EXCELing with Mathematical Modeling
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Week – 04
Lecture – 18 (Arms Race Model)

Hello, welcome to the course EXCELing with Mathematical Modelling.

In today's lecture, we will be learning about the arms race model.

So, what is this arms race?

Consider two neighbouring countries, say India and Pakistan and the amount of money that we spend on arms.

Say let $x(t)$ be that amount of money and $y(t)$ is that amount of money which is spent by our respective countries to buy arms.

So what is this arms race?

See, basically it gives you the idea that how much arms and ammunitions one country possess.

So during the Republic Day, we display our arms to the whole world.

So they get an idea, okay, this country is secured.

They have enough arms and ammunitions, enough technology, enough warfare tactics, which can be used if they suddenly attack and that some sort of create some sort of fear among the neighbouring countries and then what they do they also try to buy the spend money and buy several arms and also display them in public.

So the idea is that then there is a competition of buying arms among the two countries and hence this arms race model.

So to build the first model and as the models are based on assumptions, so the first assumption is the more one country spends on arms, it encourages the other to increase its expenditure on arms because then the country becomes a bit nervous, they see that the other countries has so much arms, so much technology and there is a mutual fear.

So, while building the model you have to assume that each country spends on arms at a rate which is directly proportional to the existing expenditure on other nation.

So, if other country is spending y amount, the country is also spending x, a proportional amount of money, which is dependent on that y.

So if we want to represent that with the form of differential equation, then

$$\frac{dx}{dt} = \alpha y, \qquad \frac{dy}{dt} = \beta x, \;\; \alpha, \beta \; > 0.$$

Where, x is the amount which the country A is spending, it is proportional to the expenditure of y. y is the amount which the country B is spending so it is proportional to what amount country A is spending, and α and β are the proportionality constant and they are taken to be positive.

Now if we solve this model, so this is very simple what you have to do is you just differentiate

$$\frac{d^2x}{dt^2} = \alpha\frac{dy}{dt} = \alpha\beta x$$

So you get

$$\frac{d^2x}{dt^2} - \alpha\beta x = 0$$

Now to solve this kind of second order differential equation you have to take, let $x = Ae^{mt}$ be a trial solution where your $A \neq 0$. So, you substitute it here

$$\frac{dx}{dt} = Ame^{mt}, \qquad \frac{d^2x}{dt^2} = Am^2e^{mt}.$$

You substitute $\frac{d^2x}{dt^2}$ and x here and you will get

$$Am^2e^{mt} - \alpha\beta Ae^{mt} = 0$$

So,

$$Ae^{mt}(m^2 - \alpha\beta) = 0.$$

$A \neq 0$ an there is no value of m for which e^{mt} is zero. So,

$$m^2 = \alpha\beta \implies m = \pm\sqrt{\alpha\beta}.$$

So, the solution to this differential equation will be

$$x(t) = A_1e^{\sqrt{\alpha\beta}t} + A_2e^{-\sqrt{\alpha\beta}t}.$$

In a similar manner, you can show

$$y(t) = B_1e^{\sqrt{\alpha\beta}t} + B_2e^{-\sqrt{\alpha\beta}t},$$

where A_1, A_2 and B_1, B_2 are arbitrary constants.

So, we have a solution.

So, I will quickly show you the numerical values of this solution and then we discuss this.

So quickly I will use the Microsoft Excel to show the solution of this particular differential equation.

Okay, so I already have done the solution just to show you.

I copy this too and I paste it here.

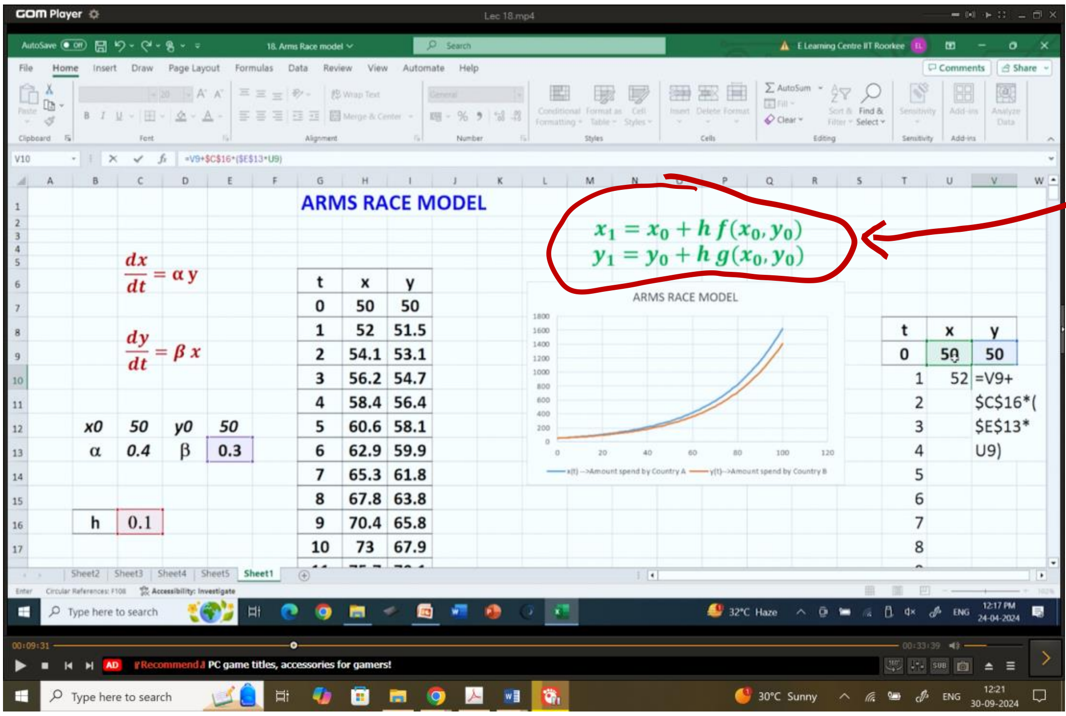
So, at time $t = 0$, I assume that let the country A spend say 50 crores on the arms and country B also spends 50 crores.

So, first is, say, 100 time steps, so this is equal to 0 plus 1 and I drag it say till 100, just increase the font size so that it is easy for you to see.

Now to calculate this we will use Euler's method. The formula is given here.

So, I have already chosen.

This is the set of equation $\frac{dx}{dt} = \alpha y$, $\frac{dy}{dt} = \beta x$. The initial values are 50 and 50, alpha is 0.4, beta is 0.3 and h is 0.1.



So, this will be equal to first x_0, the initial value, this plus h times this, now this is a constant so I will put a dollar here and I put another dollar here h times multiplied by, so you put a star when it is a multiplied in the bracket $f(x_0, y_0)$ so α times this is αy which is this.

Since α is also a constant, I put this as a dollar here or the dollar here and I close the bracket and enter.

So, I get this particular value.

Similarly, let me get this value.

This is equal to y_0 plus h which is again a constant so I put the dollar multiplied by β which is 0.3 again a constant multiplied by x_0, I close the bracket and end.

So, now what I will do is at the same time, I will press shift and the right cursor, so this two are highlighted, and you drag this to the values till the desired value which was given here, in my case it is 98.

So this is a bit time consuming actually but I mean the idea is that so that you can learn at the same time that how you are going to plot the numerical results.

So that's why this result is already here but then I just show you using this.

Now you have to plot this so select this all these three using your down cursor, you go up to the calculated value, up to here, and then you have to plot, go to insert and choose this scattered diagram and choose this one.

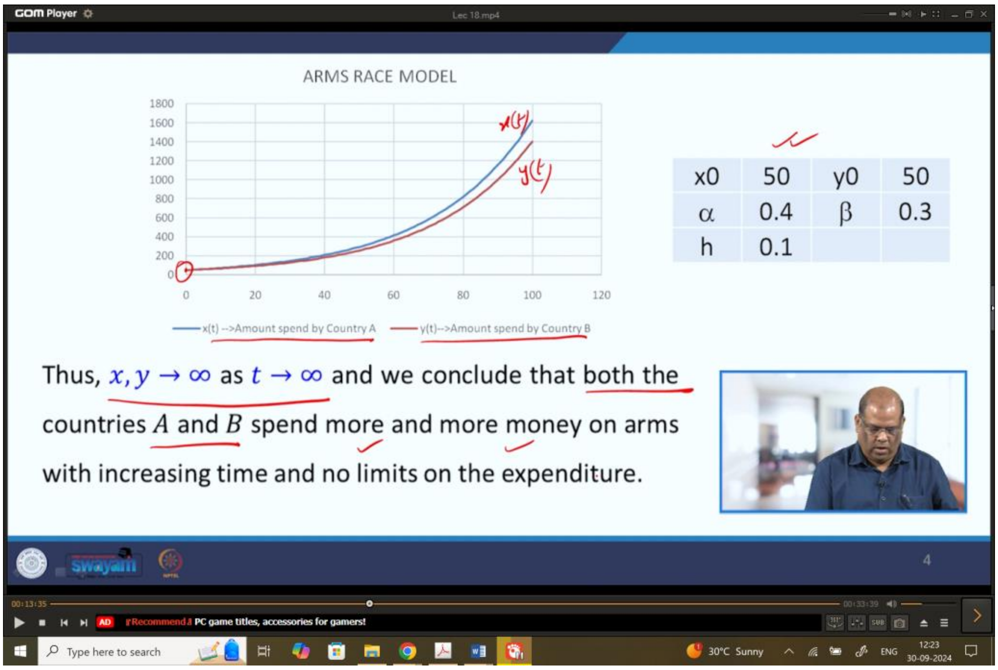
So, you get this same graph.

So, this is your series 1, the blue one, the orange one is series 2.

If you want to change the title, you just go here and type arms race model, if you want to change this series you highlight this, go to this chart design, then select data, and this will open so here is your series one highlight, go to edit and type whatever you want in this case it is $x(t)$, okay and series 2 edit and in this case it is $y(t)$, okay and okay.

So, you get this curve and this is $x(t)$ and this is $y(t)$, the amount which is spent by the respective countries.

So, let us go back to the lecture.



This particular solution you can see that as your t becomes large, then this goes to infinity this goes to zero.

So, your $x(t)$ goes to infinity and similarly as t becomes infinite, this goes to infinity, this goes to zero.

So, your $y(t)$ also goes to infinity this is exactly what is reflected in the numerical solution that we have just done.

So, your $x(t)$, which is this curve $x(t)$ and this is your $y(t)$, we have the numerical here, it starts with 50 crores from here and it goes on increasing the exponential where $x(t)$ is your amount spent by country A and $y(t)$ is the amount spent by country B. So what is the conclusion?

So the conclusion is that as t becomes large, your amount which is spent also starts growing infinitely, not at all a very good model because no country can survive that if they only spend the money on the arms, because they have to look into the infrastructure, into the growth, look into the people and other necessities of the country also.

So from this particular model, we see that both the countries A and B spend more and more money on arms with increasing times and no limit on the expenditure.

So, the economy of the country will be compromised if you follow this particular model and hence this particular model is not at all recommended.

So, once we have this conclusion from this model you have to modify the model.

So, what is the modification here?

So, initially it was assumed that each country spends at a rate which is directly proportional to the existing expenditure on the other nation.

So, now with this, one more thing is added.

So, the rate of change of countries expenditure will also be directly proportional to its own expenditure.

Then how the model changes?

So what we get is here.

$$\frac{dx}{dt} = \alpha y - \gamma x \text{ and } \frac{dy}{dt} = \beta x - \delta y \quad (\alpha, \beta, \gamma, \delta > 0)$$

So this was the initial part where this country A, the spending on the arms depends on what country B is spending.

But now at the same time, its rate of change of the country's expenditure on arms will directly be proportional to its own expenditure.

So, this is the amount which is spent on the growth, on the infrastructure of the country and hence $-\gamma x$. In the similar manner, this is the amount which has been chosen to spend on arms by country B, but then, this is the amount which is needed to its own expenditure for the growth of the country and hence $-\delta y$. So, this gives the modification of the previous model and we see that if we do a little stability analysis here, so, we can put this equation in the form say in the matrix form, say $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix}$, which is equal to, so please note that this is actually$v - \gamma x + \alpha y$, this is fine.

So x comes first, so

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\gamma & \alpha \\ \beta & -\delta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

where

$$A = \begin{pmatrix} -\gamma & \alpha \\ \beta & -\delta \end{pmatrix}.$$

So, we can see that (0,0) is the only solution here, only equilibrium solution, provided

$$\gamma\delta - \alpha\beta \neq 0$$

So, if (0,0) is the solution then what you are solving is this

$$Ax = 0 \text{ (null matrix)},$$

and for the unique solution then $det\ (A)\ \neq\ 0$.

So this is exactly the data which is the $det\ (A)\ \neq\ 0$, only then you get (0,0) to be the equilibrium solution.

Now if you want to find the Jacobian matrix in this particular case it is exactly this and if you find the eigenvalue, that is,

$$|A - \lambda I| = 0,$$

this is going to give you

$$\begin{vmatrix} -\gamma - \lambda & \alpha \\ \beta & -\delta - \lambda \end{vmatrix} = 0$$

And if you simplify this you get

$$\lambda^2 + (\gamma + \delta)\lambda + (\gamma\delta - \alpha\beta) = 0.$$

So, this is of the form I put in different form again.

I can write it in this form, also just to explain the Routh Hurwitz criteria which says if it is of the form this

$$\lambda^2 + a_1\lambda + a_2 = 0,$$

then your condition is $a_1 > 0, a_2 > 0$.

So, these rates are already the positive quantities, so clearly this gives a positive and for the system to be stable this particular thing must be greater than 0.

So, again you can put it in this form and you can write lambda square minus this thing plus this in that case your condition for stability will be this less than 0 and this greater than 0.

But as I told you, you remember only one particular thing.

So we will, from now, on we will only consider this that it can be put in the form

$$\lambda^2 + a_1\lambda + a_2 = 0$$

So both this has to be put is a positive sign and whatever is left must be greater than 0.

So, this $a_1 > 0, a_2 > 0$ and this gives the condition for stability and in this particular case the condition for stability is $\gamma\delta - \alpha\beta > 0$.

So, this particular model is stable but if you plot this.

So, what does this imply that you have this particular condition, this implies that the product of rate of depreciation, if you recall the model, so the model says that this is the amount which is spent on arms for country A and this is the amount which is spent on arms for country B and this is the amount which is spent on the country.

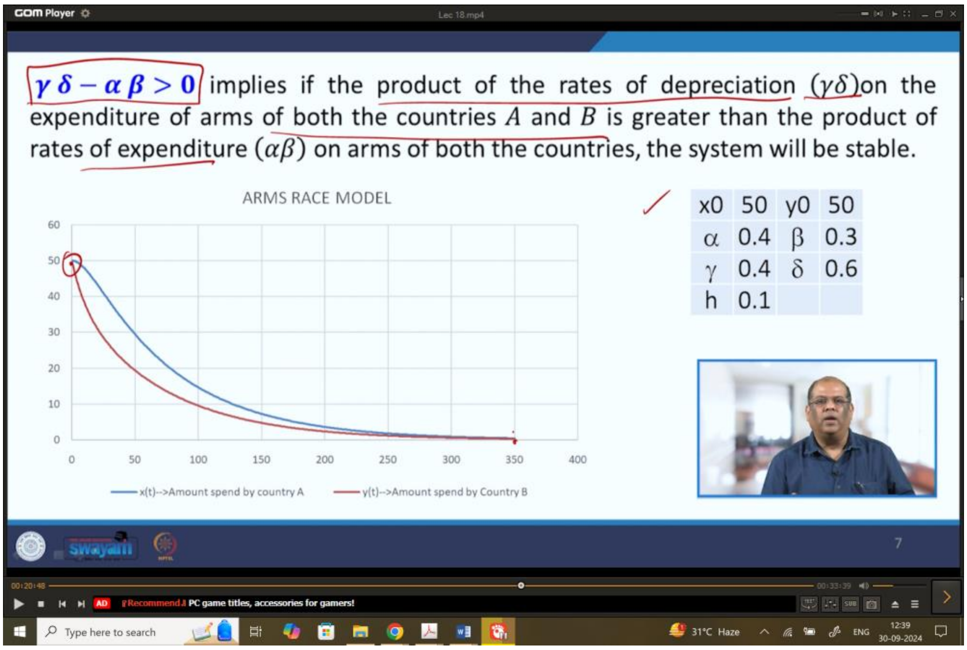
So, from the budget this amount is deducted, from the budget of the arms and similarly from the budget of the arms this amount is deducted.

So there is a depreciation and the rate of depreciation is given by this γ and this δ.

So, the interpretation, the product of the rate of depreciation which is $\gamma\delta$ on the expenditure of the arms on both the countries must be greater than the product of the rate of expenditure.

So rate is again the expenditure which is giving you the actual money to spend on the arms.

So for the model to be stable the rate of depreciation must be greater than the product of the rate of the expenditure and then your system will be stable.

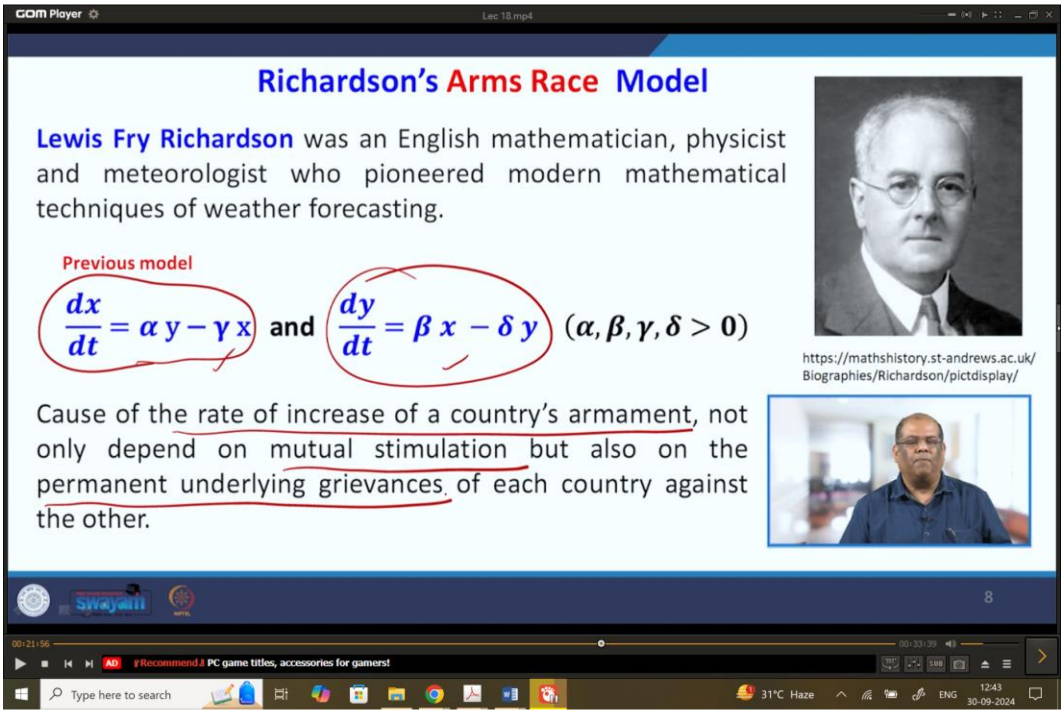


So again you start with some 50, say, crores and you see the since (0,0) is stable slowly the spend on arms the budget on the spent on arms over a long period of time becomes zero.

That means both the countries they have come to an agreement, okay, we will live in peace and hence the budget on spending money on the arms, they diminish and ultimately comes to zero.

But then there was a better refinement of this model and it is done by Lewis Fry Richardson.

So, he is an English mathematician, a physicist, a meteorologist and he come up with just two terms.



So, what he says is, this was the previous model, so, what he says the rate of increase of countries armament, that is, the amount spent on arms, they not only depend on mutual stimulation. By mutual stimulation, I mean that you have the mutual fear.

What's the other country spending?

Then you have to look into your economy so that it doesn't collapse.

And hence that give rise to this particular models for country A and country B. But then he introduced a factor which says that the permanent underlying grievance.

So between two countries there is always a tension there is a war and some war or some happenings we just could not forget.

So in our case, there is a Indo-Pakistani war in 1965 where India has the upper hand over Pakistan when the ceasefire was declared.

So Pakistan may not be able to forget that so they have an underlying grievance.

There is a Bangladesh liberation war in 1971 where, there is a, which Pakistan has to pay a heavy cost and ultimately Bangladesh was formed.

They may not be able to forget that.

We have a 2008 Mumbai attack which takes place in Taj Mahal Palace.

We cannot forget that.

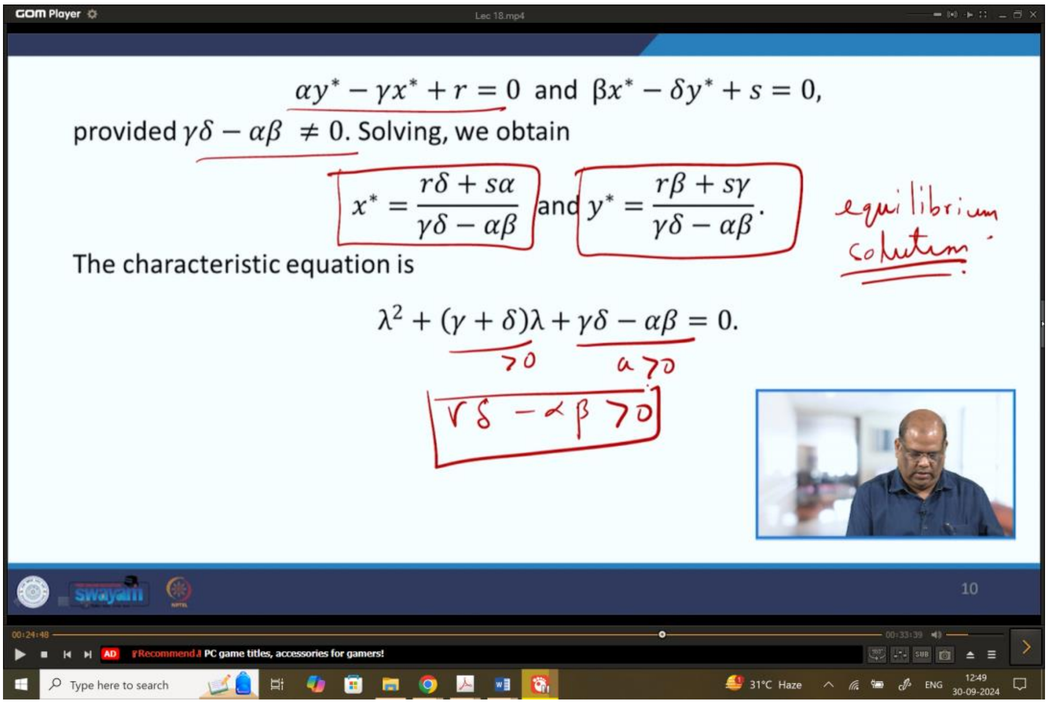
Then there is a Uri attack in 2016. We may not be able to forget that.

So, in this particular model, he just put two terms r and s.

$$\frac{dx}{dt} = \alpha\, y - \gamma\, x + r\,, \quad \frac{dy}{dt} = \beta\, x - \delta\, y + s$$

And please note that no sign has been attributed to r and s, r can be positive, r can be negative. So if r is positive, I still have the grievance. If r is negative, okay, over the time, I have learned how to forgive the neighbouring countries and my grievance becomes less and less.

With these two terms, the whole dynamics of this model changes.



So, now I solve this equation that means, I find the equilibrium solution, I put this equal to 0, I put this equal to 0, and I will find the equilibrium solution.

So now $(0,0)$ is not the equilibrium solution it is something different but what remains same is the Jacobian matrix, that is, $A = \begin{pmatrix} -\gamma & \alpha \\ \beta & -\delta \end{pmatrix}$, so the stability condition remains the same, that is, $\gamma\delta - \alpha\beta > 0$.

So now we have to see at what actually is happening just by adding these two simple looking terms r and s, whose sign can be positive as well as negative.

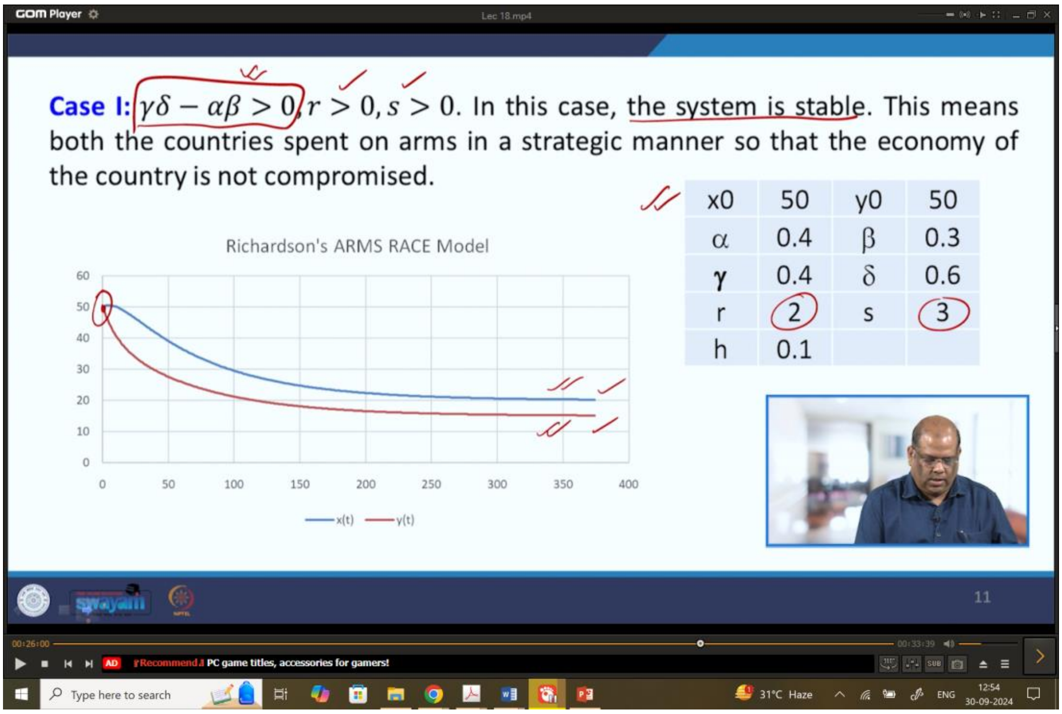
So, now you solve these two equations for the equilibrium solution provided this is not equal to 0 in fact it is greater than 0 because of the stability and you get this and this to be your new equilibrium solution.

So the characteristic equation remains same this is greater than 0 and the system has to be stable this is greater than 0 and we have the condition $\gamma\delta - \alpha\beta > 0$

Now let us take various cases.

This is needed for the stability and both the countries say have the grievance.

So I have this numerical values.



I make sure that this particular condition is satisfied and the system is stable and then I have chosen some value of r and s both are positive.

And if I plot this diagram so you get it, say, 50 crores you see that ultimately it comes to this equilibrium position, so if you interpret this that okay both the countries are spending money on arms in a strategic manner so that the economy of the country is also not compromised, that is, this solution does not go to infinite but over time it is coming to some particular value.

So, looking at what the other countries are doing, how they are building up their defence, the other country also focuses on the arms but with a strategic manner so that the budget is there and the economy of the country is not compromised.

Let us take, when both of them less than zero.

So, what is the mean, if both of them are less than zero that means there is no grievance among the countries.

So, in this particular case what happens is, that this is the equilibrium solution, now this is less than 0, this is less than 0.

This particular thing is greater than 0 because for the system to be stable, again this is less than 0, this is less than 0.

So, your equilibrium solution becomes negative.

But there cannot be a negative equilibrium solution in this case because x and y represents the amount of money that both the countries are spending.

So what actually happens?

So if your x_0 y_0, you start with initial expenditure which is 50 crores in our case, then it attends some x^* and y^* which is negative because you have taken r and s. And if it wants to attend that (x^*, y^*) which is somewhere here, it has to go through 0, obviously because the other is a negative quantity.

So, if you draw the graph, you will get something like, so as your $x(t)$ becomes 0, so what you do is, in the equation, you put $x(t) = 0$, this equation.

You put $x(t) = 0$ and you see what happens to $y(t)$

So if you do that you will get this particular equation.

It is a first order linear equation whose solution will be of the form like this.

So basically what you have to do is, you have to take this quantity here and this is equal to x. So your integrating factor equal $e^{\int \delta dt}$, so $e^{\delta t}$. So you multiply by the integrating factor both sides plus delta and this thing can be written as $\frac{d}{dt}(ye^{\delta t})$.

So you can remember this can be represented $\frac{d}{dt}(ye^{\delta t})$ so that if you just differentiate, so, u into v, so the first term differentiation of the second which gives you this the second term and differentiation of the first which is divided and this is equal to $se^{\delta t}$.

So,

$$ye^{\delta t} = \int se^{\delta t}\, dt + C_1$$

So, this is going to give you

$$ye^{\delta t} = \frac{s}{\delta}e^{\delta t} + C_1$$

And if I divide it by $e^{\delta t}$, I will get

$$y(t) = \frac{s}{\delta} + C_1 e^{-\delta t}$$

So, this is the solution and as your t becomes large, you will see your $y(t)$ is going to some $\frac{s}{\delta}$

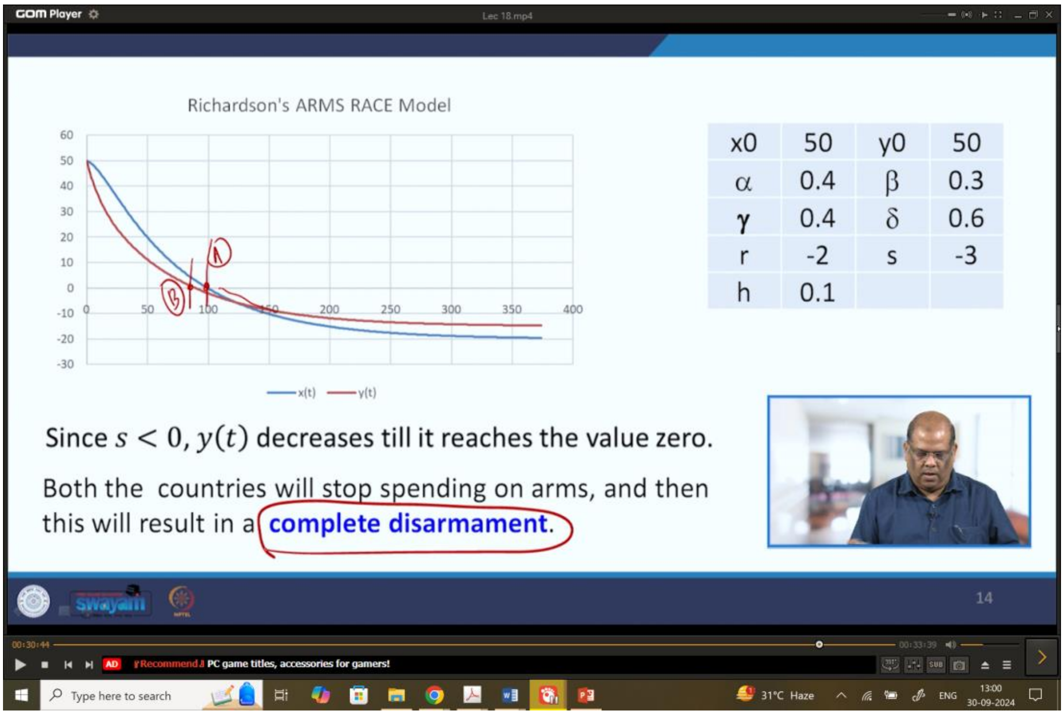
So $s < 0$, so it is going to some negative quantity.

So how do you interpret that?

So you see that at this, this is your 0.

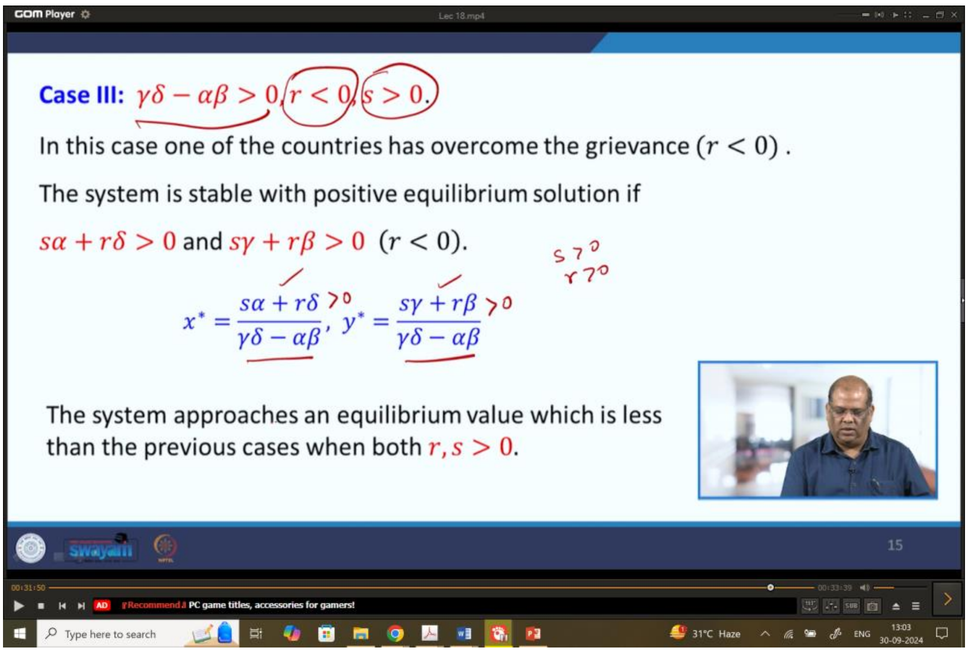
So this is this two line where your amount for country A goes to 0 and this is the amount for country B goes to 0.

So, basically in reality they cannot go here because they cannot spend a negative amount on the arms.



So, by that you will mean that both the countries once it reaches 0, has stopped spending on arms they decided okay let's live in peace whatever arms we have it's enough for us to defend and this results in something called complete , that is, they are not spending any more money on arms and decided to leave peacefully.

The next case is where one country has forgotten the grievance but one country remembers it. So what happens?



So, in this particular case, that one country has overcome the grievance and you have this particular equilibrium solution and if you want this solution to be positive, so obviously this must be greater than 0 and this must be greater than 0.

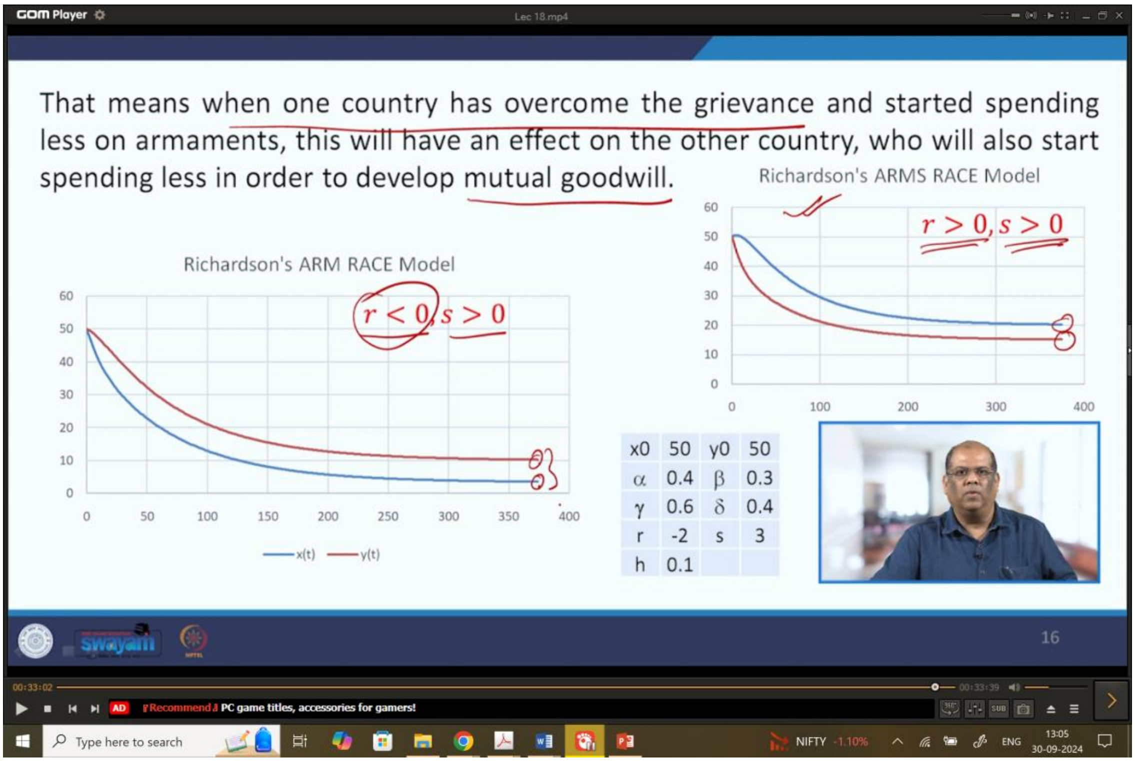
This is always greater than 0 because otherwise the system would not be stable.

So if r is less than 0, so this particular thing has to be greater than 0, this particular thing has been greater than 0.

So what you will get you will get a solution almost similar to when both s is positive and r is positive, but since in this case r is negative the value will be a bit less.

So that is why the conclusion is the system approach an equilibrium value, which is less than the previous cases, where both r and s is greater than 0.

So if you draw the graph so this particular case where r is less than 0 and s is greater than 0.



So when one country has overcome the grievance which is r less than 0, it started spending less on armaments, this will have an effect on the another country and they will also start spending less to develop a mutual good value.

So, you get some amount where the equilibrium point will take to.

So, I put a similar graph here the same when r is greater than 0 and s is greater than 0.

So, if you notice these two values the equilibrium values are larger than these two values which is obvious because your r is greater than 0 here, so it will reach a higher value of the equilibrium point and here r is less than 0 but still the system is stable so it will reach some lower value of the equilibrium point.

So that is all for this arms race model, in our next lecture we will be taking some other interesting model till then bye-bye.