EXCELing with Mathematical Modeling
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Week – 03
Lecture – 15 (Growth Models)

Hi, welcome to the course EXCELing with Mathematical Modelling.

Today we will be learning about the growth models.

So, to start with we need to know how the interaction happens. So, it happens due to the law of mass action.

Law of mass action in chemistry, it states that the rate of a chemical reaction is proportional to the product of the concentration of each reactant.

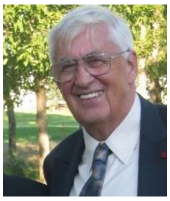
Now, this idea was interpreted in mathematical biology by the American physical chemist Lotka and Italian mathematician Volterra. This is Lotka and this is Volterra.

So, they propose that if you have a prey predator model and there is an interaction between the prey and the predator, so, that should be treated like the particle interacting in a homogeneously mixed gas or liquid and under these conditions the rate of encounter between the prey and the predator should be proportional to the product of their masses.

It means that if you have a prey and a predator, so this is a predator, a snake, and this is a prey, the frog.

So, when there is an interaction between the predator and the prey, in this case, the snake and the frog and if you denote them by, say, X and Y. So, they should be treated as the product of X and Y. So, that is what the use of law of mass action in this model.

So whenever there will be an interaction between two species, it will be treated or it would be taken as the product of the two variables.



Now Crawford Stanley Holling is a Canadian ecologist. He introduced the term called functional response.

Basically, it is introduced to describe the change in the rate of consumption of prey by a predator when the density of prey is varied.

So, the two types of function, which we will be discussing here, is Holling type I and Holling Type II.

So, if you have a prey (x) and a predator (y) and the interaction is happening at a rate, say, β.

So, this is a Holling type I function, where these two appear linearly as a product $\beta\, x\, y$, say, where x is the prey and y is the predator.

If we look into Holling type II, the function will be of the form, say, $\beta \frac{x}{x+k} y$.

Now, we generally take this kind of function when this predator (y) does not have access to the whole of the prey. It has access only to the fraction of the prey.

For example say if I take this $\frac{x}{x+k}$, this I can write as $1 - \frac{k}{x+k}$. So, if you simplify this, you are going to get this one.

So, as you can see that it is a fraction of the prey that is accessible to the predator, if I multiply it by y. So, whenever a situation arises when the predator does not have or when one species does have access to the fraction of the other species, we generally use this Holling type II functional response.

And in most of the real-life cases, it has been seen that this functional response is more close to the real-life scenario than the Holling type I.

But, again in mathematical modelling we use simple functions, and if the simple function is able to capture the dynamics of the model, we are quite happy about it.

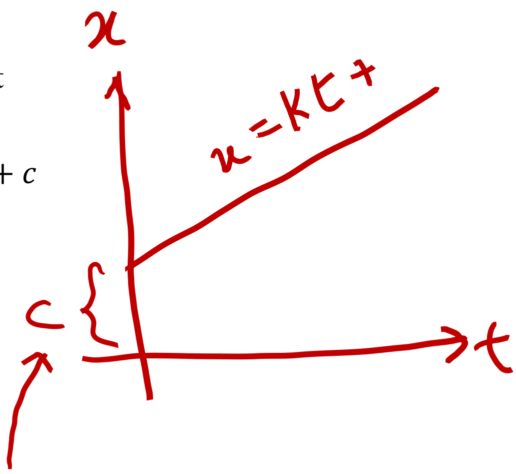
We now come to our main topic the growth models.

So, to start with we have the linear growth model.

Now in linear growth model, we assume that the growth of the population is constant.

So, if x is the population, the rate of change of growth is a constant say k,

$$\Longrightarrow \frac{dx}{dt} = k,$$



and it is a simple differential equation, if we just solve it we will get

$$\frac{dx}{dt} = k \Longrightarrow \int dx = \int kdt \Longrightarrow \quad x = kt + c$$

So if we plot this graph as you can see this is of the form

$$y = mx + c,$$

which is a straight line where m is the slope and c is the y intercept.

So if you plot the graph, you will get a straight line where this is your c, assuming that c is a positive and this is $x = kt + c$. So this is a linear growth model.

We next move to exponential growth model.

So, in exponential growth model you consider say a population with plenty of resources and there is no threat from anybody, say, any predator. So, in such case the growth rate is proportional to the population.

So, that means if x is some species then

$$\frac{dx}{dt} \propto x \implies \frac{dx}{dt} = rx$$

Now if you simply solve this differential equation you will get this

$$\int \frac{dx}{x} = \int r\, dt \implies \ln x = r\, t + \text{constant.}$$

So ln x means the base is e.

$$\text{At } t = 0, x = x_0 \implies \text{constant} = x_0$$

If you substitute it there,

$$\ln x = r\, t + lnx_0 \implies \ln\left(\frac{x}{x_0}\right) = r\, t \implies x = x_0 e^{rt}.$$

When you write this ln (log), you do not have to write base e, but just for the sake of understanding.

So, this is the solution and if we plot this solution we are going to get curves like this.

However, as you can see that this is, you know, sort of unbounded because as if your r is a positive number and as t increases your x also increases.

Generally, we do not take this kind of assumption unless needed because the growth if it is unbounded and growth unlimited it really does not capture the real life scenario.

So that is why it is the start that okay there is something called exponential growth model and unless the model demands we generally avoid this exponential growth model and move for the more realistic one, which is a logistic growth model.

So, in logistic growth model, we have the equation of the form

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right)$$

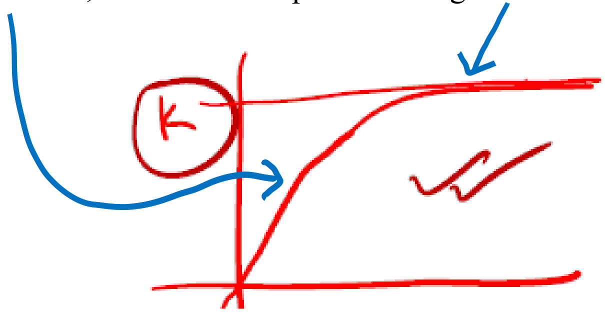
Now, this r is called the intrinsic growth rate and k is called the carrying capacity.

Now, the growth rate, you can understand the intrinsic growth rate, we use the word intrinsic because this contains

$$\text{birth rate} - \text{ death rate.}$$

So, both of them are contained in this and the carrying capacity can be understood like this.

You consider somebody is having a tumour say in the liver and it is growing. So, generally if it grows it will first start like this, but after some point it will go like this.



That is in general what happens in a growth model. So, there is some sort of exponential growth and then it ceases to grow and then comes to a constant value kind of thing.

So, this carrying capacity is something which your environment can support. For example, you consider this tumor and it is growing in the liver. So, how much that body can support or can provide support to that tumour so that he can grow substantially. So, that is what is called the carrying capacity.

We will take this carrying capacity again when we come to this prey predator model, but something that your environment can support to grow. In this particular example, suppose if your tumour is growing and it ceases and comes to a constant value which is k. So, beyond k it will not grow. So, that is what your carrying capacity is.

Now, if we try to solve this, this can be easily solved. You separate the variables

$$\int \frac{dx}{x(x-k)} = -\int r dt$$

You integrate, this is a partial fraction I can write like this.

$$\frac{1}{k}\int \left[\frac{1}{x-k} - \frac{1}{x}\right] dx = -r\,t + \text{constant} \implies \frac{1}{k}[\ln(x-k) - \ln x] = -r\,t + \text{constant.}$$

So I leave it to you how to find this solution, it is just integration, you get this result substitute at $t = 0, x = x_0,$ and simplify, you will get this result as

$$x(t) = \frac{kx_0}{x_0 + (k - x_0)e^{-rt}}$$

So from here you can see that as your t becomes large, as your t becomes large this part goes to zero and your $x(t) = \frac{kx_0}{x_0} = k.$

So, for large t as this graph suggests your value will reach the carrying capacity k.

So this is the logistic growth model.

In most of the modelling where we involve this growth, we use this logistic growth, which is more realistic than the constant growth rate.

We have a second function which also represents growth and that is called Gompertz growth model or Gompertzian growth.

In this case, your equation takes the form

$$\frac{dx}{dt} = rx \ln\left(\frac{k}{x}\right),$$

where r remains the same, it is the intrinsic growth rate and k is the carrying capacity.

Now, again we find the solution of this growth model. This can be written as

$$\frac{dx}{x\, ln\left(\frac{k}{x}\right)} = rdt \implies \int \frac{d(lnx)}{\ln(x) - \ln(k)} = -r\int dt$$

If you are not familiar with this equal to this, you can keep this whole x like this and then substitute this equal to z, you are going to get the same thing or otherwise you can do like this, you can just write this

$$\int \frac{dx}{x(\ln(x) - \ln(k))} = -\int rdt,$$

and then substitute $z = \ln(x) - \ln(k)$ your $dz = \frac{1}{x}dx$. So that is how you do the substitution.

This implies

$$\ln(\ln(r) - \ln(k)) = -rt + \text{constant} \implies \ln\left(\ln\left(\frac{x}{k}\right)\right) = rt + \text{constant}$$

At time $at\ t = 0, x = x_0$, substitute it here, and this will give you constant $= ln\left(\ln\left(\frac{x_0}{k}\right)\right)$.

So, if you substitute it here and take it to the left hand side, you will be getting

$$\ln\left(\frac{ln\left(\frac{x}{k}\right)}{ln\left(\frac{x_0}{k}\right)}\right) = -rt \implies \ln\left(\frac{x}{k}\right) = \ln\left(\frac{x_0}{k}\right)e^{-rt}$$

$$x = k\exp\left(ln\left(\frac{x_0}{k}\right)e^{-rt}\right)$$

Again, in this case, you can see that as t becomes large, then this part $\left(ln\left(\frac{x_0}{k}\right)e^{-rt}\right)$ goes to zero.

So, it is exponential 0, which is 1 and you are left with x tends to k. So, a similar kind of behaviour and in most of the cases, we use either this logistic growth expression or the Gompertzian growth.

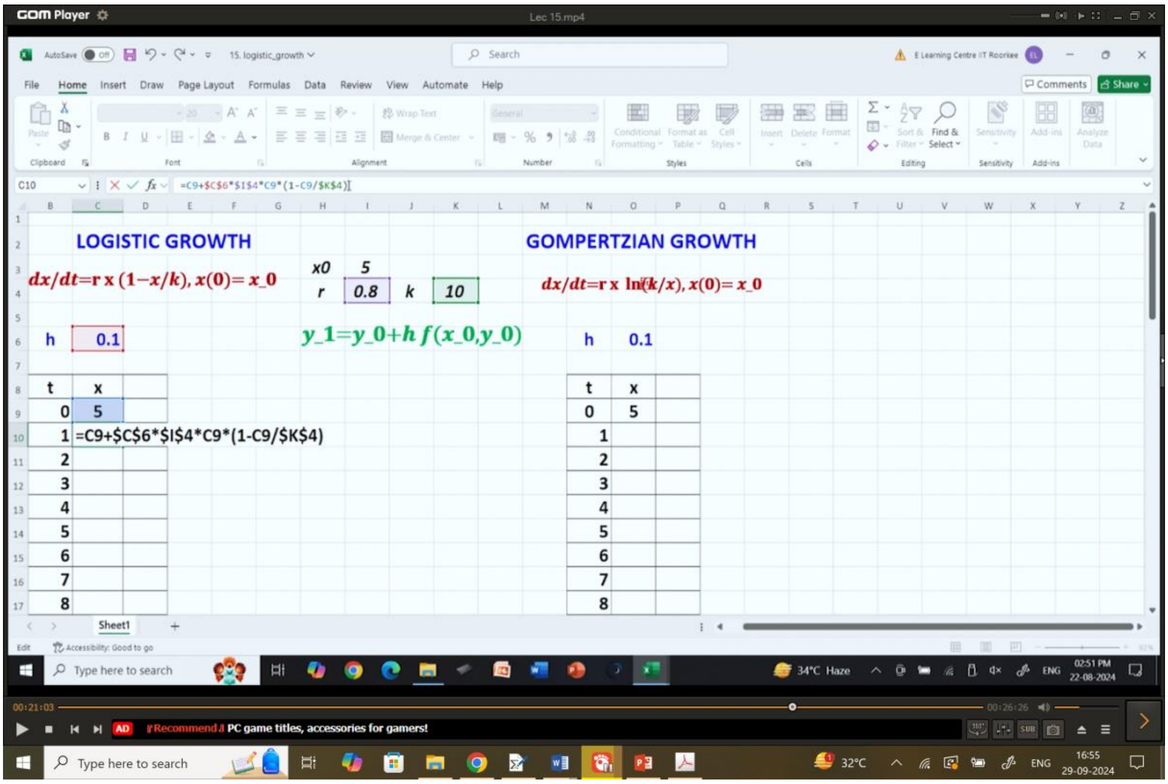
Now let us see what is the numerical solution provides you what kind of graph, both the case of logistic and the Gompertzian growth.

So we will be using Microsoft Excel to find the numerical solution of these two problems.

So as you can see, I already have the equation typed, so this is the logistic growth,

r x1 minus x by k, this is the Gompertzian growth r x log of k by x. The initial value for both of them I have taken to be 5, r is 0.8 and carrying capacity k is 10. So, I will use this formula which we have done

$$y_1 = y_0 + h\, f(x_0,y_0).$$



So, if I come here, we next calculate this x values, which is equal to x 0 plus h we took as 0.1 multiplied by r, again constant, so I put dollar, multiplied by x, which is multiplied by 1 minus x divided by k.

Since k is constant, I put a dollar and so I calculate these values up to 50 of them and I get this.

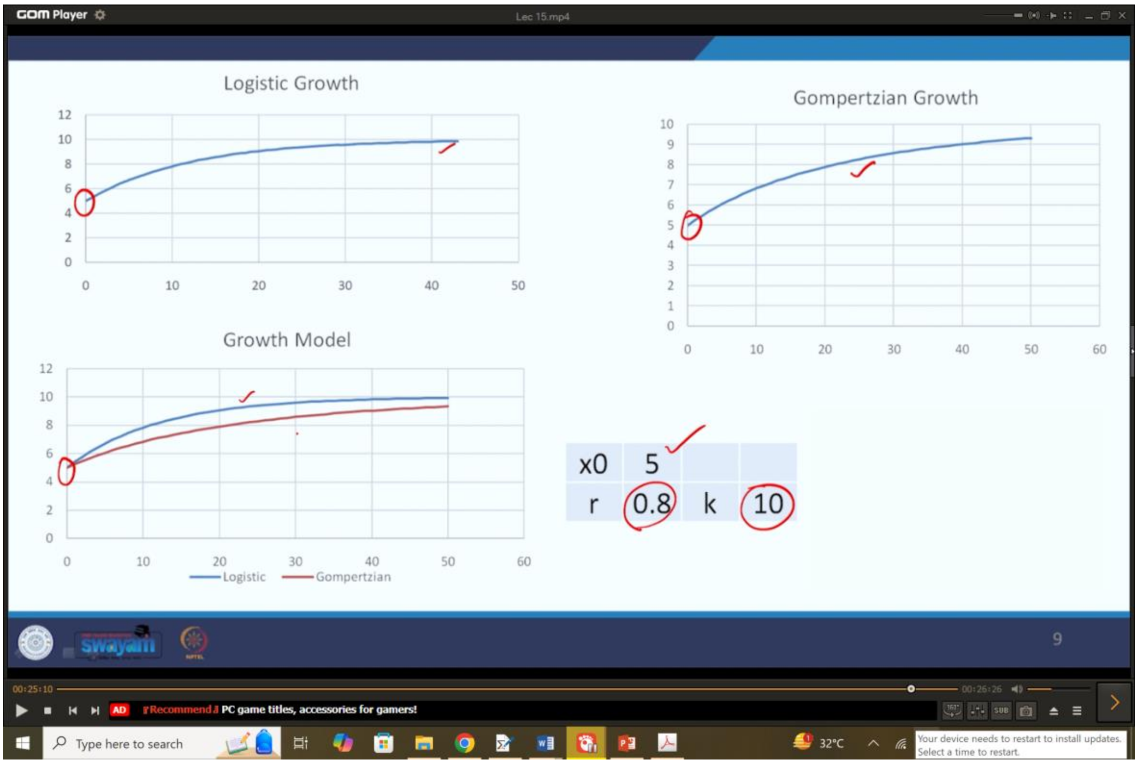
So if I want to plot this, I will come up to here, I go to insert, go to scatter and I plot this, so I get something like this, the same I can do with the Gompertzian one.

So, I will take here, this is equal to the same x0, which is this, or I put this one as x0 plus h times, which is a constant, this is your r, which is again a constant, r times x and then multiplied by log.

So here it is log, here this is log 10, so you will use log, which is log to the base e, log of k which is 10 and it is a constan,t divided by x0.

And I drag this, so if I want to plot this again, I go to insert, I go to scatter, and I click this, so you have another graph. So, numerically if you solve them, you will get these two graphs. So, we have solved these two numerically using Microsoft Excel.

Let us now compare.



So, the values as you can see, I put the initial value to be 5, the intrinsic growth rate to be 0.8 and the carrying capacity to be 10.

So, both of them have started from the value 5 and this is what the graph looks like in case of logistic growth, in case of Gompertzian growth.

Now, if I compare both the graphs, as you can see that both have started from this 5, the logistic growth is a bit higher than the Gompertzian growth with the same set of intrinsic growth rate and carrying capacity and initial value.

So, the question is which one to use? And which one is better than in a particular model.

So, it goes like this that when you model something, say, whether it is a growth of a population or whether it is a growth of a tumor, you will be going to get some data, and from that data you have to choose a particular function.

So in some data you will see that this logistic growth function, which is of the form $rx\left(1-\frac{x}{k}\right)$, this fits well than the Gompertzian function, which is $rx\ln\left(\frac{k}{x}\right)$. So that way we choose that which growth function will be better for a particular model, whether will be a logistic growth or whether it is a Gompertzian growth, both depends on the kind of data which you get from the real life scenario.

So, with this we come to the end of this growth models.

In the next lecture, we will be learning about the Prey-Predator models.

Till then bye-bye.