you see that $h_1 = 0$ that particular condition is not given by this set of equations, you can check that that $h_1 = 0$ is not given by this particular set of equation because in the previous example, $A = \{4\}$, so it says that A_4 and then 4 is basically 1 from here, $h_4^4 = 1$ and then h_2^4, I can write as $\frac{1}{2}h_1^4 + \frac{1}{2}h_3^4$ because in this case basically I have taken $i = 2 \notin A$, because $A = \{4\}$, $h_3^4 = \frac{1}{2}h_2^4 + \frac{1}{2}h_4^4$. Again in this case $i = 3 \notin A$, what about h_1^4, again $i = 1 \notin A$ but the problem with that is that from 1 to 1, I can only go, so it is basically $1 \times h_1^4$. So, this actually is not giving me any constant, it only saying that $h_1^4 = h_1^4$ so that is why this is not giving any kind of condition there, so now because h_1^4 is the probability it has to lies between these two. Now, what is the minimum value, minimal value is $h_1 = h_0$, minimal value when I call talk about minimal, minimal value is nothing but $h_1^4 = 0$ and that actually we have taken here but from another point of view that if I am in 1, I will never visit 4, from that point of view we have taken $h_1^4 = h_1^4$ but through this minimality also we can reach the same conclusion in this case.

So, with that we proceed and maybe we see now another problem where we can try to find out that absorption probability and we can do this one using the previous this stated theorem. So, this is basically the gambling model and we have already discussed the gambling model, it is basically nothing but a gambler went to a casino and start playing and initially the gambler has maybe rupees i. So, gambler has some rupees i and he went to the casino and he start playing, how the play goes, play goes like this that if the gambler wins then he will gain 1 rupee, so if the gambler win in the first game if he win he will have $i + 1$ rupee and if he lose then he will has to pay the casino 1 rupee, so now he will have $i-1$ rupee and the probability of win is p and probability is lose is q and $p+q = 1$. So, that means from i the state can go to $i+1$ with probability p that is the winning probability and from i the state can go to $i-1$ with probability q which is the losing probability, and there are two endpoint because if the gambler loses his or her all money then the gambler has to leave the game, that means gambler will be in the state 0 and on the other hand gambler fix some rupee N and if he gains the rupee N then he will stop the play, he will quit the play there and he will go back home. So, once he reached N then he will be the state N forever that is why that from 0, I can go to 0 only because he cannot play, he is forced to leave the game, on the other side he decides to leave the game once he had N rupees in his hand. So, he start with some i in between, he can go any side once he reach 0 he is forced to leave the game, once he reach N he is basically leaving the game. So, these two are basically with one probability in between that the probability of going towards right is p, probability of going towards left is q in one step, one step right, one step left, in one time step is p and q respectively and then of course, $p+q = 1$, also we take that $p > 0$, so just I mean both side you can go with some positive probability, I have written this thing here. Now, if you write in this thing in mathematical notation that $p_{00} = 1$ 00, $p_{NN} = 1$ so these two are 1, then $p_{i,i+1} = p,\ p_{i,i-1} = q$ for $i = 1, 2, \cdots, N-1$, and this is true for all i in between except 0 and N, this is true for all i, so i equals to 1 to $N-1$ and $p_{ij} = 0$ otherwise for any p_{ij} it is 0, from I, I cannot go to maybe $i-2$, I cannot go from here to here in one step. So, all the $p_{ij} = 0$ in this case. So, now one of the things which I can look into in this case is that what is the probability that gambler broke, that means gambler loses all his money, so what is the probability that the gambler is starting with an initial fortune i, here fortune means the rupees i, goes back bankrupt, so he loses all his money. So, he starts from here some i and he basically reaches this state 0 before reaching the state N, because once he reaches state N he will leave, so once he reaches N he will never reach 0, so I try to find out what is the probability that the gambler reaches the state 0 before reaching state N. And so that means gambler bankrupt, so it is a very well question to ask, suppose, I go to a casino with some money and I may ask this question that what is the probability that I lose all my money, on the other hand I can also ask this question that what is the probability I can actually reach the state N, that means I reach my target, these two probabilities of course, I can ask to myself before playing any game in the casino, if the probability of bankrupt, probability of losing everything is very very high then I should not play that game, on the other hand if the probability of losing everything is not very high, I can test my luck in that particular game. So, this kind of probabilities are very very meaningful to ask in this particular kind of setting.

So, let us see how we can solve, I can find out these probabilities and the way to do this one again writing the set of linear equations. Now in this case the set $A = \{0\}$, so again I am not writing 0 here again and again to have this simple notation, so that basically means that well this hitting probability to 0 starting from i, I write it h_i, now $h_0 = 1$ because of this condition that if $i \in A$ then $h_i^A = 1$, so $h_0 = 1$ because in this case $i = 0 \in A$ which is basically our $\{0\}$. Now, if i is anything between 0 to $N-1$, then of course, this $i \notin A$, and then I try to write the linear equation and in this case writing the linear equation is very simple because I start from h_i, now from h_i what happens that I start from i in one step I can go to $i+1$ or in $i-1$ and these two probabilities I know what is the probability of going to $i+1$ and what is the probability of going to $i-1$. So, from i, I can go to $i+1$ with probability p and once I am in $i+1$ now I need to find out what is the probability of hitting 0 starting from $i+1$ which is h_{i+1}. Similarly, from i can in one step I can go to $i-1$ which probability is q and now I need to find out starting from $i-1$ what is the probability of hitting 0, which is h_{i-1}. So, we have written the set of linear equations for $i = 1, \cdots, N-1$, what happens if I take $i = N \notin A$ and in this case basically h_N is again from N, I can be in N so it is nothing but 1 multiplied by h_N, so that means again this linear equation is not giving me anything, so I have not written this linear equation here i.e.,

$$h_0 = 1$$
$$h_i = ph_{i+1} + qh_{i-1} \text{ for } i = 1, 2, \cdots, N-1$$
$$h_N = 1 \times h_N.$$

Now, you see that how many linear equations I have here, here I have 1 which is giving me $h_0 = 1$ and here I have $N-1$. So, I have finally N linear equations, because this one I am not considering because this is not a equation as such, it is only saying $h_N = 1 \times h_N$, so this is not giving me any constraint. So, this one I am not counting except that I have N linear equations. How many h_i's are there for each state I have a h_i so how many states are there I have from starting from $0, \cdots, N$, so I have $N+1$, $h_i^{\text{'s}}$, so that means I have N equations, I have $N+1$ variables to find out. So, clearly one of the variable is a free variable, so normally what we do when we solve the set of linear equations if we have more variables than the number of equations we take some of them as a free variable like if I want to solve $X_1+X_2+X_3 = 0$ and $2X_1+3X_2+3X3 = 0$, we have basically two linear equations, three variables so we can take one variable as a free variable and we can solve other two as a function of the free variable. So, in this case also I can take one variable as a free variable that there are one extra variable compared to number of equations. So, one variable I can take extra variable and look at that I can take h_N and so again notice that h_N can take a value between 0 and 1, so if I need to take a value of h_N that has to be a minimal value because of this minimality condition here and if I take this one that means I have to take $h_N = 0$, so that is what I have written here. That since h_N lies between 0 and 1 and we are interested in minimal non-negative solution, we must take $h_N = 0$. So, now finally what I have $h_0 = 1$, I have, $h_N = 0$, I have, now I have to solve for h_i for $i = 1, \cdots N-1$.

Now let us go for this solving. What we can do from here is that, see we have the conststraint that $p+q = 1$, I can write $p + qh_i = ph_{i+1} + qh_{i-1}$. Now, you see what I can write from here, now if I keep the p term in this side it is $p \times (h_i - h_{i+1}) = q \times (h_{i-1} - h_i)$. So, what I have done $p \times h_i$, I keep in this side, this $p \times h_{i+1}$, I take to the other side, so I have $-p \times h_{i+1}$, in the similar manner I keep qh_{i-1} in this side and $q \times h_i$, I take the other side which gives me the $-q \times h_i$, so I get this particular equation and this kind of equation is generally called difference equation. So, I get this difference equation that this one is nothing but $\frac{q}{p} \times (h_{i-1} - h_i)$ or I can also write from here this $h_{i-1} - h_i = \frac{p}{q} \times (h_i - h_{i+1})$ both of them I can write. In this case what I did this p_i take the other in this side or in this case what I take is that q_i take on the other side divided by q in this side, so both of them we can write. And what we can do is that in this case we take this particular form and from this particular form, so what I can write let me write this is in the next slide that, so we get this particular expression. Now, we will put different values of i, if I put $i = N-1$ we get exactly this equation that if I put $i = N-1$ so this become $N-2$, this is $N-1$, this becomes $N-1$ and this becomes N. Then I put $i = N-2$, I get this equation, in this way I proceed, finally if I put $i = i+1$, I get this particular equation and if I put $i = 1$, I get the final equation. So, putting different values of $i = 1 \cdots N-1$ we get these equations.

That is

$$
\begin{aligned}
&h_{N-2} - h_{N-1} = \frac{p}{q}\left(h_{N-1} - h_N\right) = \frac{p}{q} h_{N-1},\ i = N-1\\
&h_{N-3} - h_{N-2} = \frac{p}{q}\left(h_{N-2} - h_{N-1}\right) = \left(\frac{p}{q}\right)^2 h_{N-1},\ i = N-2\\
&\vdots\\
&h_i - h_{i+1} = \frac{p}{q}\left(h_{i+1} - h_{i+2}\right) = \left(\frac{p}{q}\right)^{N-i-1} h_{N-1},\ i = i+1\\
&\vdots\\
&h_0 - h_1 = \frac{p}{q}\left(h_1 - h_2\right) = \left(\frac{p}{q}\right)^{N-1} h_{N-1},\ i = 1.
\end{aligned}
$$

Now, look into the first equation, we have noticed that $h_N = 0$, $h_0 = 1$ this we have, so if I put $h_N = 0$ here this quantity becomes $\frac{p}{q}h_{N-1}$, so we get this quantity same as $\frac{p}{q}h_{N-1}$. Now, look into the second equation here, second equation is that $h_{N-3} - h_{N-2} = \frac{p}{q}(h_{N-2} - h_{N-1})$. Now, this quantity is same as this quantity here, so I can just plug in this value here and I get it that it turns out to be $(\frac{p}{q})^2 h_{N-1}$, this quantity I replace by this one and I got this one. And similarly, I can proceed finally in general I get that $h_{i+1} - h_{i+2}$ turns out to be this quantity, so this power of $\frac{p}{q}$ actually increasing power 1 then 2 then so on it is increasing and along with this h_{N-1} go on and finally the term turns out to be my $\left(\frac{p}{q}\right)^{N-1} h_{N-1}$. So, we have this system of equations and we are trying to solve it. Now, what will happen if I add this equations let us see. So, if I add first notice that this term cancels with this term similarly, I have another equation here for that this one cancels with this, so on proceed and finally this one also cancels with this term here, I have a equation here the previous equation, these two terms will cancel. So, all the terms cancels except this term remains there and this term remains there. This term cancels with this, this term cancels with the similar term in the next equation, this term will cancel with this term, this term will cancel with the similar term in the previous equation. So, finally I am left with h_i here, let me write and I am left with this one $-h_{N-1}$ and what is this side I am left with, I am left with $\left(\frac{p}{q} + (\frac{p}{q})^2 + \cdots + (\frac{p}{q})^{N-i-1}\right) h_{N-1}$, that is what I am left with now and exactly that is what I have written here that $h_i - h_{N-1}$ is same as sum of this term multiplied by multiplied by h_{N-1} that is what I have written here.

Now, this h_{N-1}, I take on the other side so 1 comes here and then I have this, now this one is a geometric sum, we can do this sum but there comes two cases, first case is that if $p = q$, if $p = q$ that means this quantity is 1, this quantity is 1, because now $p = q$ this implies that that $p + q = 1$ basically implies that $p = q = \frac{1}{2}$, so that is why I have written $p = \frac{1}{2}$, if $p = \frac{1}{2}$ that means $q = \frac{1}{2}$ so in this case $p + q = 1$. So, in this case what we got is that I have $N-1$ terms are there, so if I add $N-1$ that will give me $N-i$, so I have $N-i$ here multiplied by h_{N-1}, and when $p \neq \frac{1}{2}$, what we got if $p \neq \frac{1}{2}$ we got this quantity and this comes from the geometric sum. So, we know that if I have $a + a^2 + a^3 + \cdots + a^n = \frac{1-a^{n+1}}{1-a}$. So, the power is basically number of terms here, that is basically the power so exactly that is what I have done here, here the number of term is $N-i$ so $\frac{1-(\frac{p}{q})^{N-i}}{1-(\frac{p}{q})} h_{N-1}$, so $\frac{p}{q}$ the role of a in this case. So, these two, this h_i I got this way and each h_i and this is true for any $i = 0, \cdots, N-2$ and so any h, h_0 to $hN-2$ all of them we have written in terms of $h_{N-1}, h_0, \cdots, h_{N-2}$ we have written in terms of h_{N-1}. Now, recall that we have one condition that $h_0 = 1$. Now, if I take $h_0 = 1$ that basically give me from here is that if $p \neq \frac{1}{2}$, first take the first case that $h_0 = 1$, this actually implies that $1 - (\frac{p}{q})^N$ instead of i, I put 0 here divided by $1 - (\frac{p}{q}) h_{N-1} = 1$, from that I can solve for h_{N-1} which is nothing but the reciprocal of this, so $\frac{1-\frac{p}{q}}{1-(\frac{p}{q})^N}$. So, that is what we can write and once we write this one I can plug in that value here and once I plug in this quantity cancels with this quantity, I am left with $\frac{1-(\frac{p}{q})^{N-i}}{1-(\frac{p}{q})^N}$, that is what written here. Similarly, for $p = \frac{1}{2}$ we can proceed $p = \frac{1}{2}$, h_0 is basically from here nothing but $N \times h_{N-1}$ and this has to be 1, this says that $h_{N-1} = \frac{1}{N}$, so that means if I again plug in that value here it basically turns out that when $p = \frac{1}{2}$, $h_i = 1 - \frac{i}{N}$. So, this way we can able to find these probabilities.

Now, let us see this probability little carefully. Recall that, suppose let us take about $p = \frac{1}{2}$. If $p = \frac{1}{2}$ that basically means that if you increase N what will happen, this quantity will be small, so h_i will increase. So, if I increase N then h_i also increases because this quantity will be small, so this quantity also increases, not only that h_i goes to 1 as N goes to infinity, what does this mean, what is N, N is basically the target the gambler fix, so if the gambler is very very greedy then if the winning probability is $\frac{1}{2}$ then gambler will actually finally broke, gambler will lose everything. So, when we go for gambling we should not fix our target as very high because if we fix this is very high, there is a high chance that I will go broke, I will lose my all my money, that is basically told by here when $p = \frac{1}{2}$. Moreover, when $p = \frac{1}{2}$ again you see that if I have more money, initial money is more if i increases then h_i decreases, that means if I am rich then basically there is a less chance of losing everything, if I am poor there is a more chance of losing everything. So, that are very very intuitive that if I go with lot of money there is a less chance I will basically finally I will come back with a lot of money but if I have very small money there is a chance that I lose quickly, I lose those money and I am just broke, I have to come back empty handed to my home. And this conclusions is not only true for $p = \frac{1}{2}$, even if I take $p \neq \frac{1}{2}$ you can see the same conclusion hold true in this case I write in this manner that suppose now $p < \frac{1}{2}$ which is generally case in any casino, we know that winning probability are smaller, in this case also you can see that this quantity goes to 1, why because p is less than half that basically implies that $\frac{p}{q} < 1$, so $\frac{p}{q} < 1$ and N goes to infinity so this quantity goes to 0, final h_i goes to 1. So, h_i goes to 1, so that means that when winning probability is less compared to losing probability, that if the losing probability, broking probability goes to 1 as $N \to \infty$, again the same conclusion that if I am very greedy there is a chance that I will go broke, I mean there is almost sure that I will go broke, I should not be very greedy. So, if $p > \frac{1}{2}$ then of course, I have some probability of winning even if $N \to \infty$ that h_i goes to some $l > 0$, $l \in [0, 1]$ as $N \to \infty$, that l you can easily find out from the standard limiting thing you can do and you can find it out. So, that is the final take away from this problem, that how we can solve this problem that is one takeaway and then the next takeaway from this problem is as follows that if I am greedy and if the probability of winning is less than or equals to $\frac{1}{2}$ then I will almost sure that I will go broke, I lose everything, if I am not much greedy then there is some positive probability of winning and not losing everything. And in most of the casinos that $p < \frac{1}{2}$, that means winning probability are less than $\frac{1}{2}$ and so in most of the casinos is I am too greedy, I most likely lose everything and if I have more fortune at the initial stage, I have more money at the initial stage there is a high probability that I will not go broke compared to if I have less amount of money at the initial stage when I am going to the casino. With that I stop and in next lectures we will see some more examples of such kind, thank you for listening.