EXCELing with Mathematical Modeling
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Week – 03
Lecture – 13 (Phase Plane Analysis-II)

Hello, welcome to the course EXCELing with Mathematical Modelling.

In my previous lecture, we have done phase plane analysis, which includes phase portrait, the trajectories and the classifications, mainly the node, the focus, the saddle point and the centre.

Now the question is that all these depends on the eigenvalues and its sign.

We have seen that if the eigenvalues are real and negative, then we say that you attend a stable node.

Now the question is why?

So in this particular lecture, I will give a rigorous mathematical proof that when two eigenvalues are real and negative, they will give you a stable node.

So to start with, we take the differential equation

$$\frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y) \quad \text{——— (1)}$$

This is an autonomous system, that means the right hand side does not depend explicitly on t.
I take $(0\ 0)$ to be the critical point or the equilibrium point.

Let C be the path of the system and $x = f(t)$ and $y = g(t)$, they are the parametric solutions of (1).

Now I need to explain two definitions.

We say the path approaches (0,0), the critical point, as $t \to \infty$ (or $t \to -\infty$) if

$$\lim_{t\to\infty} f(t) = 0, \qquad \text{and} \qquad \lim_{t\to\infty} g(t) = 0.$$

We use the word the path approaches (0, 0).

If along with that you have

$$\lim_{t\to\infty} \frac{g(t)}{f(t)}$$

exists or if the quotient $\frac{g(t)}{f(t)}$ becomes either positively or negatively infinite, as $t \to \infty$, we say the path enters (0,0)

So we have two new terms one is the path approaches (0 0) and another is the path enters (0 0)

So if the path approaches (0 0) we are these two limits has to be satisfied and if the path enters (0,0) along with these two, this particular condition need to be satisfied.

Now let us come to the definition of node.

So, (0 0) is your critical point of equation (1), is called a node, if there exists a neighbourhood of (0 0) such that every path P in this neighbourhood has the following properties:

Property 1: P is defined $\forall\, t > t_0$ (or $\forall\, t < t_0$).

Property 2: P approaches (0,0) as $t \to \infty$.

Property 3: P enters (0,0) as $t \to \infty$.

So, if these three conditions are satisfied, we say the critical point (0,0) is a node.

Now, let us come to the proof.

So, the hypothesis is that the roots of the characteristic equation are real, unequal and of same sign.

So, we consider a linearized version of the autonomous system, that is,

$$\frac{dx}{dt} = ax + by\,, \quad \frac{dy}{dt} = cx + dy \quad \dots(1)$$

So, we assume let us the solution of this be of the form

$$x = Ae^{\lambda t}, \qquad y = Be^{\lambda t},$$

where your A, B, λ are constants.

So if you assume this to be the solution of this system of linearized equation, you just substitute them and you get

$$A\lambda e^{\lambda t} = aAe^{\lambda t} + bBe^{\lambda t}$$

Similarly, you will get

$$B\lambda e^{\lambda t} = cAe^{\lambda t} + dBe^{\lambda t}$$

Now since, $e^{\lambda t} \neq 0$, you cancel it from the left hand side and the right hand side and you will get after simplification

$$(a-\lambda)A + bB = 0, \quad cA + (d-\lambda)B = 0$$

So here your A *and* B, they are the variables for this equation.

I mean, do not confuse with these constants. So these constants are here, but when you have substitute, now you are solving for this A and B. Let us say they are unknowns.

Now if I consider that

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} \neq 0,$$

Then, clearly, $A = 0,\ B = 0$ is the unique solution and this is known as also the trivial solution.

But we are interested in the non-trivial solution and for that this determinant must be equal to zero.

So, you have $\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$, for the non-trivial solution.

Now, if you expand the determinant you get the characteristic equation, which is

$$\lambda^2 - (a+d)\lambda + ad - bc = 0,$$

.

and you say let λ_1 and λ_2 be the roots of this characteristic equation.

We choose $\lambda = \lambda_1$ and substitute in this particular equation, let us name it as (1).

So let me rewrite this substitute in

$$(a-\lambda)A + bB = 0$$

$$cA + (d-\lambda)B = 0$$

So, you substitute $\lambda = \lambda_1$ and obtain the non-trivial solution which will be of the form

$$x = A_1 e^{\lambda_1 t}, \qquad y = B_2 e^{\lambda_1 t}\,.$$

So basically you substitute $\lambda = \lambda_1$ here and obtain some solution of A and B, which you name it as A_1 and B_1.

And once you get it you substitute it back $x = Ae^{\lambda t}$ that was the form of the solution.

So you substitute in place of $A = A_1$ which is obtained from this equation and the value of $\lambda = \lambda_1$. So that is how you get this particular expression and this particular expression.

In the similar manner if you put $\lambda = \lambda_2$ you will get the solution to be

$$x = A_2 e^{\lambda_2 t}, \qquad y = B_2 e^{\lambda_2 t}\,.$$

So again in the similar way you substitute it here and get another solution which you name it $A = A_2,\ B = B_2$ for $\lambda = \lambda_2$ and you substitute back here and you get again this solution.

So, here your A_1, A_2, B_1 and B_2 they are called definite constants. These solutions are linearly independent. So, your general solution will be of the form

$$x = c_1 A_1 e^{\lambda_1 t} + c_2 A_2 e^{\lambda_2 t}, \qquad y = c_1 B_1 e^{\lambda_1 t} + c_2 B_2 e^{\lambda_2 t}.$$

So, this is now the general solution of the differential equation $\frac{dx}{dt} = ax + by$, $\frac{dy}{dt} = cx + dy$.

Let me rewrite the equation

$$x = c_1 A_1 e^{\lambda_1 t} + c_2 A_2 e^{\lambda_2 t},$$

$$y = c_1 B_1 e^{\lambda_1 t} + c_2 B_2 e^{\lambda_2 t}$$

Here, c_1 and c_2 they are arbitrary constants whereas your A_1, A_2, B_1 and B_2, they are definite constants.

Now, let us choose $c_2 = 0$.

So, if you choose $c_2 = 0$, you get $x = c_1 A_1 e^{\lambda_1 t}$, $y = c_1 B_1 e^{\lambda_1 t}$ and if I divide, I get

$$\frac{y}{x} = \frac{B_1}{A_1} \qquad \ldots (2). \qquad \text{This is one set.}$$

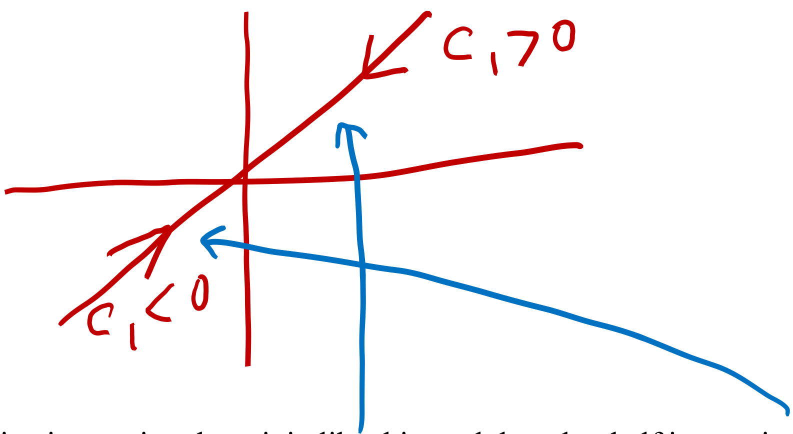
Again if I choose $c_1 = 0$, you get $x = c_2 A_2 e^{\lambda_2 t}$, $y = c_2 B_2 e^{\lambda_2 t}$, then again

$$\frac{y}{x} = \frac{B_2}{A_2}, \qquad \ldots (3). \qquad \text{My another set.}$$

Now, I say let λ_1 and λ_2 are negative.

So by taking that I will show that this represents a stable node.

So, if your $c_1 > 0$ the solution which is represented here, they will consist of a half straight line, which is of the form $y = \frac{B_1}{A_1} x$ with slope $\frac{B_1}{A_1}$
.
And if your $c_1 < 0$, then again you will get the other half of the straight line, which passes through the origin as you can see here and with the slope $\frac{B_1}{A_1}$. So, the two paths enters (0 0) with slope $\frac{B_1}{A_1}$. So, what you actually getting is, that you have something like this.



So, one half of the line is entering the origin like this, and the other half is entering like this, it is the same line. This is, for say $c_1 > 0$, and this is, for say $c_1 < 0$.

And the properties which you need to see is, we have taken $x = f(t)$ and $y = g(t)$.

So, if I express that this is my $f(t) = c_1 A_1 e^{\lambda_1 t}$ and this is my $g(t) = c_1 B_1 e^{\lambda_1 t}$, then I need to show

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} c_1 A_1 e^{\lambda_1 t} = 0.$$

.

Since, $\lambda_1 < 0$, this will be equal to zero, because there will be an exponential decay.

It is more clear, if you can just put a negative sign and you can just visualize as $t \to +\infty$, this becomes less and less and ultimately goes to zero.

In the similar manner, $\lim_{t\to\infty} g(t) = \lim_{t\to\infty} c_1 B_1 e^{\lambda_1 t} = 0$, which means the path approaches (0,0) along with that if you now show that

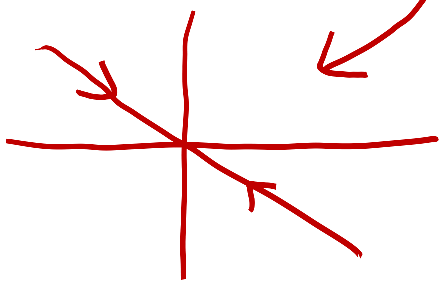
$$\lim_{t\to\infty} \frac{g(t)}{f(t)} = \lim_{t\to\infty} c_1 A_1 e^{\lambda_1 t} = \frac{B_1}{A_1}$$

So, a constant value. So, this implies that the path enters (0 0).

So, both the properties are satisfied and we get that when $c_1 > 0$, you get a half straight line and $c_1 < 0$, you get the another half and together they consist of this particular straight line.

The same thing holds for c_2 that if your $c_2 > 0$, in that particular case you will get your $x = c_2 A_2 e^{\lambda_2 t} = f(t), y = c_2 B_2 e^{\lambda_2 t} = g(t)$, and your

$$\frac{y}{x} = \frac{B_2}{A_2}$$



.

So if $c_2 > 0$, you get a half line and if $c_2 < 0$ you get another half of the line both are approaching and entering the equilibrium (0,0). From here you can see that since, $\lambda_2 < 0$,

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} c_2 A_2 e^{\lambda_2 t} = 0.$$

With the similar logic, you will get

$$\lim_{t\to\infty} g(t) = \lim_{t\to\infty} c_2 B_2 e^{\lambda_2 t} = 0.$$

Also,

$$\lim_{t\to\infty} \frac{g(t)}{f(t)} = \lim_{t\to\infty} \frac{B_2}{A_2} = \frac{B_2}{A_2}, \text{it is a constant and gives you a finite limit.}$$

So that means it approaches and enters the point (0, 0). So these two are rectilinear cases by rectilinear cases means when you get a straight line.

Now let us move to the non-rectilinear cases.

Suppose you have $c_1 \neq 0, c_2 \neq 0$. In that case what you will get?

So, if I write the equation one more time this is

$$x = c_1A_1e^{\lambda_1 t} + c_2A_2e^{\lambda_2 t}, \qquad y = c_1B_1e^{\lambda_1 t} + c_2B_2e^{\lambda_2 t}.$$

So, here $\lambda_1 < \lambda_2 < 0$. So, both $\lambda_1,\ \lambda_2$ are negative values.

So, if we now check

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty}(c_1A_1e^{\lambda_1 t} + c_2A_2e^{\lambda_2 t}) = 0\,,\text{and}$$

$$\lim_{t\to\infty} g(t) = \lim_{t\to\infty}(c_1B_1e^{\lambda_1 t} + c_2B_2e^{\lambda_2 t}) = 0.$$

So, this is because this is negative, it is easy to visualize if you are directly put a negative sign here.

So this goes to zero, this goes to zero, and you are getting this value to be zero.

So why we are doing this? Because in the definition of the node, if you recall, it says that the path approaches zero and the path enters (0, 0).

So, these two property will ensure that the path approaches the equilibrium point (0,0) and the third property, which says $\lim_{t\to\infty}\frac{g(t)}{f(t)}$, if this also gives you a finite limit or tends to plus infinity or minus infinity, then we say that the path enters (0, 0).

So, here, we find this

$$\frac{y}{x} = \frac{(c_1B_1e^{\lambda_1 t} + c_2B_2e^{\lambda_2 t})}{(c_1A_1e^{\lambda_1 t} + c_2A_2e^{\lambda_2 t})} = = \frac{\frac{c_1B_1}{c_2}e^{(\lambda_1-\lambda_2)} + B_2}{\frac{c_1A_1}{c_2}e^{(\lambda_1-\lambda_2)t} + A_2}.$$

Now,

$$\lim_{t\to\infty}\frac{g(t)}{f(t)} = \lim_{t\to\infty}\frac{y}{x} = \lim_{t\to\infty}\frac{(c_1B_1e^{\lambda_1 t} + c_2B_2e^{\lambda_2 t})}{(c_1A_1e^{\lambda_1 t} + c_2A_2e^{\lambda_2 t})} = \lim_{t\to\infty}\frac{\frac{c_1B_1}{c_2}e^{(\lambda_1-\lambda_2)} + B_2}{\frac{c_1A_1}{c_2}e^{(\lambda_1-\lambda_2)t} + A_2}.$$

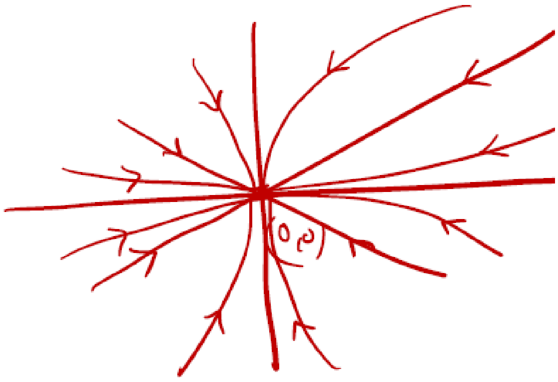
So, you can see that this tends to zero because you are already you have taken that $\lambda_1 < \lambda_2 < 0$, so, $\lambda_1 - \lambda_2 < 0$.

As $t \to \infty$, $\frac{c_1B_1}{c_2}e^{(\lambda_1-\lambda_2)} \longrightarrow 0, \frac{c_1A_1}{c_2}e^{(\lambda_1-\lambda_2)t} \longrightarrow 0$, and the limiting value

$$\lim_{t\to\infty}\frac{\frac{C_1B_1}{C_2}e^{(\lambda_1-\lambda_2)} + B_2}{\frac{C_1A_1}{c_2}e^{(\lambda_1-\lambda_2)t} + A_2} = \frac{B_2}{A_2}$$

So, now this ensures that the path approaches (0 0) and this ensures the path enters (0 0).

So, for both the rectilinear case and non-rectilinear case, we see that the path approaches (0,0) as well as the path enters (0 0), and by definition of the node, this (0,0) will give you a node, when both of the eigenvalues are negative and if you draw the figure you will get something like this.



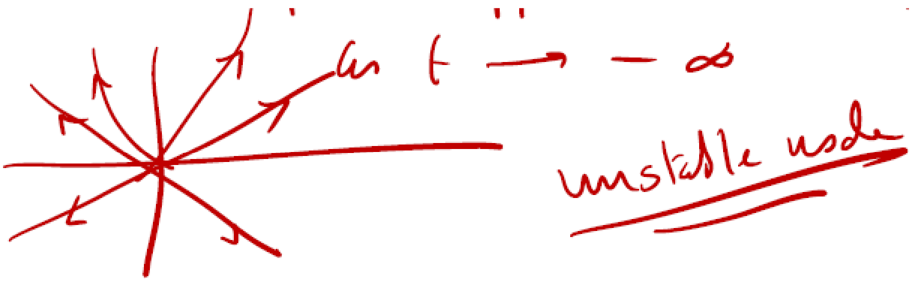
The first one is a straight line for the rectilinear. The another one is also another straight line, and the rest are the non-rectilinear paths.

So, you can just draw something like this, and hence this represents a node and as you can see that they approach and enters the origin (0,0), hence an asymptotically stable node.

So, when $\lambda_1 < \lambda_2 < 0$, you see that the path represents a node, I mean the trajectories, they becomes an asymptotically stable node about the equilibrium point (0,0).

If your $\lambda_1, \lambda_2 > 0$, then everything remains the same, but only thing that you have to derive the whole thing when $t \to -\infty$, and you will see that the all the paths whether it is rectilinear or whether it is non-rectilinear, this will approach and enters (0,0) as $t \to -\infty$ and because it tends to $-\infty$, the diagram will be exactly the same only the direction of the arrow will change.

So in this particular case, this is the straight line and this is the straight line your arrow will be outwards along with the paths and this represents an unstable node.



So, with this, I assume that you get a clear idea that when two eigenvalues the real and negative, they will represent a stable node, asymptotically stable node.

In a similar way you can prove for a spiral or for a saddle point or even for a centre.

In the next lecture, we will be taking some typical examples involving this Lyapunov stability.

Till then bye-bye.