EXCELing with Mathematical Modeling
Prof. Sandip Banerjee
Department of Mathematics
Indian Institute of Technology Roorkee (IITR)
Week – 03
Lecture – 14 (Phase Plane Analysis-III)

Hello, welcome to the course EXCELing with Mathematical Modelling.

In this particular session, we will be solving some typical problems on stability analysis, namely, using this Lyapunov function.

So, to start with, let us consider the dynamical system

$$\frac{dx}{dt} = y, \frac{dy}{dt} = -a\, sinx - by, \qquad a, b > 0.$$

.

So, if you recall the Lyapunov function which we have taken to be V(x,y).

So, if you can find a Lyapunov function V(x,y) such that this is positive definite, V(0,0) has to be zero assuming (0,0) to be the equilibrium point or critical point or fixed point.

If you assume that to be x^* and y^*, generalize it, so you can put them as $V(x^*,y^*)$ and $\dot{V}$, this has to be negative semi definite.

So, by this I mean this has to be

$$V(x, y) > 0, \forall\ (x, y) \neq (0,0) \qquad \dots (i)$$

$$\dot{V}(x, y) \leq 0, \forall\ (x, y) \neq (0,0) \qquad \dots (ii)$$

$$\dot{V}(x, y) < 0, \forall (x, y) \neq (0,0) \quad \dots. \qquad (iii)$$

for (ii) it is Lyapunov stable and for (iii) it is asymptotically stable.

Now, in this particular problem you have to choose and again I say that there is no hard and first rule you have to figure out by yourself and it will come with practice.

So, in this particular case it cannot be $x^2 + y^2$ because we have some sin x here.

So, you have to choose it in such a clever way that

$$V(x, y) = a(1 - cosx) + \frac{1}{2}y^2$$

You will understand why this $\frac{1}{2}$ has come.

If you do not take $\frac{1}{2}$ here, you will face some problem and ultimately you will bring that value to half, so which you will realize soon.

So, $a(1 - cosx) + \frac{1}{2}y^2 > 0$ for all $(x, y) \neq (0,0)$.

This is obviously true because it contains a square and we know that the value of $-1 \leq cos \leq 1$

So, this value is positive also for $(x, y) \neq (0,0)$, a > 0.

Now, if I put $x = 0$ and $y = 0$, I see that the right hand side has been satisfied.

So, (0 0) is my equilibrium point.

So, I find V(0,0).

$$V(0,0) = a(1 - \cos(0)) + \frac{1}{2}0^2 = 0$$

Now let us calculate $\dot{V}$.

$$\dot{V} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} = a \ \sin x \,(y) + \frac{1}{2} \ 2y\,(-a \ \sin x - by)$$
$$= a\,y \ \sin x - \ a\,y \ \sin x - by^2 = -by^2$$

So, the coefficient of this particular term becomes (-1) only if you choose this value to be $\frac{1}{2}$.

So if you choose it something else suppose, I keep it only y^2 this becomes only 2y so $-2ayx$, then I would not be able to prove that this is negative definite.

So to make this one and this one cancel this particular constant has been chosen.

So, as I have told you that there is no hard and first rule, it will come with practice and at the same time, this Lyapunov function is not unique, it can take any form, for one form you may get the conditions to be satisfied, for other forms you may not. So, you have to keep on playing with it. So, this you get

$$V(\dot{x}, y) = -by^2 < 0 \ \forall \ (x, y) \neq (0,0).$$

So, this satisfies negative definite and you can say that the origin is asymptotically stable.

So this is one particular example where we have non-algebraic term, mainly, the trigonometric term as $\sin x$.

Let us move to the next example

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -x - y$$

.

So let us choose

$$V(x, y) = 2x^2 + y^2 > 0, \forall (x, y) \neq (0,0)$$

$$V(0,0) = 0.$$

So, from here, I will get (0,0) as my equilibrium point.

And now comes

$$\frac{dV}{dt} = \dot{V} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} = 4xy + 2y\,(-x - y)$$

$$= 4xy - 2xy - 2y^2 = 2xy - 2y^2.$$

So, the conclusion is with this form of Lyapunovn function, your definiteness cannot be determined. Hence, I cannot conclude about stability with this form of Lyapunov function.

Now, to make this two terms to cancel each other, if I just take this to be $V(x, y) = x^2 + y^2$, then you see that here your

$$\dot{V} = 2xy\,(y) + 2y(-x - y) = 2xy - 2xy - 2y^2 = -2y^2 < 0.$$

So this is exactly what I was telling, that with one form of Lyapunov function you may get to prove the stability of the system, and with another form you may not.

So, you have to choose your Lyapunov function cleverly, such that everything all the conditions are satisfied.

Now, let us take an example where if it is an algebraic terms on the right hand side like

$$\frac{dx}{dt} = -3x^3 - y, \qquad \frac{dy}{dt} = x^5 - 2y^3.$$

Now this right hand side you can see there are higher degrees of polynomials.

In that particular case an attempt can be made like this, that you choose your Lyapunov function in this form

$$V(x, y) = ax^{2m} + by^{2n} \quad ,(a, b > 0)$$

and from here you can see that (0 0) is your equilibrium point.

Now, you have to find a suitable m and n here. So, what you have to do is, so the first two properties are satisfied that this will always be positive for non-zero x and y and at the point (0,0), they become zero.

So you start with

$$\dot{V} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} = 2amx^{2m-1}(-3x^3 - y) + 2bny^{2n-1}(x^5 - 2y^3)$$

So, instead of guessing the values of m and n, this is a systematic manner by which you can find the value of m and n. So,

$$\dot{V} = -6\,amx^{2m+2} - 2amx^{2m-1}y + 2bnx^5y^{2n-1} - 4bxy^{2n+2}$$

So, this term is fine, this term is fine, problem is with these terms which are called indefinite terms.

So, you have to choose in the values of m and n in such a manner that these two term cancels out and for that what you have to do, the power of x's and the power of y's has to be same.

So, you choose

$$x^{2m-1} = x^5 \implies 2\,m - 1 = 5 \implies m = 3.$$

Similarly, you can choose

$$y^{2n-1} = y^1 \implies 2n - 1 = 1 \implies n = 1$$

So, m = 3 and n = 1 gives you the Lyapunov function

$$\dot{V} = -18ax^8 - 6ax^5y + 2bx^5y - 4by^4.$$

So, you have got the powers of x and y is same, but now you have to choose your values of a and b in such a manner that these two cancels out. So, if you choose $a = 1, b = 3$,

$$\dot{V} = -18x^5 - 6x^5y + 6x^5y - 12y^4 = 18x^5 - 12y^4 < 0, \forall (x,y) \neq (0,0).$$

Hence the system is asymptotically stable.

If you want to check the global stability, so you have to show that the system is radially unbounded.

So, what does that mean?

It says that

$$V(x,y) \to \infty \quad as\ \|(x,y)\| \to \infty.$$

So, by norm you mean the distance and here it is the distance of (x,y) from the origin, from the equilibrium point.

So, if you use the distance formula

$$\|(x,y)\| = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{(x)^2 + (y)^2}$$

$$V(x,y) = x^6 + 3y^2.$$

$$\text{As } x \to \infty, y \to \infty, \ \|(x,y)\| \to \infty \text{ and } V \to \infty.$$

So, you show that the system is radially unbounded and hence it is also globally asymptotically stable.

So, this is another technique of finding this expression for Lyapunov function, but it works on only certain sets of problem, where the right hand side are higher powers of polynomial.

In our next example, we take an equation and prove where you have an equilibrium point $(x^*, y^*) \neq (0,0)$. So,

$$\frac{dx}{dt} = x - xy = f(x, y).$$

$$\frac{dy}{dt} = -\gamma y + xy = g(x, y).$$

So, let us take this particular example.

$$x - xy = 0 \Longrightarrow x(1 - y) = 0. \quad \ldots (i)$$

$$\Longrightarrow \; x = 0, y = 1.$$

My second equation

$$-\gamma y + xy = 0 \; \Longrightarrow y(-\gamma + x) = 0. \quad \ldots (ii)$$

$$\Longrightarrow \; y = 0, x = \gamma.$$

If I put x = 0 in (ii), it implies y = 0. So, one of the equilibrium point is (0,0).

If I put y =1 in (ii), then x = γ. So, $(\gamma, 1)$ is another equilibrium point.

This time we will be checking the stability at the point $(\gamma, 1)$ where $\gamma > 0$.

So, if I use Lyapunov's indirect method, then I have to write the matrix

$$A = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

So if I calculate this,

$$A = \begin{pmatrix} 1 - y & -x \\ y & -\gamma + x \end{pmatrix}_{(x^*, y^*) = (\gamma, 1)} = \begin{pmatrix} 0 & -\gamma \\ 1 & 0 \end{pmatrix}.$$

So, the eigenvalues can be calculated from this characteristic equation

$$|A - \lambda I| = 0 \; \Longrightarrow \; \begin{vmatrix} -\lambda & -\gamma \\ 1 & -\lambda \end{vmatrix} = 0 \; \Longrightarrow \; \lambda^2 + \lambda = 0 \Longrightarrow \lambda = \pm i\sqrt{\gamma}\,.$$

So, this gives a centre, because it has only imaginary roots, which is stable but not asymptotically. Now, this is Lyapunov indirect method.

Let us now use the Lyapunov's direct method where we use the function $V(x, y\,)$ and instead of taking (0,0) we take $(\gamma, 1)$ to be our equilibrium point.

As I told you Lyapunov function they becomes typical for a particular problem.

Like in this problem let us define a function

$$H(x,y) = x^* \ \ln x - x + y^* \ \ln y - y. \quad \ldots (iii)$$

Now obviously you will be wondering that how suddenly this function has come, as I told you it will come with practice and with various examples, where you will see many many forms of this Lyapunov function and when you take an unknown problem you will be able to figure out.

So,$(x^*, y^*) = (\gamma, 1)$. So, if I put it (iii) that is

$$H(x,y) = \gamma \ln x - x + \ln y - y.$$

This is not my Lyapunov function the notation remains the same, it is

$$V(x,y) = H(x^*, y^*) - H(x,y).$$

Now why this has been done, because otherwise if you substitute the value $(\gamma, 1)$ here, this will not be equal to 0 and that is one of the condition.

So this will be

$$V(x,y) = \gamma \ \ln x^* - x^* + \ln y^* - y^* - (\gamma \ \ln x - x + \ln y - y)$$

$$= (\gamma \ \ln \gamma - \gamma + \ln 1 - 1) - (\gamma \ \ln x - x + \ln y - y)$$

$$\Longrightarrow \ V(x,y) = \gamma(\ln \gamma - \ln x) - \ln y + (x - \gamma) + y - 1.$$

So this is your Lyapunov function.

Now, if you put

$$V(\gamma, 1) = \gamma(\ln \gamma - \ln \gamma) - \ln y + (\gamma - \gamma) + 1 - 1 = 0$$

So at the equilibrium point, the function vanishes.

Now let us calculate $\dot{V}$. Before that, let me quickly do it here itself.

So, if you want to calculate this

$$\dot{V} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y}.$$

So, if I substitute it, I will get

$$\dot{V} = -[\left(\frac{\gamma}{x} - 1\right)(x - xy) + (\frac{1}{y} - 1)(-\gamma y + xy)]$$

So, if you simplify this you will get

$$\dot{V} = -[\gamma - \gamma y - x + xy - \gamma + x + \gamma y - xy] = 0$$

which means that your $(\gamma, 1)$ is Lyapunov stable or stable, which coincides with the Lyapunov indirect method.

So, whether you use the indirect method or whether you use the direct method in both the cases you get the same answer.

The important question here is that you have a system which is non-linear

$$\frac{dx}{dt} = ax + by + P(x, y)$$

$$\frac{dy}{dt} = cx + dy + Q(x, y) \qquad \text{....}(1)$$

However, you are linearizing the system, you are doing some analysis and you are concluding on the linearized system.

You say this linear system is a node or this linear system is a stable spiral and then you conclude that on the non-linear system.

So, the question is that how sure you are that that is going to work, because all your analysis on the linear system, we say it is a linear stability analysis, whereas our equation may be linear may be non-linear, but we keep the result as the same, if it is linear no problem, but if it if it is non-linear we linearize it first then do our analysis and then conclude on the linear system and then we say that our non-linear system also follows the same kind of dynamics.

Now, the question is why that happens?

So, here what you see that I have written the differential equation of the form that, say, this is the linear part $ax + by$, $cx + dy$ and this is the non-linear part ($P(x, y)$, $Q(x, y)$).

Now, the hypothesis is that this

(i) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0,$

(ii) P_1 and Q_1, they have continuous first partial derivatives, for all x such that

$$\lim_{(x,y)\to(0,0)} \frac{P_1(x, y)}{\sqrt{x^2 + y^2}} = 0 \quad \text{and} \quad \lim_{(x,y)\to(0,0)} \frac{Q_1(x, y)}{\sqrt{x^2 + y^2}} = 0.$$

So, if this condition is satisfied then we have the following conclusions:

So, here I write the linear system

$$\frac{dx}{dt} = ax + by$$

$$\frac{dy}{dt} = cx + dy. \qquad \text{...}(2)$$

Let $\lambda_{1,}$ and λ_2 be the eigenvalues of the system (2). Then, the conclusion is

(i) If $\lambda_{1,}$ and λ_2 are real,unequal and of same sign, then (0,0) is node for (2) and also for (1).

So, that is why it happens that you have linearized it, but you did not check this property which is automatically fulfilled and then you conclude that (0,0) is a node for my linear system and therefore, it is also a node for my non-linear system.

The same thing is true if

(ii) $\lambda_{1,}$and λ_2 are real, unequal and opposite signs.

So, in that case you get a saddle for (2) and hence a saddle for (1).

(iii) If $\lambda_{1,}$ and λ_2 they are complex conjugate with nonzero real part, so complex conjugate real part not equal to zero, then you get spiral or focus.

So if the system (2) shows that it is a spiral or a focus, then system (1) will also will have a spiral or a focus at the equilibrium point (0, 0).

(iv) If $\lambda_{1,}$ and λ_2 real and equal with $a, d \neq 0, b = c = 0$, so, if this happens this one is a with conditional that you have to remember this, then if your system (2) is a node, your system (1) will also be a node.

(v) If $\lambda_{1,}$ and λ_2 is purely imaginary, then (0,0) is center for (2), then for (1), it can be a center or a spiral.

So, this is one deviation that you get, that if it is purely imaginary root we know that this is a centre but for (1), it can be a centre or it can be a spiral and similarly if

(vi) if your roots are real and equal, with the condition, $a = d \neq 0, b = c = 0$, then (0,0) is node for (2) but for (1), it can be a node or spiral.

So, basically I need you to remember this theorem, that yes we have a theorem which tells you that though we are doing our stability analysis on the linear system but the result is also true for the non-linear system because of this following properties and hence what we can say on the linear system is also holds true for the nonlinear system.

Quickly we just take an example such that it is clear.

$$\frac{dx}{dt} = x + 2y + x^2$$

$$\frac{dx}{dt} = -3x - 4y + 2y^2$$

So, now verify step by step all properties:

(i) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ -3 & -4 \end{vmatrix} = -4 + 6 = 2 \neq 0$. So, the first property is satisfied.

(ii) $\lim_{(x,y)\to(0,0)} \frac{P_1(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}} = 0$

We put $x = r\cos\theta, y = \cos\theta$, such that $x^2 + y^2 = r^2$, such that $as\ x \to 0, y \to 0,$ then $r \to 0.$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{P_1(x,y)}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \mathrm{r}^2 \frac{\cos^2\theta}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}}$$

$$= \lim_{r\to 0} r^2 \frac{\cos^2\theta}{r} = \lim_{r\to 0} rcos^2\theta \ \le\ \lim_{r\to 0} r = 0.$$

In the similar manner you can prove that second part

$$\lim_{(x,y)\to(0,0)} \frac{Q_1(x,y)}{\sqrt{x^2+y^2}} = 0.$$

So, all properties have been satisfied and if you take the linearized system, that is,

$$\frac{dx}{dt} = x + 2y,$$

$$\frac{dy}{dt} = -3x - 4y,$$

I find the eigenvalue at the point (0, 0),

$$\begin{vmatrix} 1-\lambda & 2 \\ -3 & -4-\lambda \end{vmatrix} = 0 \ \Rightarrow \lambda = \frac{5 \pm i\sqrt{15}}{2}.$$

So, this is imaginary values with positive real part and hence it is a spiral and an unstable one, because this real value is positive.

So, this linear system gives you an unstable spiral and we conclude that the non-linear system will also give you an unstable spiral.

So, that is how your theorem has been explained with the help of an example.

So, in my next lecture, we will be talking about some growth models.

Till then bye-bye.