EXCELing with Mathematical Modeling
Prof. Sandip Banerjee
Department of Mathematics
Indian Institute of Technology Roorkee (IITR)
Week – 02
Lecture – 10 (Linear Stability Analysis-II)

Hello, welcome to the course EXCELing with Mathematical Modeling.

Continuing with our previous lecture on stability analysis, I will now extend it to a system of differential equations, mainly, two differential equations.

So, let us consider the differential equation of the form

$$\frac{dx}{dt} = f(x,y), \quad \text{and} \quad \frac{dy}{dt} = g(x,y)$$

So, if (x^*, y^*) is the steady state solution, so obviously, it is going to satisfy this equation and this equation. So, to find the steady state solution we have to put

$$\frac{dx}{dt} = 0, \quad \text{and} \quad \frac{dy}{dt} = 0,$$

which will imply $f(x,y) = 0$ and $g(x,y) = 0$.

And, if we say that (x^*, y^*) is our steady state solution, then obviously this is going to satisfy this equation, and hence if you substitute those two values, then obviously

$$g(x^*, y^*) = 0 \text{ and } f(x^*, y^*) = 0.$$

We will now be finding the condition that when this system of differential equation is stable.

So, as defined before, now it is two variables one is x, another is y, we give a small perturbation $x = X + x^*$, $y = Y + y^*$ about the equilibrium point (x^*, y^*). So, if you do that, you will be getting

$$\frac{dx}{dt} = f(X + x^*, Y + y^*)$$

and

$$\frac{dy}{dt} = g(X + x^*, Y + y^*)$$

Now, we will be using the Taylor series expansion and if you recall the Taylor series expansion is

$$f(x,y) = f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b) + \text{higher order terms},$$

and since we are interested on linear stability analysis, we have only taken the linear terms.

So, this was in the form, let me rewrite them

$$\frac{dx}{dt} = f(x, y) \quad \text{and} \quad \frac{dy}{dt} = g(x, y)$$

So, I will be using this formula or Taylor series expansion on this. So, if I take the first equation, this will give me

$$\frac{dx}{dt} = f(x^*, y^*) + (x - x^*) f_x(x^*, y^*) + (y - y^*) f_y(x^*, y^*) + \text{higher order terms.}$$

In the similar manner,

$$\frac{dy}{dt} = g(x^* y^*) + (x - x^*) g_x(x^*, y^*) + (y - y^*) g_y(x^*, y^*) + \text{higher order terms.}$$

So, we neglect the higher order terms and we put the differential equation. So, from here, you can see that if I differentiate $x = X + x^*$,

$$\frac{dx}{dt} = \frac{dX}{dt},$$

as your x^* is a constant, and also I can replace $x - x^* = X$. So, we will be using this two, and if I substitute here, my

$$\frac{dX}{dt} = X\, f_x(x^*, y^*) + Y f_y(x^*, y^*),$$

because (x^*, y^*) is an equilibrium point, this will be equal to zero, this will be equal to zero, as stated earlier. Similarly,

$$\frac{dY}{dt} = X\, g_x(x^*, y^*) + Y g_y(x^*, y^*)$$

Now, this can be put in the matrix form also. So, if I write it in the matrix form, then

$$\frac{d\bar{x}}{dt} = A\bar{x},$$

where my $\bar{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$ at the point (x^*, y^*).

Now let $\bar{x} = \hat{v}e^{\lambda t}$ $(\hat{v} \neq 0)$ be a trial solution. So, I am going to substitute it here. So,

$$\frac{d\bar{x}}{dt} = A\hat{v}e^{\lambda t}$$

So, if I substitute it here, what I get is

$$\hat{v}\lambda e^{\lambda t} = A\hat{v}e^{\lambda t} \implies (A\hat{v} - \lambda\hat{v})e^{\lambda t} = 0, \quad e^{\lambda t} \neq 0$$

Hence, I get

$$A\hat{v} = \lambda\hat{v}.$$

If you recall this is just the definition of the eigenvalue. So, basically this λ here is the eigenvalue of this matrix A. Now to find this eigenvalue what we do we take this to the left hand side and you take the determinant of that which will give us

$$\begin{vmatrix} f_x - \lambda & f_y \\ g_x & g_y - \lambda \end{vmatrix} = 0$$

If you simplify this, then you get it is as

$$\lambda^2 - (f_x + g_y)\lambda + f_x g_y - g_x f_y = 0.$$

Now if you see this one it is just the addition of off diagonal which is known as the trace of the matrix A and this is just the product or value of Determinant A,

$$\lambda^2 - \text{Trace}(A)\ \lambda + \text{Det}\ (A) = 0$$

So this is basically $Det(A)$. Now for the system to be stable both the eigenvalues needs to be negative. And if both the eigenvalues need to be negative so then sum of the eigenvalues, that is, $\text{Trace}(A)$ must be negative, because if λ_1 and λ_2 are the eigenvalues, this generally gives the sum of the roots and this gives the product of the roots.

So the sum of the roots in both λ_1 and λ_2 are negative, must be less than 0 which implies that the $\text{Trace}(A) < 0$ and if λ_1 and λ_2 are negative, their product must be positive, which implies that the $\text{Det}(A) > 0$.

So the condition that the system of two equations will be stable is that the

$$\text{Trace}(A) < 0 \text{ and } \text{Det}(A) > 0.$$

.

You can remember them as a formula also or you can straight away derive, because anyway you have to find this matrix A and find the eigenvalues. If you can straight away find the eigenvalues just see whether both of them are negative or not.

If both of them are negative, then your system is stable. If it is not, that means if one of them is negative, one of them is positive, or both are positive, in either cases, the system is unstable.

Now this criterion or this condition is known as Routh-Hurwitz's criteria.

So, for two equations, if your characteristic equation is of the form

$$\lambda^2 + a_1\lambda + a_2 = 0,$$

then the condition that the system is stable, is $a_1 > 0$ and $a_2 > 0$.

Please note that this equation, and this equation, there is a tiny difference is of this negative sign.

So, if I bring this negative sign inside, then I put a positive sign here, this negative sign is absorbed in this a_1, and hence here $a_1 > 0$ and $a_2 > 0$.

So, either you write this equation as a negative one, or you write it in the positive one then this is easy to remember.

If the equation is, suppose three, then you get a cubic characteristic equation, then it is

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0.$$

In this case, the Routh-Hurwitz criteria that the system will be stable is

$$a_1 > 0, \qquad a_2 > 0, \qquad a_3 > 0,$$

and with that

$$a_1 a_2 - a_3 > 0.$$

So, if all these four conditions are satisfied we say the system is stable.

So, let us move on to some examples. So, you have to find the stability of the system

$$\frac{dx}{dt} = -x + y$$

and

$$\frac{dy}{dt} = x\,y - 1$$

So, the first thing is, you have to find the points of equilibrium and to do that you put

$$\frac{dx}{dt} = 0 \quad \text{and} \quad \frac{dy}{dt} = 0.$$

This gives you

$$-x + y = 0 \Rightarrow x = y$$

and this gives you

$$xy - 1 = 0 \Rightarrow x.x - 1 = 0 \;\Rightarrow\; x^2 = 1 \Rightarrow x = \pm 1$$

So we have

$$x = \pm 1.$$

So, when $x = 1$, $x\,y - 1 = 0$ gives

$$y = 1$$

and when

$$x = -1, y = -1$$

So we get that

$$(-1,-1) \text{ and } (1,1)$$

to be the steady state solutions.

So, once you get the steady state solution, now you calculate that matrix A. So, the matrix A is you take

$$f(x,y) = -x + y$$

and take

$$g(x,y) = xy + 1$$

So,

$$A = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ y & x \end{pmatrix}$$

at that point x^* and y^*.

So one of the set is $(-1,-1)$ another set is $(1,1)$. So now we have to calculate the eigenvalues.

So for $(1,1)$. So, we see for $(1,1)$ you are getting

$$|A - \lambda I| = 0,$$

this will give you

$$\begin{vmatrix} -1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0.$$

So, if I solve this, this is

$$(1+\lambda)(\lambda - 1) - 1 = 0.$$

So, I get

$$\lambda^2 - 2 = 0 \Longrightarrow \lambda = \pm\sqrt{2}\,.$$

So, one of the eigenvalue is positive, another eigenvalue is negative implies the system is unstable.

At this point, I must notify you, that there are other classifications also, which we will be coming in my next lectures.

For the time being, you just need to know whether the eigenvalues are negative or positive.

If both of them are negative, then the system is stable.

If that criterion is not satisfied, that is if one of them is positive, one of them is negative or both are positive, then the system is unstable.

Let us check what happens for $(-1,-1)$

So, in that case

$$A = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}.$$

Your

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = 0$$

So, we have $(\lambda + 1)^2 + 1 = 0$

So, in that case you get $(\lambda+1)^2 = -1$. We write this as

$$(\lambda+1)^2 = i^2$$

and your

$$\lambda = -1 \pm i\,.$$

So, I specifically choose this example to show that what happens when the value of λ is imaginary.

In that particular case, you have to look at the real part. So, if the real part is negative, then your system is stable. If your real part is positive, then your system is unstable.

So, in that particular case, the system is stable at the point $(-1,-1)$.

And as I told you before, we have more classifications on this stability, and which we will be learning later.

Let us take another example where we introduce the parameters $\alpha, \beta,\ \gamma,\ \delta$, and k and how do you check the stability of this particular example.

$$\frac{dx}{dt} = \alpha x\left(1-\frac{x}{k}\right) - \beta xy, \qquad \frac{dy}{dt} = -\gamma y + \delta xy$$

So, what we generally do is you put $x = X + x^*$ and $y = Y + y^*$, and then you retain only the first order terms and you ignore the higher order terms.

So, if I use that, then I will substitute it here, I will get

$$\frac{d(X+x^*)}{dt} = \alpha(X+x^*)\left(1-\frac{X+x^*}{k}\right) - \beta(X+x^*)(Y+y^*).$$

Now, why we are doing this, because I want to show that by using this method and by using the formula which we just derived, we both get the same answer.

So you just remember any one of them, the previous one is will take less time, this will take more time but this is for the sake of the understanding.

So, here (x^*, y^*) are the equilibrium points and it is also a constant, so in this particular case it is

$$\frac{dX}{dt} = \alpha X + \alpha x^* - \frac{\alpha}{k}(X^2 + x^{*2} + 2x^*X) - \beta(XY + x^*Y + y^*X + x^*y^*)\,.$$

Let us retain only the first order terms and I will ignore or discard the second order terms. So, here it is

$$\frac{dX}{dt} = \alpha X + \alpha x^* - \frac{\alpha}{k}x^{*2} - 2\frac{\alpha}{k}x^*X - \beta x^*Y - \beta y^*X - \beta x^*y^*.$$

So, let us me write the constant term first. So, from this two, if I take $\alpha\, x^*$ common,
I get

$$\frac{dX}{dt} = \alpha x^*\left(1-\frac{x^*}{k}\right) - \beta x^* y^* + \left(\alpha - 2\frac{\alpha}{k}x^* - \beta y^*\right)X - \beta x^* Y$$

Now, if you look into this term, this is nothing but this term where you put $x = x^*$ and $y = y^*$.

And since (x^*, y^*) is the equilibrium solution, obviously this is going to be zero and hence this term

$$\alpha x^*\left(1-\frac{x^*}{k}\right) - \beta x^* y^* = 0.$$

So you are left with

$$\frac{dX}{dt} = \left(\alpha - 2\frac{\alpha}{k}x^* - \beta y^*\right)X - \beta x^* Y.$$

Now, let us simplify the second equation. So for the $\frac{dx}{dt}$ you are left with

$$\frac{dX}{dt} = \left(\alpha - 2\frac{\alpha}{k}x^* - \beta y^*\right)X - \beta x^* Y$$

If I now take

$$\frac{dy}{dt} = -\gamma y + \delta xy.$$

So, you put $x = X + x^*$ and $y = Y + y^*$ like before.

So, this will be

$$\frac{d(Y + y^*)}{dt} = -\gamma(Y + y^*) + \delta(X + x^*)(Y + y^*)$$

So, I simplify

$$\frac{dY}{dt} = -\gamma Y - \gamma y^* + \delta XY + \delta x^* Y + \delta y^* X + \delta x^* y^*$$

And if you write this in the form

$$\frac{dY}{dt} = (-\gamma y^* + \delta x^* y^*) + \delta y^* X + (\delta x^* - \gamma)Y$$

So, we have x term, we have y term and we have the constants.

So, with the same logic if (x^*, y^*) is our equilibrium solution it is going to satisfy this.

So, if I put an x^* here y^* here, this is going to be zero and hence this part

$$(-\gamma y^* + \delta x^* y^*) = 0.$$

And we are left with

$$\frac{dY}{dt} = \delta y^* X + (\delta x^* - \gamma)Y$$

So, this is our linearized form and if we want to construct the matrix A from here, it will be

$$A = \begin{pmatrix} \alpha - \frac{2\alpha x^*}{k} - \beta y^* & -\beta x^* \\ \delta y^* & \delta x^* - \gamma \end{pmatrix}$$

So, we just take the coefficients from here and we put it here.

Now, what I am trying to show you here is that we have the original equation

$$\frac{dx}{dt} = \alpha x\left(1 - \frac{x}{k}\right) - \beta xy,$$
$$\frac{dy}{dt} = -\gamma y + \delta xy.$$

Our previous derived formula says that you take this to be some

$$\alpha x\left(1 - \frac{x}{k}\right) - \beta xy = f(x, y),$$

this to be some

$$-\gamma y + \delta xy = g(x, y)$$

and our matrix

$$A = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

at the point (x^*, y^*).

So, if I find f_x from here, that will give me

$$f_x = \alpha - \frac{2\alpha x}{k} - \beta y.$$

if I differentiate it with respect to x.

If I differentiate it with respect to y, it is just $f_y = -\beta x$. Differentiate this with respect to x, you get $g_x = \delta y$ and with respect to y, it is $g_y = -\gamma + \delta x$. Then,

$$A = \begin{pmatrix} \alpha - \frac{2\alpha x}{k} - \beta y & -\beta x \\ \delta y & -\gamma + \delta x \end{pmatrix}$$

and this need to be calculated at (x^*, y^*),

So I just put x^* and y^* here and you can see that both these matrices are same.

So, whether you use this method or whether you use previous method, you get the same answer but then this is time saving, so we use this.

So, you now have calculated the matrix A, we now have to look into the stability of the system and for that you need to find the equilibrium solution.

So to find the equilibrium solution you have to put

$$\frac{dx}{dt} = 0 \quad \Longrightarrow \alpha x\left(1 - \frac{x}{k}\right) - \beta xy = 0,$$

and

$$\frac{dy}{dt} = 0 \quad \Longrightarrow -\gamma y + \delta xy = 0.$$

So, from this equation you get

$$y(-\gamma + \delta x) = 0 \;\Longrightarrow y = 0 \text{ and } x = \frac{\gamma}{\delta}.$$

So if you put y=0 here, in this particular equation, you get

$$\alpha x\left(1 - \frac{x}{k}\right) = 0 \;\Longrightarrow x = 0 \text{ and } x = k.$$

So for y = 0, you get two values of x. Hence, (0,0) and (k,0) are your steady state solutions.

Now you substitute $x = \frac{\gamma}{\delta}$ here. If you do that, you get

$$\alpha\frac{\gamma}{\delta}\left(1 - \frac{\gamma}{\gamma k}\right) - \beta\frac{\gamma}{\delta}y^* = 0,$$

y^* which I need to find. This is the common part which goes off, and you are left with

$$y^* = \frac{\alpha}{\beta}\left(\frac{\delta k - \gamma}{\delta k}\right).$$

So, another non-zero equilibrium solution is

$$\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\left(\frac{\delta k - \gamma}{\delta k}\right)\right).$$

Now, before that if this equation represents a population. In that particular case, all the solutions have to be positive.

So, we have to assume that all the parameters are positive, that is, $\alpha, \beta, \gamma, \delta$ and k, they are all positive and at the same time the equilibrium solution is positive.

So, this is (k,0), this is, (0,0) and $\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\left(\frac{\delta k - \gamma}{\delta k}\right)\right)$ is positive, for this has to be positive, this part has to be greater than zero.

So, the condition that the non-zero equilibrium point will exist if we consider this as a population then it is $\delta\, k - \gamma > 0$.

So, the point is when you have the equation with parameters, you will see that either it is satisfying directly or you will get some condition for which need to be satisfied for finding the positive equilibrium solution and also for the stability of the system.

Now let us look into the stability of the system, say, at the point (k,0).

So, if you want at the point (k,0) then you find what is your matrix

$$A = \begin{pmatrix} \alpha - \frac{2\alpha x^*}{k} - \beta y^* & -\beta x^* \\ \delta y^* & -\gamma + \delta x^* \end{pmatrix}.$$

So if you want to find at the point (k,0), so I have to substitute $x^* = k$ and $y^* = 0$ and your matrix

$$A = \begin{pmatrix} \alpha - \frac{2\alpha k}{k} - 0 & -\beta k \\ 0 & -\gamma + \delta k \end{pmatrix}.$$

So, if I simplify, I get this

$$A = \begin{pmatrix} -\alpha & -\beta k \\ 0 & -\gamma + \delta k \end{pmatrix}$$

If I want to find the eigenvalues, then it is

$$|A - \lambda I| = 0,$$

which will give me

$$|A - \lambda I| = \begin{vmatrix} -\alpha - \lambda & -\beta k \\ 0 & -\gamma + \delta k - \lambda \end{vmatrix} = 0$$

So, from here I get $(\lambda + \alpha)(\lambda - \delta k - \gamma) = 0$. So, one of the value you can see $\lambda = -\alpha$ and another value is

$$\lambda = \delta k - \gamma.$$

This is clearly negative because α is a positive constant, but this because of the existence of positive equilibrium, this is positive.

So, one of the λ is one of the eigenvalue is negative, one of the eigenvalue is positive. So, at (k,0) system is unstable.

Similarly, you can check for $(0,0)$ and this non-zero equilibrium point, which I leave for practice for you.

In my next class, I will be again discussing about the stability of the system but Lyapunov stability.

Till then bye-bye.