EXCELing with Mathematical Modeling
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Week – 04
Lecture – 16 (Prey-Predator models)

Hello, welcome to the course Excelling with Mathematical Modelling.

In today's lecture, we will learn about the Prey-Predator models.

So, already I have given you a little hint about this Lotka and Volterra Prey-Predator model.

The first Prey-Predator model was initially proposed by Alfred Lotka.

So, he used this equation to analyze the Prey-Predator interaction in his book on biomathematics.

The same set of equation was published in 1926 by Vito Volterra and who also become interested in mathematical biology.

So, the history goes like this that Volterra's son-in-law, his name is Umberto D` Alcona.

So, he was studying this fish, he was a marine biologist

So, he was studying the behaviour of the fish and during the World War in the year 1914 to 1918, he noticed that in the Adriatic Sea, the percentage of fish that were caught were not what he has expected.

So, for example, the locals want a particular kind of fish that they generally accustomed to eat.

But then during this period, what he finds that that some different fish was caught which was a predatory fish which means the fish which used to eat the fish that was a choice of the locals.

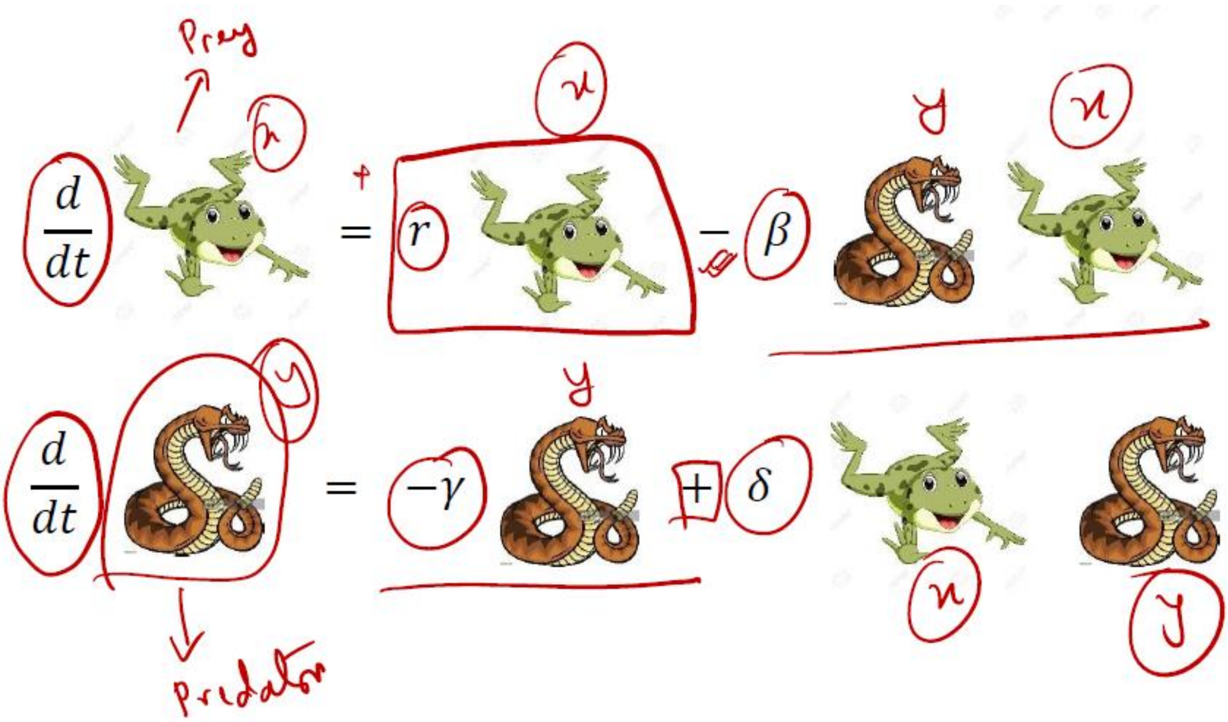
So that was the point of discussion between him and Volterra and listening to that Volterra created this model based on his observation and independently he created this same set of equation that was created by Alfred Lotka, though obviously he acknowledged Alfred's work and then this equation is came to known as Lotka-Volterra prey-predator model.

So, that is the history behind this model.

So, if you consider the prey predator model as you can see in the video that the predator here is the cheetah and the prey is the deer.

So, it is a common thing that the predator will hunt down its prey for food only and it is going to benefit in its biomass the food which we all need.

So, to model this, let us take this set of equations.



So, this frog is your prey and the snake is your predator.

So, this is my prey and this is my predator.

This gives you the rate of growth or rate of change of growth of the prey and this $\frac{d}{dt}$ gives you the rate of growth of the change of predator.

So, if there is no prey suppose this part is not there then what happens is the prey grows freely at a rate say r but when it comes in contact with the predator obviously the predator eats the prey at a rate β and since it is eating the prey at a rate β its population decreases that is why this negative sign is here.

So, the growth increase in population that is why positive sign, the death because it is interaction with the predator that is why this negative sign.

In the similar manner if your predator does not get any food then obviously it is going to die.

So, that explains the first part that suppose the predator is not getting any food it is going to die at a rate γ and since it is dying the population is decreasing we have a negative sign.

And since once it comes in contact with the prey that is if it gets food obviously its population will increase say at a rate δ and hence the positive sign.

So this explains this particular cartoon.

Now if I replace this by the variable x by the variable y, this is x, this is x, this is y, this is y, this is x and this is y. What we get is called the Lotka-Volterra equation and it will look like this.

$$\frac{dx}{dt} = r\,x - \beta\,x\,y$$

$$\frac{dy}{dt} = -\gamma\,y + \delta\,x\,y$$

Now, next comes the analysis of this model.

So, as you know that first you have to find the equilibrium solution, for which you put the values

$$\mathrm{r\,x} - \beta\,\mathrm{x\,y} = 0$$

and

$$-\gamma\,y + \delta\,xy = 0$$

If you solve these two equations, you will get the values to be (0,0) and another value is

$$\left(\frac{\gamma}{\delta}, \frac{r}{\beta}\right)$$

Now if you want to calculate the eigenvalue, obviously you have to calculate the matrix which is

$$A = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

So if we take this to be our f(x,y) and this to be our g(x,y) and then you calculate $|A - \lambda I| = 0$, to calculate the eigenvalues and if you do that you will get something like

$$\begin{vmatrix} 0-\lambda & -\beta\frac{\gamma}{\delta} \\ \delta\frac{r}{\beta} & 0-\lambda \end{vmatrix} = 0$$

And, if we calculate our eigenvalues, it is going to be

$$\pm\, i\,\sqrt{r\,\gamma}$$

So purely imaginary eigenvalues.

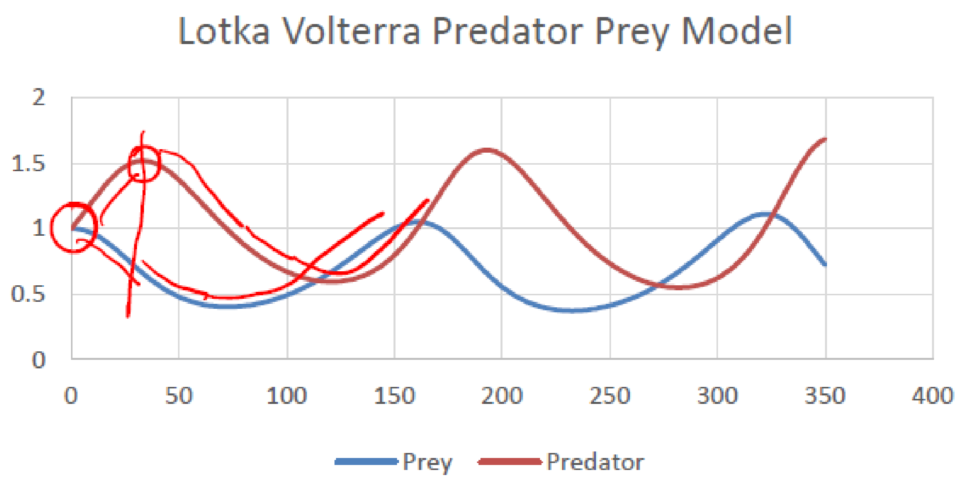
If we now look into the numerical solution, so the initial values were $x_0 = 1$, $y_0 = 1$, your r = 0.4, $\beta = 0.4$, $\gamma = 0.4$ and $\delta = 0.6$.

So if we put this value and solve this in Microsoft Excel, we will get the curve like this where your blue is your prey and your red is the predator.

Now what is the interpretation of this graph?

So, both of them starts with the initial value 1.

Now as you can see that the predator increases.



The predator, as the predator increases so if there is an interaction between the prey and the predator, so the predator is going to eat the prey and hence the population of the predator increases.

So you can see that the population of the predator increases whereas the population of the prey goes down and this is the maximum value where your predator has increases but then the number of prey falls short and the predator is not getting enough food.

So then what happens?

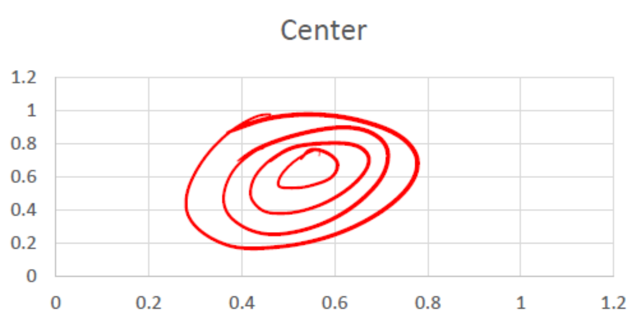
The population of the predator starts going down.

This is going down but at one point when it happens that the predators are also becoming less and with the number of preys are there. So the predator is not able to catch the prey.

So, then the prey again starts growing and once they are in abundance that means lot of prey again the predator starts eating them and again it starts growing.

So, there is a cycle between these dynamics of this particular prey predator model and if you plot the phase portrait your eigenvalues are going to be if I substitute the values here it is purely imaginary and you will be getting a centre.

So, basically you will be getting a curve like this.



So, that is the dynamics of this Lotka-Volterra prey predator model.

Let us now modify this model.

$$\frac{dx}{dt} = \mathrm{r\,x}\left(1 - \frac{x}{k}\right) - \beta\, x\, y \qquad \frac{dy}{dt} = -\gamma\, y + \delta\, \mathrm{x\,y}$$

So, the modification is that now you have instead of r x, I have taken the logistic growth where your r is the intrinsic growth rate and k is your carrying capacity.

So, again if you want an explanation of this carrying capacity, consider this pond.

So, this pond has lot of fish and they are all surviving.

So, then you decided to put some, add some extra number of fish. So, you put them, say, you put 100 more fishes.

Now, it is fine, 100 more fishes, again after a few days you want to push again 200 number of fishes, but then in the next day you find that some dead fish floats up.

That means the environment cannot support that much amount of fish.

So, that is exactly what this carrying capacity k does.

So, it is that number which will tell you okay this is the number up to which the environment can support this particular species.

So, that is this carrying capacity k.

β is the rate at which your predator is killing the prey.

Here, this equation remains same.

If the predator does not get any food, then it dies at a rate γ and if this gets food, it grows at a rate δ.

If you want the analysis of this model, as usual, you take this to be f(x,y), this to be g(x,y) and then you find the equilibrium solution.

If you put this equal to 0 and this equal to 0, your equilibrium solution will be (0,0), (k,0) and

$$\left(\frac{\gamma}{\delta}, r\frac{k\delta - \gamma}{k\beta\delta}\right)$$

Now, since it is a population, it cannot be negative.

So, for the existence of this particular equilibrium point, if this equilibrium point to exist that this part has to be positive which means that $k\delta - \gamma > 0$.

So, this is the condition which you have to state, such that this equilibrium point exists.

Then, you have to find the Jacobian matrix, which is

$$A = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix},$$

and we have done this before and you calculate this at the point (x^*, y^*).

These two are simple you will be able to do them. If I want these values, then after differentiating with respect to x if I substitute these values, on simplification this becomes

$$\begin{pmatrix} -r\gamma & \dfrac{\beta\gamma}{\delta} \\ \dfrac{r(k\delta - \gamma)}{k\beta} & 0 \end{pmatrix}$$

Now, if you want to find the eigenvalue, write the characteristic equation which will look like

$$\lambda^2 + \frac{r\gamma}{k\delta}\lambda + r\gamma\left(1 - \frac{\gamma}{k\delta}\right) = 0$$

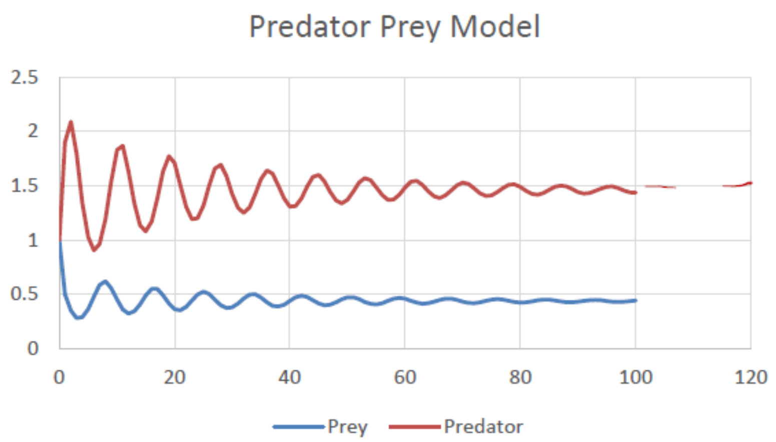
So, by Routh Hurwitz criteria, if the equation is of the form

$$\lambda^2 + a_1\lambda + a_2 = 0,$$

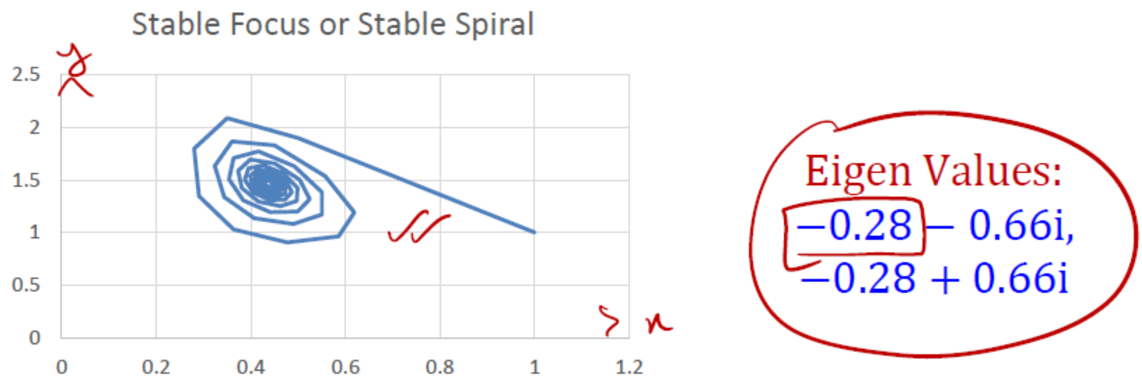
then the system will be stable if both a_1 is positive and a_2 is positive, as you can see these are already positive because the constants are positive. So, you have to take this $r, k, \beta, \gamma, \delta$, they are all positive constants. Because as they give you the rate at which the prey is catching the predator the carrying capacity all are positive quantities.

So, now if this system has to be stable this has to be greater than 0, but this is obviously greater than 0 because you have $\frac{k\delta-\gamma}{k\gamma}$. If you simplify this, so this is always positive because of the existence of this equilibrium point. So this system is always stable.

Now if you want to look into the numerical solution, so your x_0 again I take keep the same initial condition that your x_0 is 1, your y_0 is 1, your r is 1.3, your k is 1, β is 0.5, γ is 0.7, δ is 1.6. If you substitute these values, you will get a curve like this.



So, if you calculate the eigenvalues, you will see that they give you imaginary one with conjugate pair with negative real part and we know from before that if your eigenvalues are imaginary with positive real part, then you get a stable focus or a stable spiral.



This is exactly what you got.

So, this is your x axis and this is your y axis and this is for the prey and the predator.

So, the dynamic says that this will be a damping kind of circle and after the damping, it reaches this equilibrium point.

So, this is an example of a modified prey predator model. Sometimes it is also called modified Lotka-Volterra model.

Let us move to a next example of the prey predator model.

So, this is a pelican and fish.

So, this bird pelican can see that how it dives and it catches the fish, and they have this mechanism the throat is so big that the fish is caught as you can see in the video that it is struggling and then it passed down through this particular hole in the mouth inside and that is how the pelican eats the fish. It hunts and eats the fish.

So, in this particular case as you can see the pelican is a bird, it flies it dives down to the lake and catches the fish but then the pelican does not have access to all the fishes. It has an access to a fraction of the fishes and that is where we use the Holling Type II functional response to explain it more.

So, now your model is this is the logistic growth which I have explained before.

Now this is the part of the prey.

This is the Holling Type II functional response and y is your predator, in this case, the pelican.

And as I told you before that $\frac{x}{x+\alpha}$ can be written as $1-\frac{\alpha}{x+\alpha}$.

So, you can see that this represents the this represents actually a fraction of the fish, that is accessible to the predator, in this case the pelican.

So, this is the logistic growth, this is the rate at which the predator is eating the prey, β is the rate, this is the functional response how they interact and sorry this will be δ, and since the predator is eating the prey, it will add to its population and hence plus $\delta \frac{xy}{x+\alpha}$ and if the pelican does not get any fish, it will die at a rate γ.

$$\frac{dx}{dt} = \mathrm{r}\,\mathrm{x}\left(1 - \frac{x}{k}\right) - \beta\,\frac{x\,y}{x+\alpha} \qquad \frac{dy}{dt} = -\gamma\,\mathrm{y} + \beta\,\frac{x\,y}{x+\alpha}$$

So, this is your prey predator model, where the predator does not have access to the whole of the preys but only have access to the fraction of the prey.

So, we have to do the similar analysis, you take this to be f(x,y), this has to be g(x,y) and then you calculate the equilibrium solution in this case it will be (0,0), (k,0) and this becomes

$$\left(\frac{kr\delta(\delta - 2\gamma)}{\beta(\delta-\gamma)^2}, \frac{k\gamma}{\delta-\gamma}\right)$$

.

So, this is the equilibrium point and now for the existence of this equilibrium point because this is a population which cannot be negative, so $\delta > \gamma$, and $\delta > 2\gamma$.

So, if I take this the value is γ, this value is 2γ. So, both of them need to be satisfied.

So, this holds that if $\delta > 2\gamma$ then obviously it is greater than γ.

So, we will say that if this is the condition then you have always this equilibrium point as a positive equilibrium point and this equilibrium point will exist.

So, the condition we will only pose here is δ has to be greater than 2γ.

So, when you choose your numerical values you have to be careful that this inequality holds the δ has to be greater than 2γ.

So, once you get your equilibrium point then you calculate your f_x, your f_y, your g_x and your g_y.

So, in this particular case at the point (x^*, y^*) your f_x is

$$k^3 r - 3krx^{*2} - 2\gamma x^{*3} - k^2\beta y^*.$$

This will be

$$f_y = -\frac{x^*\beta}{k + x^*}.$$

This will be

$$g_x = \frac{k\delta y^*}{(k + x^*)^2}$$

and this will be

$$g_y = -\gamma + \frac{\delta x^*}{k + x^*}$$

and you have to put either (0,0) or (k,0) or the other equilibrium point to see whether they are stable or whether they are unstable.

So, if I choose the other equilibrium point, it is going to be $\frac{k\gamma}{\delta-\gamma}$ that is your y and your x is equal to

$$\frac{kr\delta(\delta - 2\gamma)}{\beta(\delta - \gamma)^2}.$$

So I want to check the equilibrium at this point x^* and y^* and if you substitute here your Jacobian matrix A is going to give you

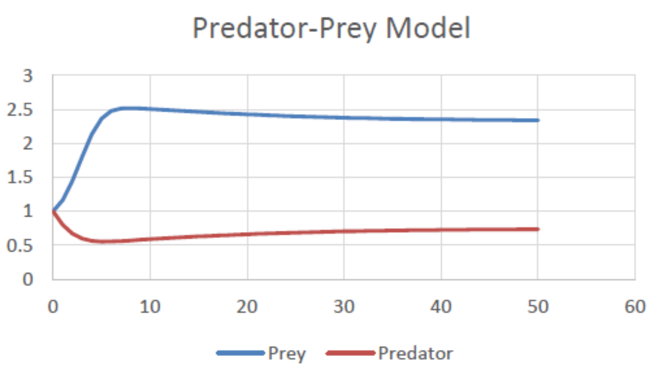
$$\begin{pmatrix} 2r\gamma^2 & \frac{-\beta\gamma}{\delta} \\ \frac{r(-2\gamma + \delta)}{\beta} & 0 \end{pmatrix}$$

Then, you find the characteristic equation and in this case, it is going to be

$$\lambda^2 + \frac{2r\gamma^2}{\delta(\delta - \gamma)}\lambda + \frac{r\gamma}{\delta}(\delta - 2\gamma) = 0.$$

Now, because of the existence of this equilibrium point, we must have $\delta > 2\gamma$, which will imply this to be positive and this also to be positive. So by Routh-Hurwitz criteria, the equilibrium point (x^*, y^*) where x^* is this, and y^* is this, is asymptotically stable.

Now, whether it is going to be, I mean what kind of dynamics it will give, we can only verify it with the numerical solution. So, if you solve it numerically, with these numerical values, you will get a graph like this.



x0	1	y0	1
r	1.3	k	1
a	1	b	0.5
g	0.7	d	1.6

So, which says that in this particular case your prey grows and then comes to a constant value, which is the equilibrium solution, and the predator first goes down because it would not be able to catch the prey, and then when it starts catching the prey, it maintains a steady population.

So, in a prey-predator model, it is always expected that there will be a coexistence.

The prey will also exist. The predator will also exist.

One may dominate the other but if one of the species goes to zero, then this is something against the nature because then that particular species is endangered, means, they just go extinct.

So, that generally does not happen in a real life situation. So, this kind of behaviour or this kind of dynamics is expected and if you want to see the phase portrait.

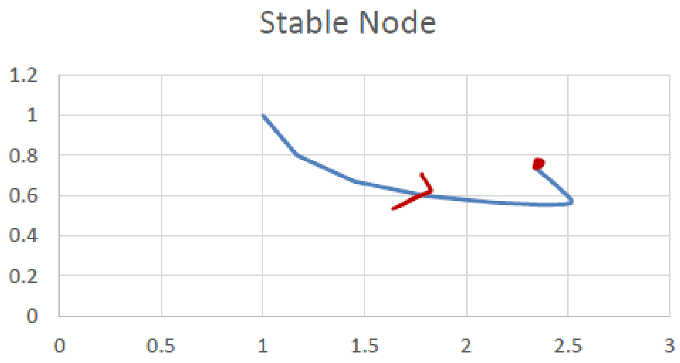
So you can see that this is the eigenvalue here with this particular numerical values.

So, both are real and negative and we know that we get a stable node.

So only thing is I do not have the arrow here.

So if this is the equilibrium point, so it is a stable node.

So I have to put the arrow like this because it is stable.



Eigen Values:
$-0.73, -0.16$

So, with this we come to the end of this lecture, which discussed three kinds of prey-predator model and depending on the situation you just go on modify these models.

In our next lecture we will be looking into some competition models.

Till then bye-bye.