EXCELing with Mathematical Modeling
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Week – 03
Lecture – 11 (Lyapunov Stability)

Hello, Welcome to the course EXCELing with Mathematical Modelling.

Today is my third lecture on this stability of the dynamical system and today we will be discussing about Lyapunov stability.

Before that, let me recall this quadratic forms, what is quadratic form, what is the positive semi-definite, what is positive definite, negative definite and so on.

We will be using this in the definition of Lyapunov stability.

So, start with an expression of the form

$$\sum_{i,j=1}^{n} a_{ij}x_i x_j ,$$

where a_{ij} is are real numbers. So, an expression of this form, is called a real quadratic form.

For example, if I write $ax^2 + 2hxy + by^2$, where your a, h and b are real numbers, then this is a quadratic form in two variables. Variables are x and y.

Similarly, $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$, this is a quadratic form in three variables.

I can put numeric of this a, b, and c, so, I can write say $4x^2 + 5xy + 9y^2$. This is a quadratic form of two variables.

If I denote this by

$$\emptyset = \sum_{i,=1}^{n}\sum_{j=1}^{n} a_{ij}x_i y_j,$$

then there exists a matrix, a unique symmetric matrix, such that

$$\emptyset = X^T BX,$$

where B is this unique and symmetric matrix and we say B to be the matrix of the quadratic form.

So how this $(X^T BX)$ is equal to this $(\sum_{i,=1}^{n}\sum_{j=1}^{n} a_{ij}x_i y_j)$?

Well, we can easily see it with the help of this example.

Say, suppose I take this quadratic form $4x^2 + 5xy + 9y^2$ and

I write $B = \begin{pmatrix} a & h \\ h & b \end{pmatrix} = \begin{pmatrix} 4 & 5/2 \\ 5/2 & 9 \end{pmatrix}$.

If I compare it with $ax^2 + 2hxy + by^2$ and $4x^2 + 5xy + 9y^2$, I will get a = 4, b = 9 and h = 5/2, $\Rightarrow B = \begin{pmatrix} a & h \\ h & b \end{pmatrix} = \begin{pmatrix} 4 & 5/2 \\ 5/2 & 9 \end{pmatrix}$, where my X is $\begin{pmatrix} x \\ y \end{pmatrix}$.

Generally, it is always customary to write the vector as a column, so this is $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $X^T = (x \;\; y)$. So, if I now find what is my $X^T BX$ here, so this will be

$$(x\ y)_{1\times 2} \begin{pmatrix} 4 & 5/2 \\ 5/2 & 9 \end{pmatrix}_{2\times 2} \begin{pmatrix} x \\ y \end{pmatrix}_{2\times 1}$$

$$= (x\ y)_{1\times 2} \begin{pmatrix} 4\mathrm{x} + 5\mathrm{y}/2 \\ 5\mathrm{x}/2 + 9\mathrm{y} \end{pmatrix}_{2\times 1}$$

$$= \mathrm{x}\left(4\mathrm{x} + \frac{5\mathrm{y}}{2}\right) + \mathrm{y}\left(\frac{5\mathrm{x}}{2} + 9\mathrm{y}\right),$$

and if we simplify this, it is $4x^2 + 5xy + 9y^2$ and we get this particular quadratic form.

So, $\emptyset = \sum_{i,=1}^{n} \sum_{j=1}^{n} a_{ij} x_i y_j$ and $\emptyset = X^T BX$, they are the same.

Now, let us move on to the definition of positive definite.

We now write $\emptyset = X^T BX$ which is equal to $\sum_{i,=1}^{n} \sum_{j=1}^{n} a_{ij} x_i y_j$ Now, this quadratic form is

(i) positive definite, if $\emptyset > 0$, for all $X \neq \mathbf{0}$ (null matrix),
$= 0$, for $X = \mathbf{0}$

(ii) positive semi definite, if $\emptyset \geq 0$, for all $X \neq \mathbf{0}$,
$= 0$, for $X = \mathbf{0}$

(iii) negative definite, if $\emptyset < 0$, for all $X \neq \mathbf{0}$,
$= 0$, for $X = \mathbf{0}$

(iv) negative semi-definite, if $\emptyset \leq 0$, for all $X \neq \mathbf{0}$,
$= 0$, for $X = \mathbf{0}$

and finally, this quadratic form is called

(v) indefinite, if $\emptyset < 0$, for some $x \neq \mathbf{0}$, and $\emptyset > 0$ for some $X \neq \mathbf{0}$.

With this definition, we now move on.

If we take an example, $5x^2 + y^2 + 5z^2 + 4xy - 8xz - 4yz$, and we have to check under what category it falls.

So the trick is that you try to make this a whole square.

So if I want to make this a whole square, so instead of taking say root 5 whole square, I will prefer, I will write it as

$$4x^2 + y^2 + 4z^2 + x^2 + z^2 + 4xy - 8xz - 4yz$$

$$= (2x)^2 + y^2 + (2z)^2 + 2\,(2x)\,(y) - 2\,(2x)\,(2z) - 2\,(y)\,(2z) + x^2 + z^2$$

$$= (2x + y - 2z)^2 + x^2 + z^2.$$

So, you can see that this is always positive for all nonzero x, y and z and this is only equal to zero, when x =0, y = 0, z = 0, which satisfies this definition of positive definite and hence this quadratic form is positive definite.

So, that is how you just prove.

Now, let us move to this Lyapunov's stability.

So, this is another way of checking the stability of the system without explicitly integrating the differential equation.

So, the man who gave this idea is Alexander Mikhailovich Lyapunov and his concept is that if you have a differential equation

$$\frac{d\tilde{x}}{dt} = f(\tilde{x}), x(t_0) = \tilde{x}_0 \quad where\ f: R \to \ R$$

with an initial condition, then he use an idea that if the potential energy has a relative minimum at the equilibrium point, then the equilibrium point is stable.

So, basically if you have a minimum potential energy, then your equilibrium point is stable, otherwise it is unstable.

So, let us check that what is its relation between the stability and this potential energy. I mean, how do you relate? How when potential energy decreases, you get more stability?

So, this can be explained by this example, which I saw in YouTube and it goes like this.

You consider two electrons, say these two electrons, and I put them in a vacuum. This is my setup 1.

Then after some time, I move it here and since they are both negative, obviously they will move apart. So, this is my second setup.

Call this setup 2.

Now I say that setup 2 is more stable than setup 1.

Now how it is that?

So I say fig 1 is stable and fig 2 is unstable.

So it is like this, that this arrangement 2, it is more preferable by nature than arrangement 1.

And why is that?

Say, you start from an initial condition, say, from here and if you drop a particle so obviously it will come here oscillate a bit and ultimately it will settle here.

So the nature wants that if you drop anything, a particle from here so it will settle down here.

So in the similar manner that if you keep these two particles as an initial condition and then after some time I move it here and I saw that they moved apart, which is obvious because you have two negative electrons and they will move apart.

So, hence for nature, this is more convenient than this one.

And hence, we say that this system 2 is more stable than system 1.

So, unless we have these two of them, I cannot compare.

So, if only this is there or this is there, then it does not make any sense.

So, I have to have a second option or second diagram, through which I can compare that which one is more stable than the other.

So, in this particular case, your setup 2 is more more stable than setup 1.

So, I say this one is stable and this one is unstable.

Say if I take another, one is positive, another is negative, put them in vacuum, so, this is again the setup 1 and after some time, I will see because one is positive, another is negative,

They will attract each other, so again, I say that this is more stable than this one.

So I categorize this as stable and this as unstable.

Now, I have moved this from here to here. So there is some change in energy.

So the question is that what happens to this total energy? And as we know that energy is neither created nor destroyed. It was changes to one form to another form. And because it is inside the vacuum, this holds true.

So moving this from here to here, there is a change in energy and we know that is the kinetic energy.

So I say that let us here the kinetic energy is zero, and here, say, the kinetic energy is 10.

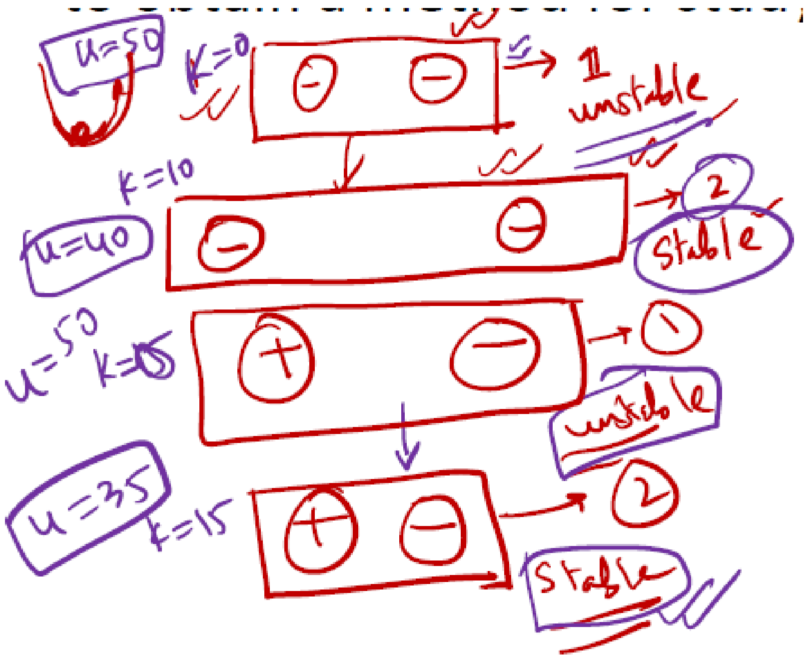
Now the question is where the initial energy comes from and that is where the potential energy comes. So here inside there is some potential energy.

So this configuration already has an inbuilt potential energy say U = 50, and I moved it from here to here, which generates this kinetic energy of unit 10.

And since the total energy remains constant, in this particular case, your U will be 40.

So what do you notice? That from here to here, your potential energy decreases and your stability is increasing. So from unstable or rather this setup is more stable than this one.

So as your potential energy decreases, your stability increases.



The same thing is here.

From here to here, I have moved this frame. So there is a kinetic energy. Say that kinetic energy is 15.

From moving to here to here, initially it was zero. Here it was 15.

U, I take that to be again 50. So, here U will be 35.

So, again there is a change in the potential energy which decreases from its initial point, from its initial configuration and you move from unstable to stable or rather this configuration is more stable than this one.

So, with this example, it is clear that as your potential energy decreases, you attain more stability.

So, this Russian mathematician Alexander Mikhailovich Lyapunov, he generalized this principle and he figured out a method for studying the stability of this autonomous system.

So, what is the condition for this Lyapunov stability? So, he says that he defines an autonomous system. So, this is the autonomous system with the initial condition

$$d\tilde{x}/dt = f(\tilde{x}), \qquad x(t_0) = x_0 \text{ where } f: R \rightarrow R,$$

having isolated critical point at the origin $\tilde{x} = 0$ and $f(\tilde{x})$ has continuous partial derivatives for all $\tilde{x}$.

Here origin has been taken the critical point if any other if there is an $\tilde{x}$ star as the critical point the same thing holds true or you can shift the origin to $\tilde{x}$ star.

So, this system has a critical point $\tilde{x} = 0$, and this function has a continuous partial derivative for all $\tilde{x}$. Then he defined a function $V(\tilde{x})$ which he called as the Lyapunov function and this function must be positive definite in the neighborhood of the origin.

And the derivative of this function, which is $\dot{V}(\tilde{x})$ of $V(\tilde{x})$ with respect to the system is negative semi definite in the neighborhood of $\tilde{x} = 0$.

So, if you get such a function, then this $V(\tilde{x})$ is called the Lyapunov function of the system.

If you want to look at mathematically, so this $V(\tilde{x})$ is called a Lyapunov function if

(i) $V(\tilde{x}) > 0$ in the neighborhood of the origin $\tilde{x} = 0$,

(ii) $V(\tilde{x}) = 0$ for all $\tilde{x} = 0$,

(iii) $\dot{V}(\tilde{x}) \leq 0$ in the neighborhood of the origin $\tilde{x} = 0$,

(iv) $\dot{V}(\tilde{x}) = 0$ for all $\tilde{x} = 0$.

So, if all these four conditions satisfies for a function $V(\tilde{x})$, we call this a Liopunov function.

So, if there exists a Lyapunov function $V(\tilde{x})$ if you can find such a function, which is not unique again, and there is no hard and fast rule that how you will find that function but if you can find that function in the neighborhood of the equilibrium point in this case it is origin x = 0 then the steady state x = 0 solution is called Lyapunov stable.

However, if $\dot{V}(\tilde{x}) < 0$ in the nbd. of the origin $\tilde{x} = 0$, then the steady- state solution $\tilde{x} = 0$, it is called asymptotically stable.

So, we have to just check that whether a $V(\tilde{x})$ exists, then this function has to be positive definite, $V(0)$ has to be 0 and the derivative has to be negative semi-definite.

If it is semi-definite, if it is Lyapunov stable, if it is negative definite it is asymptotically stable.

So, this is the result for this Lyapunov stability.

Let us check with examples, before that this is Lyapunov condition for global stability.

So, whatever we have studied previously, those are local stability by local stability I mean that if you have this as your equilibrium point, and you start somewhere in the neighborhood of this point, then this is going to reach the equilibrium point is the system is stable.

But this global stability you consider the entire domain and no matter from where you start in the whole domain, this is going to reach this particular equilibrium point if the system is globally stable about that equilibrium point.

So, this is the basic difference between this local stability and the global stability.

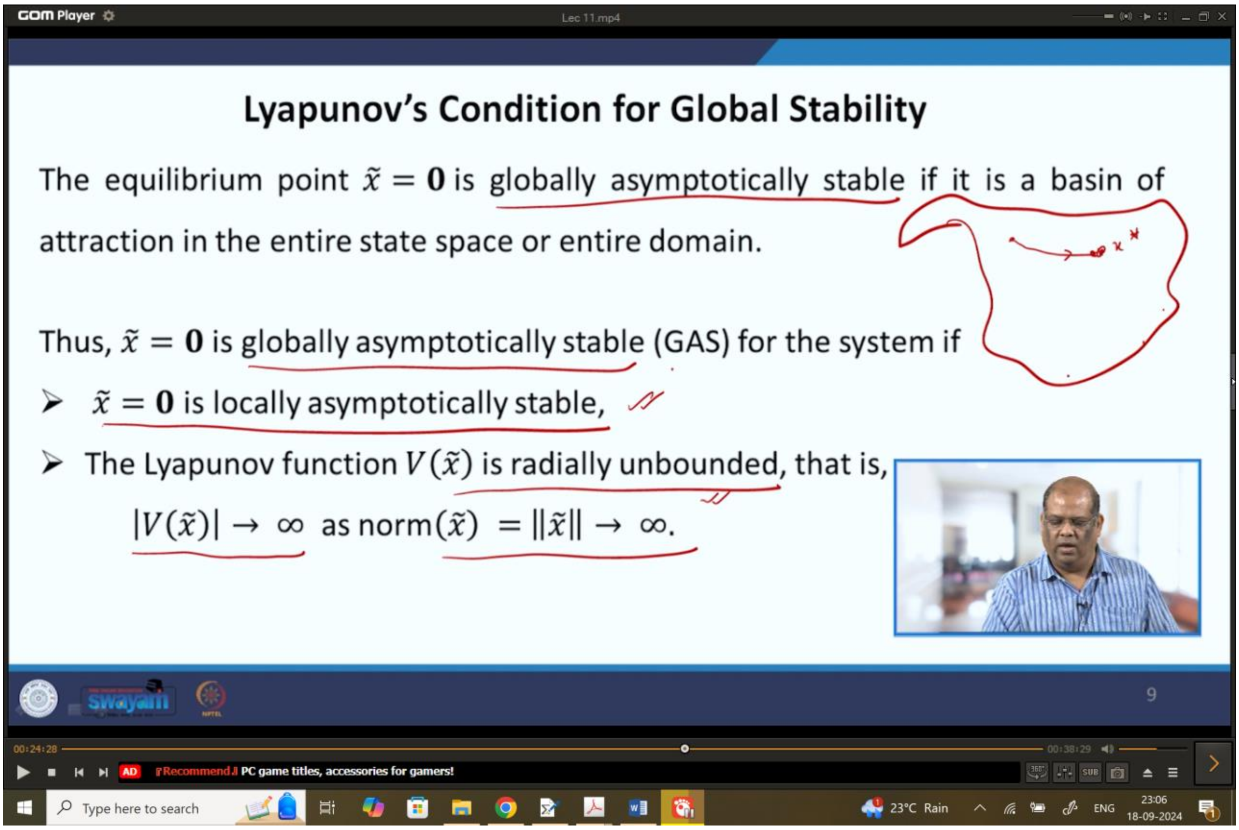
So, local stability is in the neighborhood and the global stability is in the entire domain.

So, for $\tilde{x} = 0$ to be globally asymptotically stable, first it has to be locally asymptotically stable, that is a Lyapunov function must exist, that function has to be positive definite and the derivative has to be negative definite not semi-definite because it is locally asymptotically stable.

And once you prove that it is locally asymptotically stable you see that whether this mod of the Lyapunov function it goes to infinity as the norm goes to infinity which is known as radially unbounded.

, $$|V(\tilde{x})| \to \infty \text{ as norm } (\tilde{x}) = ||\tilde{x}|| \to \infty.$$

If along with locally asymptotically stable, the function is also radially unbounded, then we say that $\tilde{x} = 0$ is globally asymptotically stable, in short, GAS.



Now, let us take examples. So, the first example that we take

$$\frac{dx}{dt} = -x - y, \qquad \frac{dy}{dt} = x - y^3$$

So, if you want to find the fixed points, so from here, so you put

$$-x - y = 0,\ x - y^3 = 0$$

and one of the fixed point is (0,0), which you can see from here because it can satisfy the equation. So, you find the matrix A

$$A = \begin{pmatrix} -1 & -1 \\ 1 & -3y^2 \end{pmatrix}_{(0,0)} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

If you want to calculate the eigenvalue

$$\begin{vmatrix} -1-\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}.$$

Well, this is what we have done before and this is also called Lyapunov's first method.

What we learned about this Lyapunov function is called Lyapunov's second method or direct method.

So, here we define the function $V(x, y) = ax^2 + by^2 \quad (a, b > 0)$.

Well, obvious question is how I choose this function. As I told you, there is no hard and first rule, after doing looking into examples, few examples, you will understand.

First of all, this function is positive definite because you can see that other than x = 0 and y = 0, this function is always positive. So, it is positive definite.

$$V(x, y) > 0 \ \forall \ (x, y) \neq (0,0) \text{ and also, } V(0,0) = 0.$$

So, very important that you check these two properties.

Next comes

$$\dot{V}(x, y) = \frac{dV}{dt} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} = 2ax(-x - y) + 2by(x - y^3)$$

Now you simplify. So, if you simplify it is

$$\dot{V}(x, y) = -2ax^2 - 2by^4 + 2xy(b - a)$$

So, I will choose the a and b in such a manner, such that this is less than zero, either less than zero or less or equal to zero. If it is less or equal to zero, then this will be negative semi-definite. If it is only less than zero, then it will be negative definite.

So it is clear that if I choose b = a, and whatever may be the value, if I choose b = a, equal to either 1, 2, 1/2, whatever the value you choose, this going to vanish and you are left with

$$\dot{V}(x,y) = -2ax^2 - 2by^4,$$

Which is always less than zero for a = b = 1 (say),

$$\dot{V}(x,y) = -2x^2 - 2y^4,$$

and which is always less than zero for non-zero x and y. So, this becomes negative definite.

So, your V is positive definite, your $V(0,0) = 0$ and your $\dot{V}(\tilde{x}) < 0$, which means that the system is asymptotically stable.

So, this is locally stable about the origin or rather asymptotically stable about the origin.

Now let us see whether your $V(x,y)$ is radially bounded or not.

So, I have to find the norm of $(x,y) = \sqrt{x^2+y^2} \to \infty$ as $(x,y) \to \infty$, so

$$\|X\| = \sqrt{x^2+y^2} \to \infty \text{ and } V(x,y) = x^2 + y^2 \to \infty.$$

Hence $V(x,y)$is radially unbounded.

So, your condition for global stability is fulfilled and you say the system is globally asymptotically stable.

So, to sum up what you have to do is you first find the equilibrium point, then you have to write the Lyapunov function, I mean you have to write the function $V(x,y)$ and show that it is a Lyapunov function.

This is totally will come from practice, there is no hard and first rule.

by doing few problems you will understand what kind of function you have to take and you have to show that this function is a Lyapunov function that means it has to be positive definite, it has to be zero at the equilibrium point, in this case it is origin, and the derivative V has to be negative definite or negative semi-definite in this case it is negative definite if you choose b = a, equal to any value, in this case I have chosen 1, and once it is done, you prove that you show that the system is asymptotically stable.

Then you have to show that it is radially unbounded for that you take the norm and as (x,y) tends to infinity this norm also goes to infinity and this $V(x,y)$ also goes to infinity.

So, by the condition of globally asymptotically stable, the Lyapunov's condition, so as your $V(x,y)$ tends to infinity as your norm goes to infinity and that condition is fulfilled and you say the system is globally asymptotically stable.

We take another example say

$$\frac{dx}{dt} = -x + y^2, \qquad \frac{dy}{dt} = -2y + 3x^2$$

So, as usual we define our $V(x, y) = ax^2 + by^2$, clearly,

$$V(x, y) > 0 \; \forall \; (x, y) \neq (0,0)$$

$$V(0,0) = 0.$$

Clearly, here also, x = 0, y = 0 is your steady state solution or equilibrium solution because this satisfies this equation.

So, I now find what is my $\dot{V}$. So, your

$$\dot{V}(x, y) = \frac{dV}{dt} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y}$$

$$= 2ax(-x + y^2) + 2by(-2y + 3x^2)$$

$$= -2x^2(a - 3by) - 2y^2(2b - ax)$$

So the reason I choose minus because I have to show this is less or equal to zero.

Now if I want this to be less or equal to zero, so clearly I must have

$$a - 3by > 0, \quad 2b - ax > 0$$

So if these two are positive quantities clearly then this will be less than zero.

So, the condition becomes

$$a > 3by$$

$$y < \frac{a}{3b}$$

Similarly,

$$2b - ax > 0$$

$$2b > ax$$

$$x < \frac{2b}{a}$$

So, if I choose again my a= b = 1, then I get $x < 2, y < \frac{1}{3}$.

So, if this condition holds, then your $\dot{V}(x, y) < 0$, hence the system is locally asymptotically stable.

But, if I choose my a, to be say, $a = \frac{1}{2}$, and $b = \frac{1}{4}$, I get $x < 1$ and $y < \frac{2}{3}$.

So, the condition changes, but you still attend the asymptotic stability.

So, this is how your Lyapunov function is being used to calculate the stability of the system.

In the next lecture, we will be talking about the phase portrait and the phase plane analysis.

Till then, bye-bye.