EXCELing with Mathematical Modeling
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Week – 04
Lecture – 17 (Competition Model)

Hello, welcome to the course EXCELing with Mathematical Modelling.

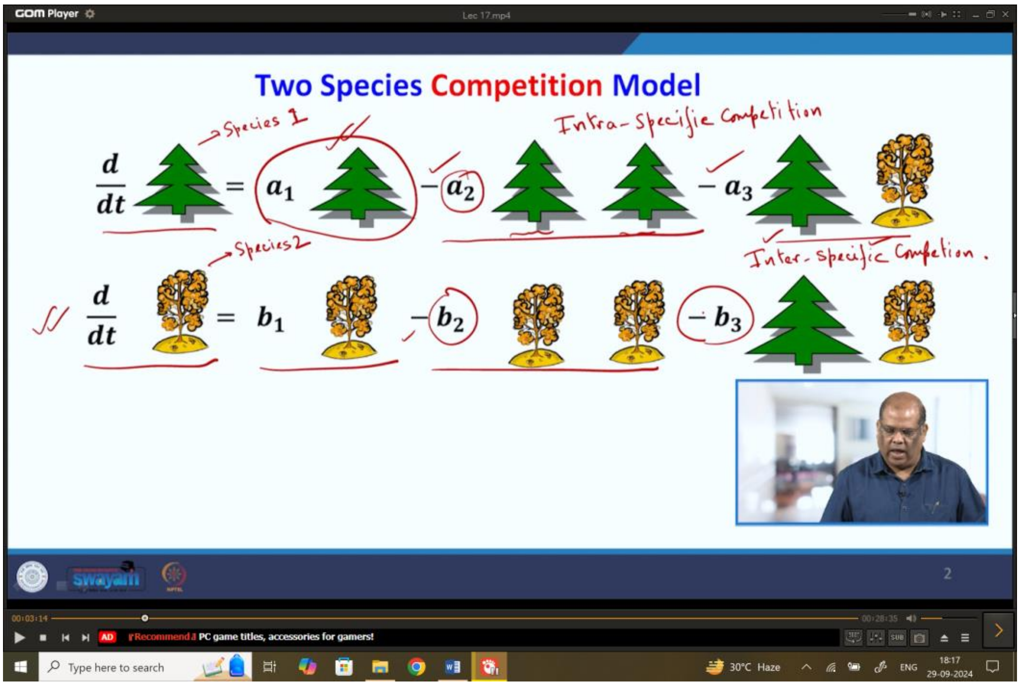
Today we will be discussing about competition model.

Competition is the way of life with any species. So, competition among the species will be about food, about space and any other needs.

So, we will be starting with competition between two plant species.

So, say this plant is some species 1 and this plant is some species 2.

So, in plants they compete for space, compete for nutrition, complete for sunlight, and hence this competition model.



So, the left hand side which you see is the rate of growth of species 1, it depends on exponential growth that is a_1 times the species.

Also, two plants of the same species, they also compete for the same resources which is water, sunlight, nutrient, space and hence there is a competition between the plants of the same species and it is called intra-specific competition.

Also, there is a competition between the species of two different species and hence this into this and since this competition will take a toll on the growth of the plant, this is a negative sign, this is also a negative sign.

So, basically this represents the growth of the plant, this represents the competition due to same species plant at a rate a_2 and this will represent the competition of different species plant and it is called inter-specific competition.

The same thing for the species 2, this is the rate at which species 2 grows follows an exponential growth and intraspecific competition at a rate b_2 that is why this negative sign and an interspecific competition at a rate b_3 again with the negative sign.

So, if you replace them by the variables x and y, this will look,

$$\frac{dx}{dt} = a_1x - a_2x^2 - a_3xy$$

$$\frac{dy}{dt} = b_1y - b_2y^2 - b_3xy$$

So, this is a typical competition model between two species.

Now, let us do a little analysis of this.

So, we have this two models x and y, they are the number or they can be the densities, generally we take them as the densities.

So, the very first thing that you have to do is find the equilibrium solution and for that you put this equal to 0 and this equal to 0.

So, I have

$$x(a_1 - a_2x - a_3y) = 0$$

$$y(b_1 - b_2y - b_3x) = 0$$

So, the solution is, from this equation,

$$x = 0 \quad \text{and} \quad a_1 - a_2x - a_3y = 0$$

So, also from here

$$y = 0 \quad \text{and} \quad b_1 - b_2y - b_3x = 0$$

So when $x = 0$, if you substitute this value here, you will get

$$y(b_1 - b_2y) = 0$$

This will give y $= 0$ and $y = \frac{b_1}{b_2}$

So when $x = 0, y = 0$ when $x = 0, y = \frac{b_1}{b_2}$, that is (0,0) and $(0, \frac{b_1}{b_2})$.

So, these two are the equilibrium solutions when x = 0, and please note all these constants a_1, a_2, a_3 they are positive and so are b_1, b_2, b_3 they are also positive because they generally represent the rate.

Similarly, if you have $y = 0$, if I substitute it here, I will get

$$x = 0 \quad \text{and} \quad a_1 - a_2 x = 0$$

which will give me $x = 0$ *and* $x = \frac{a_1}{a_2}$.

So when $y = 0$, $x = 0$ and when $y = 0, x = \frac{a_1}{a_2}$.

So (0,0) is common, $(\frac{a_1}{a_2}, 0)$ is one equilibrium point, and $(0, \frac{b_1}{b_2})$ is another equilibrium point.

And finally, if we solve this equation and this equation, that is,

$$a_1 - a_2 x - a_3 y = 0 \quad \text{and} \quad b_1 - b_2 y - b_3 x = 0,$$

so, simultaneous equation, linear equation, you can use any method, say method of cross multiplication and if you solve it, your answer is going to be

$$x = \frac{a_1 b_2 - a_3 b_1}{a_2 b_2 - a_3 b_3}$$

Similarly,

$$y = \frac{a_2 b_1 - a_1 b_3}{a_2 b_2 - a_3 b_3}$$

To find this solution, I leave this to you. It is quite simple.

You just have to solve this simultaneous equation.

So, we have four equilibrium points.

One is (0,0)

One is $\left(\frac{a_1}{a_2}, 0\right)$

One is $(0, \frac{b_1}{b_2})$ and the other is some (x^*, y^*) where x^* is given by this and y^* is given by this.

So once you get the equilibrium solution you now have to check for stability.

So to check for stability we have to find Jacobian matrix, if I write the equations

$$\frac{dx}{dt} = a_1 x - a_2 x^2 - a_3 xy = f(x, y)$$

$$\frac{dy}{dt} = b_1 y - b_2 y^2 - b_3 xy = g(x, y)$$

So, the Jacobian matrix

$$A = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

at the point (x^*, y^*).

And you will get this value to be

$$\begin{pmatrix} a_1 - 2a_2x^* - a_3y^* & -a_3x^* \\ -b_3y^* & b_1 - 2b_2y^* - b_3x^* \end{pmatrix}$$

Now, let us take each of the equilibrium points, say, (I) $(0,0)$.

So, I substitute $(0,0)$ here and I get the value of

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}$$

If I want to find the eigenvalue that is and this will give me

$$\begin{vmatrix} -\lambda + a_1 & 0 \\ 0 & b_1 - \lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda = a_1, b_1 > 0.$$

Both are positive because they are the rates, so real and positive eigenvalues, system is unstable and to be specific, it will be an unstable node, which we have done before.

This is some sort of justified because $(0,0)$ means that both the species becoming zeros, which means they go to extinct, so it is not good for nature that the two particular species, whether it is a plant species or whether it is an animal species, they compete against one another and then both of them dies out.

So, nature wants coexistence that both the species must survive and hence this is quite justified that your $(0,0)$ is unstable.

Let us move to the second one, which is $\left(\frac{a_1}{a_2}, 0\right)$, that means the first species is there, but the second species did not survive goes to extinct.

So, if I substitute it you will get the value of A. So, you have to take this matrix you have to substitute $x^* = \frac{a_1}{a_2}$ and $y^* = 0$.

So if you do that you will get

$$\begin{pmatrix} -a_1 & -\frac{a_1a_3}{a_2} \\ 0 & b_1 - \frac{a_1b_3}{a_2} \end{pmatrix}$$

If you want to find the eigenvalue of this

$$|A - \lambda I| = 0$$

which will imply

$$\begin{vmatrix} -a_1-\lambda & -\dfrac{a_1a_3}{a_2} \\ 0 & b_1-\dfrac{a_1b_3}{a_2}-\lambda \end{vmatrix} = 0 \implies \lambda = -a_1\,, b_1-\frac{a_1b_3}{a_2}.$$

.It is clear that this is less than zero.

Now your second value if you want the system to be stable then this has to be less than zero. So that you get a condition from here that if you want the system where the first species exist where the second species goes to 0. Then

$$b_1-\frac{a_1b_3}{a_2}<0.$$

So you can get a condition that

$$b_1<\frac{a_1b_3}{a_2}.$$

You can write it in any form. You can write it as $b_1a_2<a_1b_3$.

What you have to take care is when you are doing the numericals, you have to be sure that if you want a stable system for this particular equilibrium point, this inequality must be satisfied.

Our third equilibrium point is $(0,\frac{b_1}{b_2})$.

In the similar manner, we get this

$$A=\begin{pmatrix} a_1-\dfrac{a_3b_1}{b_2} & 0 \\ \dfrac{-b_1b_3}{b_2} & -b_1 \end{pmatrix}.$$

Again, if you want to find the eigenvalue, you will get

$$\begin{vmatrix} a_1-\dfrac{a_3b_1}{b_2}-\lambda & 0 \\ \dfrac{-b_1b_3}{b_2} & -b_1-\lambda \end{vmatrix} = 0$$

And the solution is going to give you

$$\lambda=-b_1, \quad a_1-\frac{a_3b_1}{b_2}.$$

So if you again want this system to be stable, this is clearly less than zero.

And if you want the system to be stable, this has to be less than zero,

that is,

$$a_1<\frac{a_3b_1}{b_2}.$$

So, for $(0,0)$, you find that the system is unstable for $(\frac{a_1}{a_2},0)$ you find the system to be stable provide an inequality holds, same thing for $(0,\frac{b_1}{b_2})$ you find that the system is stable if this particular inequality holds.

Now, let us go for the non-zero equilibrium point, this will be

$$\left(\frac{a_1b_2-a_3b_1}{a_2b_2-a_3b_3},\frac{a_2b_1-a_1b_3}{a_2b_2-a_3b_3}\right)$$

So this has to be I mean a bit laborious, you take that A, you substitute the values (x^*,y^*) and after simplification you will get something like this,

$$A=\begin{pmatrix}\frac{a_2(a_3b_1-a_1b_2)}{a_2b_2-a_3b_3} & \frac{a_3(a_3b_1-a_1b_2)}{a_2b_2-a_3b_3}\\ \frac{b_3(a_1b_3-a_2b_1)}{a_2b_2-a_3b_3} & \frac{b_2(a_1b_3-a_2b_1)}{a_2b_2-a_3b_3}\end{pmatrix}.$$

So, once you get this, you have to find the characteristic equation, which is

$$|A-\lambda I|=0.$$

If you want to find the eigenvalues in the form of $a_i's$ and $b_i's$, it will be a bit complicated.

So, you find the characteristic equation, which will be of the form

$$\lambda^2+\frac{a_2(a_1+b_1)b_2-a_1b_2b_3-b_1a_2a_3}{a_2b_2-a_3b_3}\lambda+\frac{(a_1b_2-a_3b_1)(a_2b_1-a_1b_3)}{a_2b_2-a_3b_3}=0.$$

Now, one thing you need to notice that this equilibrium point, they represent the density or the number of species.

Let us call, say, it is a density and it has to be positive and hence for this to be positive, $a_2b_2-a_3b_3$ has to be greater than zero, $a_1b_2-a_3b_1$ has to be greater than zero.

Similarly, $a_2b_2-a_3b_3$has to be greater than zero, $a_2b_1-a_1b_3$ has to be greater than zero.

So for the existence of (x^*,y^*) where $x^*\neq 0, y^*\neq 0$, both $x^*>0, y^*>0$ and condition for that is the numerator and the denominator has to be positive.

Previously, we have something like $(\frac{a_1}{a_2},0)$, both a_1 and a_2 are positive.

So, this is obvious that this is a positive quantity, but here we have to follow the condition that

$$a_1b_2-a_3b_1>0,\qquad a_2b_2-a_3b_3>0,\quad \text{and}\quad a_2b_1-a_1b_3>0.$$

Now, when you look into the characteristic equation, the Routh-Hurwitz criteria gives you that if it is

$$\lambda^2+d_1\lambda+d_2=0,$$

then the system will be stable if $d_1>0, d_2>0$.

There is another version you can write it as

$$\lambda^2 - (-d_1)\lambda + d_2 = 0.$$

So, this represents the sum of the roots, this represents the product of the roots.

So we know the system is stable if both the roots, if real, their sum has to be negative and their product will be positive. So this part has to be less than zero this part has to be greater than zero. Already there is a $(-d_1)$ here, so it is less than zero.

Now, if you want this to be positive, that is, in this particular case, so you take this minus sign out, this becomes positive. So, in that particular case, this has to be positive.

So, there are two ways of representing the equation.

You just remember one way, otherwise, sometimes it can be confusing.

Say, let us remember this only,

$$\lambda^2 + d_1\lambda + d_2 = 0,$$

So, both the signs has to be positive and that will make d_1 and d_2 will be positive that is the condition.

Now if we look into this particular thing, the product of the rules we see that this is greater than zero from here, this is greater than zero, from here and this is greater than zero from here.

So the existence of the equilibrium point makes that this particular expression is positive.

So it satisfies $d_2 > 0$ automatically.

Now come to this particular expression, say, this particular expression.

$$\frac{a_2(a_1+b_1)b_2 - a_1 b_2 b_3 - b_1 a_2 a_3}{a_2 b_2 - a_3 b_3}\lambda$$

The denominator is positive because of this condition.

Now if I want the numerator to be positive, so that $d_1 > 0$ is satisfied, so my condition should be that

$$a_2(a_1 + b_1)b_2 - a_1 b_2 b_3 - b_1 a_2 a_3 > 0$$

So this is the condition that the non-zero equilibrium point (x^*, y^*) will be stable, if this particular condition is to be satisfied.

So, when you do the numerical solution, when you choose your $a_i's$ and $b_i's$, you have to be careful that this particular condition is satisfied, only then you get the system to be stable.

Now, so much about the theory, let us look into the numerical solutions.

So, we have this is for species A. So, this is species A or species 1 and this is species 2.

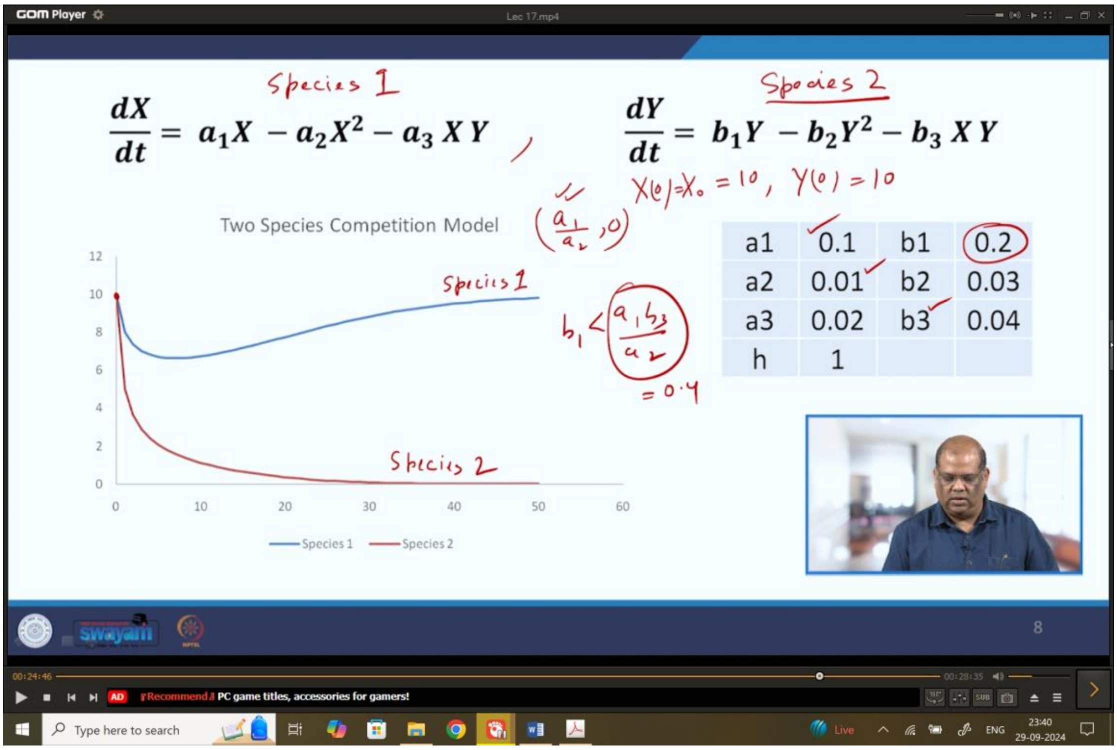
So, I will not go for $(0, 0)$ because I know that there is unstable.

So, I choose the values in such a manner that I get this particular curves.

Now, what does this say?

This is for the species 1 and this is for the species 2.

So, for $\left(\frac{a_1}{a_2}, 0\right)$, this is the case where your species 2 goes to zero, that is, extinction, and your species 1 approaches this equilibrium point.



Now, when this is going to approach this equilibrium point, we have already shown that

$$b_1 < \frac{a_1 b_3}{a_2}$$

Now, if I pluck the values from here a_1 b_3 a_2, I will see that this value is coming to be 0.4 and b_1 has to be less than 0.4. So, b_1 is 0.2 which is less than 0.4 and hence the condition for stability has been satisfied the numerical values are chosen in that manner.

So, in this particular case your species 1 survives and your species 2 goes to extinct, both of them start with the same initial condition, that is, 10.

So, we have taken that $x_0 = 10$ and $y_0 = 10$.

So, x_0 means the value of x at time $t = 0$ and starting with the same density or the same number of population we find that, in this particular case, the species 1 survives and the species 2 goes to extinction, maybe they do not get enough food or sunlight or space and hence they cannot survive, they dies.

The second example is, where your species 1, they go to extinct and your species 2, they survives So, I have to change this particular set a little and in this particular case you have $(0, \frac{b_1}{b_2})$.

So, your species 2 goes to this equilibrium point $\frac{b_1}{b_2}$ and your species 1 goes to zero and the condition for that is

$$a_1 < \frac{a_3 b_1}{b_2}.$$

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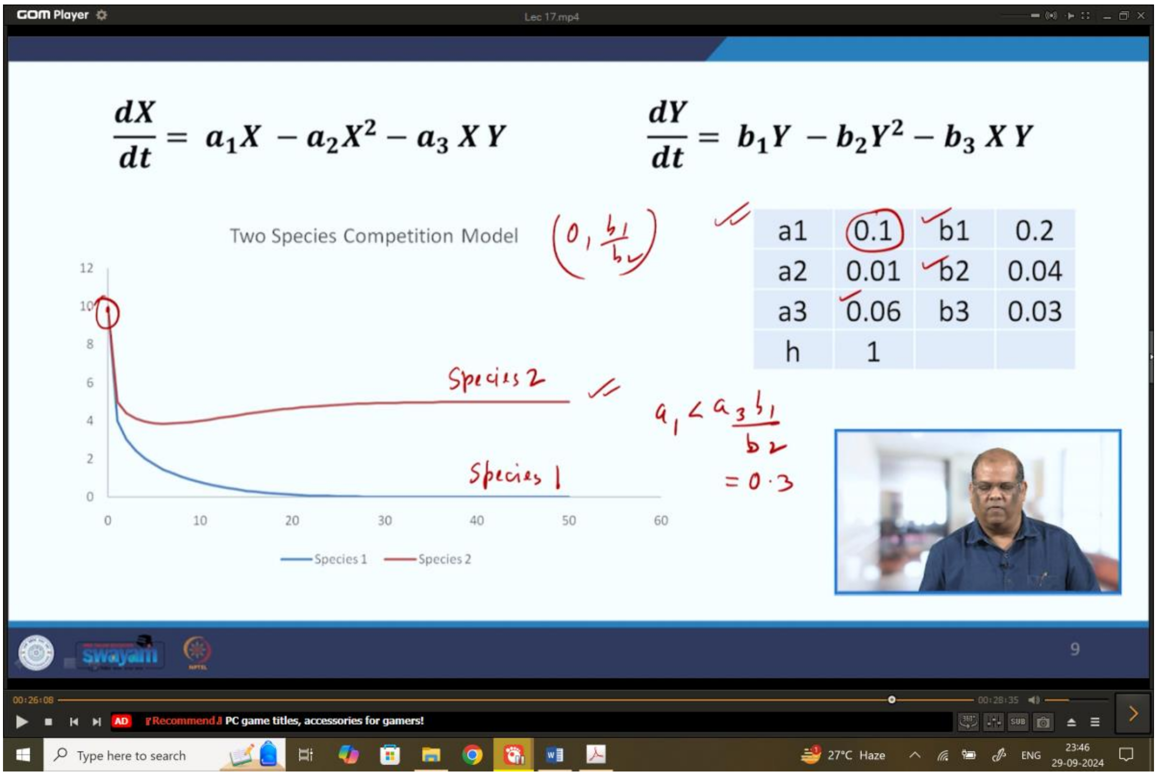


So, here is your a_3, here is your b_1, here is your b_2. If you substitute those values, you will get that this value is 0.3 and the value of a_1 is 0.1, which is definitely less than 0.3, and hence we have a stability.

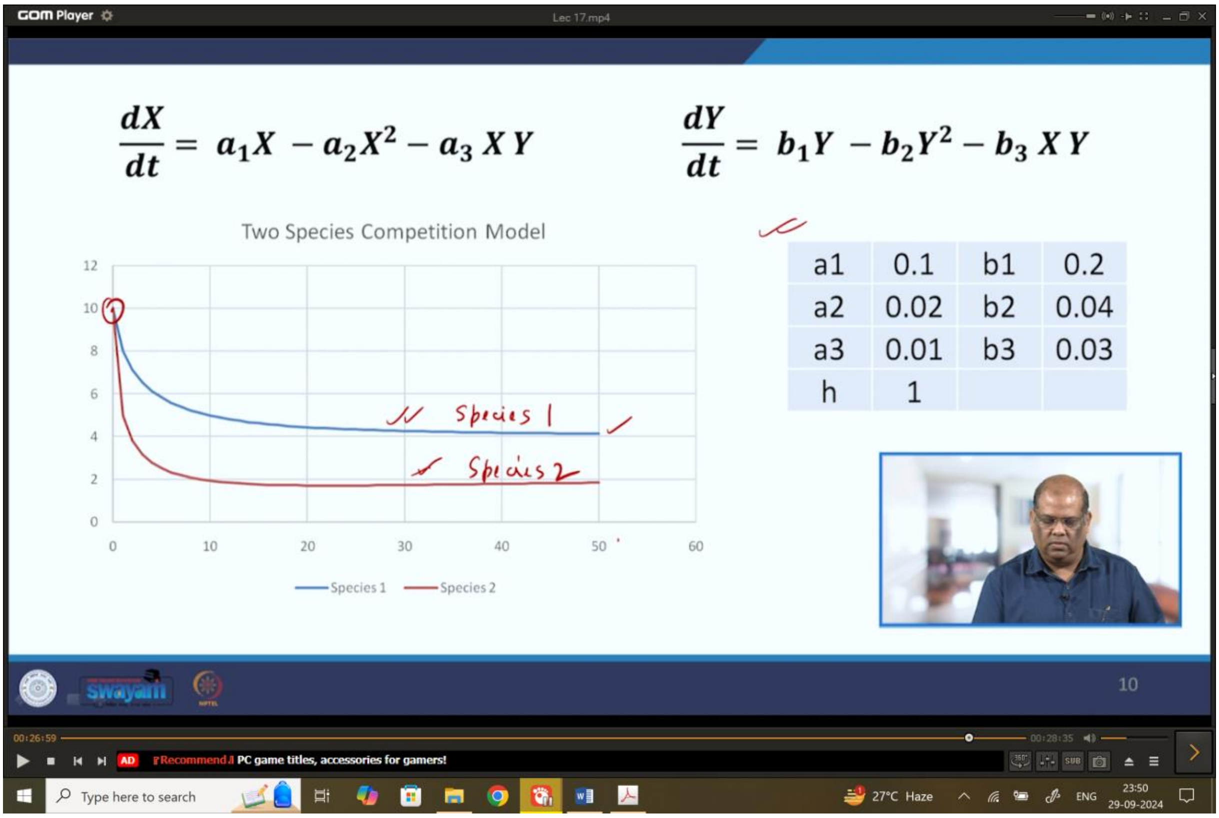
So, since the system is stable, again you have started from the same initial condition $x_0 = 10$, $y_0 = 10$, and you see that your species 1 did not survive whereas your species 2 survives.

Our third example is where both of them survives.

So, we have already checked the stability condition that we have derived and using this particular numerical values we find that these are the curves which they are giving.

Now what does that mean?

It means that yes both of them survives both of them coexisting but species 1 dominates over species 2.



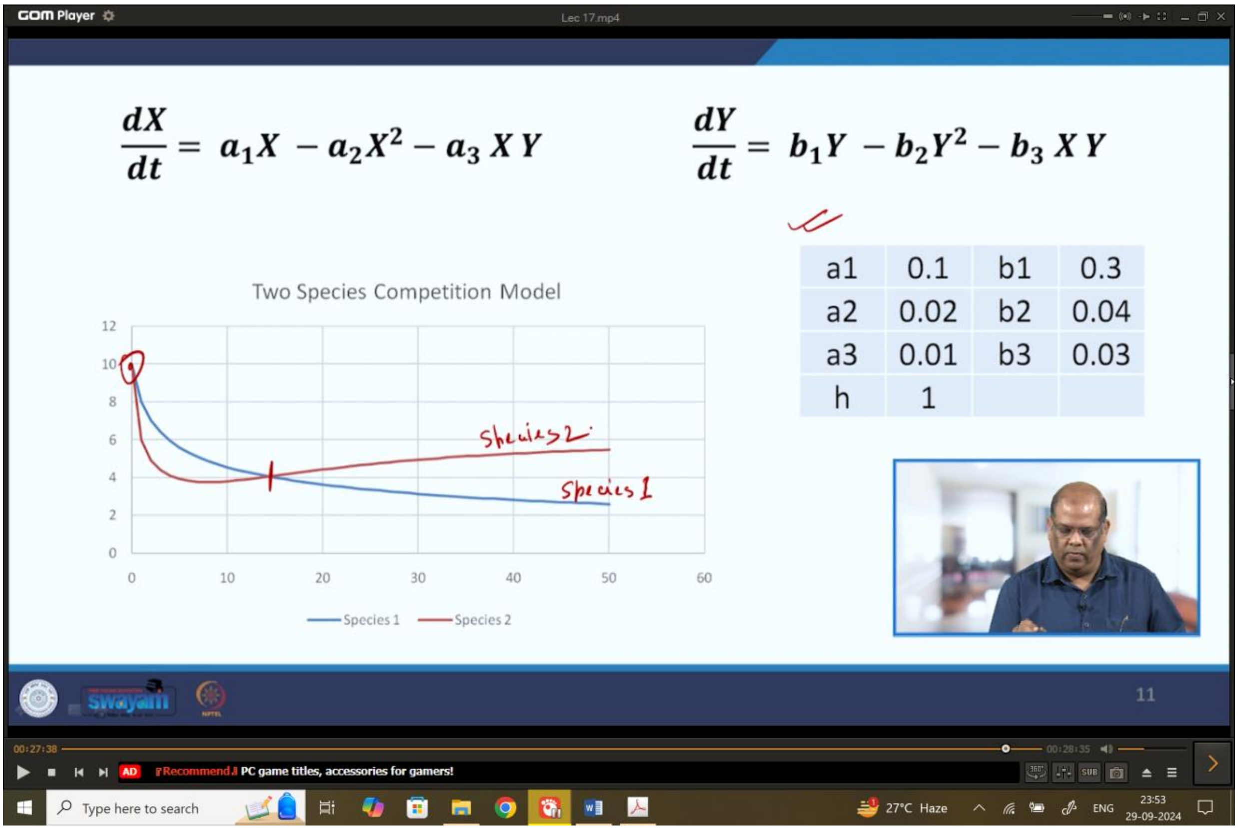
And why it dominates? Because, somehow it managed to get more of the resources than species 2 and hence its growth is much more than this particular species.

So this curve is better, I mean gives a better value than this particular curve.

The final example is again I have taken a different set. So what you see here is this is species 1 and this is species 2, both started with the same value 10.

So, initially species 1 was dominating over species 2 till this point, and then the species 2 takes over and it is now dominating species 1.

So, this is an interesting example and like this you can change your numerical set of values for $a_i{}'s$ and $b_i{}'s$ and get different kind of results.



So, with that we come to an end of the two species competition model.

In our next lecture, we will be learning about the arms race model.

Till then, bye-bye.