EXCELing with Mathematical Modeling
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Lecture – 19 (Combat model-I)

Hello, welcome to the course EXCELing with Mathematical Modelling.

Today we will be learning about the combat models.

So, what is a combat?

So, in the time of war when two enemies face each other, there are various strategies by which they fight with one another.

One of them is while taking them down facing one to one and that particular kind is called the conventional combat model.

So, here we will be learning today about this conventional combat technique.

The person who is responsible for this mathematical model is F. W. Lanchester. He is a British engineer and he used this modelling technique during the First World War.

So, he developed a model, which can be used to predict the outcome and the surviving of how many soldiers during the battle that can be calculated or that can be predicted. So, that model helped in better planning prediction of the battles and their possible outcomes.

Also, that model can be manipulated to some other areas like this guerrilla war tactics, which we will be discussing, then a combination of guerrilla army or a conventional army.

So, today our discussion will be on only this conventional combat model.

Before that, so, Lanchester used this simple technique, that is,

$$\text{rate of change} = \text{rate in} - \text{rate out}.$$

So the idea is that if your model can be simple and can capture all the dynamics why make it complicated.

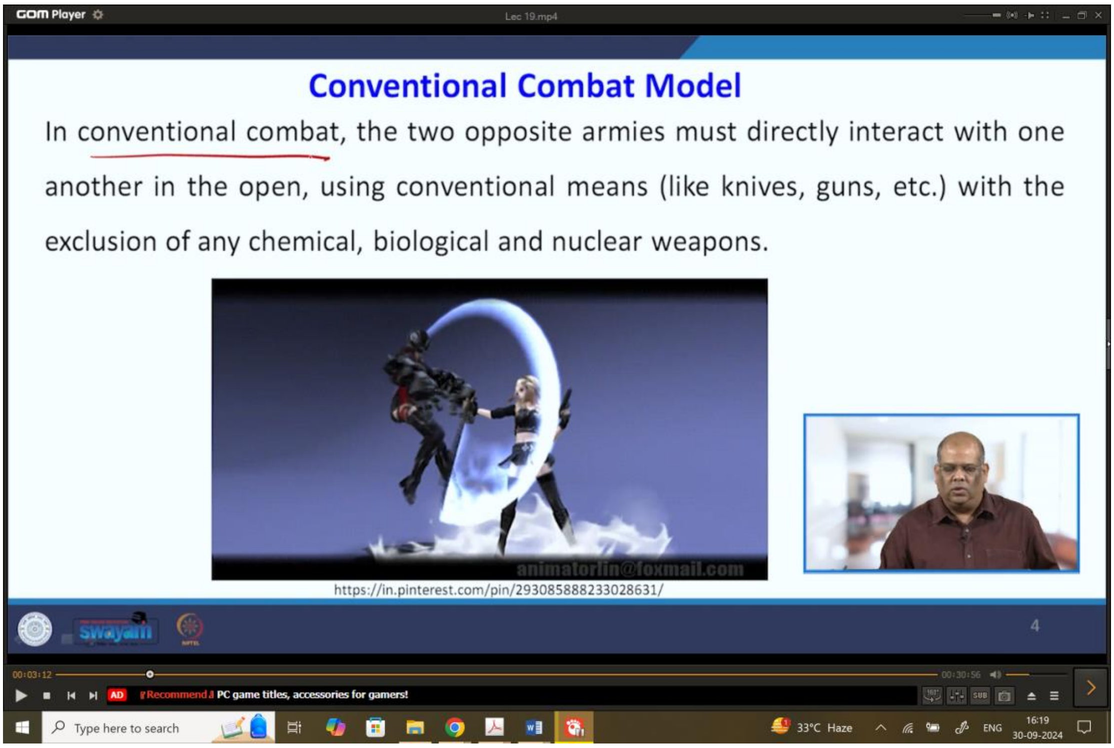
So it is a very simple equation that the rate of change will be the rate in and the rate out, the difference between them, where this rate-in means that the total number of troops that has been supplied, which we called reinforcement, and rate-out is the number of troops not available for the fight.

So, either they are dead or they are sick or injured that they cannot participate in the battle.

So, just how many soldiers have been supplied and how many soldiers cannot take part, their difference is the rate of change.

So, what is this conventional combat model?

So, in this combat model which is a conventional one, so the two opposite armies they directly interact with one another.



So, as you can see that they use either knives or you can use guns or sometimes even hand to hand combat.

And that will not include any kind of bombing or chemical weapons or nuclear weapons or fighting with each another with guns from long range. That is not permissible.

The combat is that, okay, you have to face your enemy one to one and fight with them.

So, if your $x(t)$ and $y(t)$, they denote the number of troops, or in another word, combatants, that is the persons who are fighting each another.

This is for Army A that is $x(t)$ and Army B this is the $y(t)$. It is total number of troops.

Now we have to make the assumptions.

The very first assumption is that the operation losses are neglected. That is number one assumption.

There is no new supply of troops. That is, once there is an $x(t)$ number of troops available for a certain army A, there won't be any addition of troops. So, that is your no reinforcement.

And number three, the most important one, the combat loss rate of the conventional army A is proportional to the size of the opposing army B. So, if your $x(t)$ is your number of troops for army A, the rate of change, so their loss must be proportional to the army B. And since it is a loss

$$\frac{dx}{dt} = -\alpha y,$$

So, y is the number of troops for the army B, α is their fighting technique and with that technique how much troops of army A, they are able to kill or they are able to take down or they are able to injure that depends on this α, their fighting skills and since it is a loss, this is a negative sign.

The same thing will happen for army B. So

$$\frac{dy}{dt} = -\beta x,$$

where this β is the rate of the fighting skill for this for the troop of army A and with that skill how much loss they can impose on army B that is determined by this equation.

So, if we now look into the model, it is

$$\frac{dx}{dt} = -\alpha y \text{ and } \frac{dy}{dt} = -\beta x,$$

$x(t)$ and $y(t)$ are the number of troops, x for army A and y for army B and this α and β which are both positive. They are the fighting skills or we say fighting effectiveness coefficients of the army B. So this is for the army B and this is for the army A.

There should be some initial troops for both sides, that is, the initial started with x_0 and army B started with some y_0 at time $t = 0$.

So now we have to solve this differential equation and let us see what is the conclusion you get from here. So, we start with

$$\frac{dx}{dt} = -\alpha y$$

You differentiate this one more time and you get

$$\frac{d^2x}{dt^2} = -\alpha\frac{dy}{dt} = -\alpha(-\beta x)$$

So, you get

$$\frac{d^2x}{dt^2} - \alpha\beta x = 0$$

And as you know, this is a second order homogeneous differential equation and you take

$$x = Ae^{mt}$$

to be the trial solution. You substitute it here and you get

$$\frac{dx}{dt} = Ame^{mt}, \qquad \frac{d^2x}{dt^2} = Am^2e^{mt}$$

You substitute them in this equation and you will get

$$Am^2e^{mt} - \alpha\beta Ae^{mt} = 0.$$

So, here you have to consider $A \neq 0$. So, you take

$$Ae^{mt}(m^2 - \alpha\beta) = 0.$$

Since $A \neq 0$ and there is no value of m for which $e^{mt} = 0$, the only possibility is

$$(m^2 - \alpha\beta) = 0.$$

And, this implies $m = \pm\sqrt{\alpha\beta}$, where both $\alpha, \beta > 0$.

So, now your solution is going to be

$$x(t) = A_1e^{\sqrt{\alpha\beta}t} + A_2e^{-\sqrt{\alpha\beta}t}$$

Now, the initial conditions are given, I have to find the values of A_1 and A_2.

So, you have calculated

$$x(t) = A_1e^{\sqrt{\alpha\beta}t} + A_2e^{-\sqrt{\alpha\beta}t}$$

I just differentiate it once, then I get

$$x'(t) = \sqrt{\alpha\beta}A_1e^{\sqrt{\alpha\beta}t} - \sqrt{\alpha\beta}A_2e^{-\sqrt{\alpha\beta}t}$$

So the reason I have to differentiate it because there are two arbitrary constants A_1 and A_2 and I need two equations.

So, now you have the equation

$$\frac{dx}{dt} = x'(t) = -\alpha y(t)$$

and

$$\frac{dy}{dt} = y'(t) = -\beta x(t)$$

So, if I now put $t = 0$ here, so I get

$$x(0) = A_1 + A_2 = x_0 \qquad \text{———} (1)$$

because this becomes 1, this also becomes 1 and it is given that $x(0) = x_0$ and $y(0) = y_0$

So, this value is x_0 and if I write

$$x'(0) = \sqrt{\alpha\beta}(A_1 - A_2)$$

Now, this I have to find from here, you have

$$x'(t) = -\alpha y(t) \implies x'(0) = -\alpha y(0) = -\alpha y_0 \,.$$

In the similar manner

$$y'(0) = -\beta x_0 \,.$$

But here I will need only $x'(0)$ so I will be using this particular value, so this is $-\alpha y_0$. So it gives me

$$A_1 - A_2 = -\frac{\alpha}{\sqrt{\alpha\beta}} y_0 = -\sqrt{\frac{\alpha}{\beta}} y_0 \qquad (2)$$

So, I just solve for A_1 and A_2 from (1) and (2).

So, if I just add them,

$$2A_1 = x_0 - \sqrt{\frac{\alpha}{\beta}} y_0 \implies A_1 = \frac{1}{2}\left(x_0 - \sqrt{\frac{\alpha}{\beta}} y_0\right),$$

and if I just subtract them,

$$2A_2 = x_0 + \sqrt{\frac{\alpha}{\beta}} y_0 \implies A_2 = \frac{1}{2}\left(x_0 + \sqrt{\frac{\alpha}{\beta}} y_0\right)$$

So, I will substitute this A_1 and A_2 in this x and I get my

$$x(t) = \frac{1}{2}\left(x_0 - \sqrt{\frac{\alpha}{\beta}} y_0\right) e^{\sqrt{\alpha\beta}t} + \frac{1}{2}\left(x_0 + \sqrt{\frac{\alpha}{\beta}} y_0\right) e^{-\sqrt{\alpha\beta}t}$$

So, that is the solution of x and in the similar manner I can calculate $y(t)$.

So, there are two ways of calculating this $y(t)$ either you have to do this whole solution or $y(t)$ can be calculated directly from here that is your

$$x'(t) = -\alpha y(t) \implies y(t) = -\frac{x'(t)}{\alpha}.$$

So, you just have to differentiate this and divide by α, to get this $y(t)$.

And if you do that you will get

$$y(t) = \frac{1}{2}\sqrt{\frac{\beta}{\alpha}}\left[-\left(x_0 - \sqrt{\frac{\alpha}{\beta}} y_0\right) e^{\sqrt{\alpha\beta}t} + \frac{1}{2}\left(x_0 + \sqrt{\frac{\alpha}{\beta}} y_0\right) e^{-\sqrt{\alpha\beta}t}\right]$$

So basically I have this equation for $x(t)$ and I have this equation for $y(t)$.

So now from these two equations, I will try to give you the conclusion, of what is the outcome of this combat model.

So please note that this is a positive one, but this one will give you a variable answer, as I can say the condition that if $x_0 - \sqrt{\frac{\alpha}{\beta}}y_0 > 0$, I will get one thing. If $x_0 - \sqrt{\frac{\alpha}{\beta}}y_0 < 0$, I will get another thing. If $x_0 - \sqrt{\frac{\alpha}{\beta}}y_0 = 0$, I may get any other solution.

So let us check that.

So the very first thing is when your $x_0 - \sqrt{\frac{\alpha}{\beta}}y_0 > 0$.

So I already give you the solution here.

$$x(t) = \frac{1}{2}\left(x_0 - \sqrt{\frac{\alpha}{\beta}}y_0\right)e^{\sqrt{\alpha\beta}t} + \frac{1}{2}\left(x_0 + \sqrt{\frac{\alpha}{\beta}}y_0\right)e^{-\sqrt{\alpha\beta}t} \qquad (5)$$

So this particular thing has been taken greater than zero.

So, if this is greater than zero, then from here, what we can see is this whole thing is positive

$e^{-\sqrt{\alpha\beta}t} > 0$, this thing is positive and $e^{\sqrt{\alpha\beta}t} > 0$.

So, $x(t)$ is always positive.

So if your $x_0 - \sqrt{\frac{\alpha}{\beta}}y_0 > 0$, then it is clear from this particular equation, say, (3) and this equation, say, (4), implies $x(t) > 0, \forall\, t \geq 0$.

So, even if you put $t = 0$, you will get the value is coming to be x_0, which is your initial value and that is also a positive quantity.

So, $x(t) > 0$, when $x_0 - \sqrt{\frac{\alpha}{\beta}}y_0 > 0$.

Now, either your $y(t) > 0$ or $y(t)$ will go to zero.

So, suppose I assume, say, $y(t) = 0$ for some time $t = T$. So, let us see what is that capital T here. So, I will substitute this value here and I will get

$$\frac{1}{2}\sqrt{\frac{\beta}{\alpha}}\left[-\left(x_0 - \sqrt{\frac{\alpha}{\beta}}y_0\right)e^{\sqrt{\alpha\beta}T} + \frac{1}{2}\left(x_0 + \sqrt{\frac{\alpha}{\beta}}y_0\right)e^{-\sqrt{\alpha\beta}T}\right] = 0, \qquad \text{for some T.}$$

So, this will imply that, $\frac{1}{2}\sqrt{\frac{\beta}{\alpha}}$ is a non-zero quantity, so, it will imply that

$$\left(x_0 + \sqrt{\frac{\alpha}{\beta}}y_0\right)e^{-\sqrt{\alpha\beta}T} = \left(x_0 - \sqrt{\frac{\alpha}{\beta}}y_0\right)e^{\sqrt{\alpha\beta}T}$$

So, basically, what I have done is, that this particular thing is equal to zero, and I have taken this to the right hand side.

And once you do that, you just divide, because I have to calculate this capital T, so,

$$\frac{x_0 + \sqrt{\frac{\alpha}{\beta}}y_0}{x_0 - \sqrt{\frac{\alpha}{\beta}}y_0} = \frac{e^{\sqrt{\alpha\beta}T}}{e^{-\sqrt{\alpha\beta}T}} = e^{2\sqrt{\alpha\beta}T}.$$

Take log on both sides and you can calculate

$$\ln\left[\frac{x_0 + \sqrt{\frac{\alpha}{\beta}}y_0}{x_0 - \sqrt{\frac{\alpha}{\beta}}y_0}\right] = 2\sqrt{\alpha\beta}T$$

$$\Rightarrow T = \frac{1}{2\sqrt{\alpha\beta}}\ln\left[\frac{x_0 + \sqrt{\frac{\alpha}{\beta}}y_0}{x_0 - \sqrt{\frac{\alpha}{\beta}}y_0}\right].$$

So, we get a positive time T, because this value is positive for which the value of $y(T)$ can be equal to zero.

So, the conclusion is that your $x(t)$ is always positive, no matter what the value of t is, and your y(t), it gives a finite T for which your $y(T)$ is zero. So, the troops of army B become zero in some finite time whereas the troop for army A is still positive.

So, army A wins the battle.

Once again you have calculated you have shown that this $x(t)$ is always positive because this particular expression is always positive and if this particular expression is always positive, then this is positive, this is surely positive, exponential values are always positive, so $x(t)$ is always positive.

Then what you have shown is that, let us see if you can find any finite time for which this $y(t)$ is equal to zero because $x(t)$ is always positive, and you have shown that, okay there exists a T which is finite, and your $y(t)$ becomes zero.

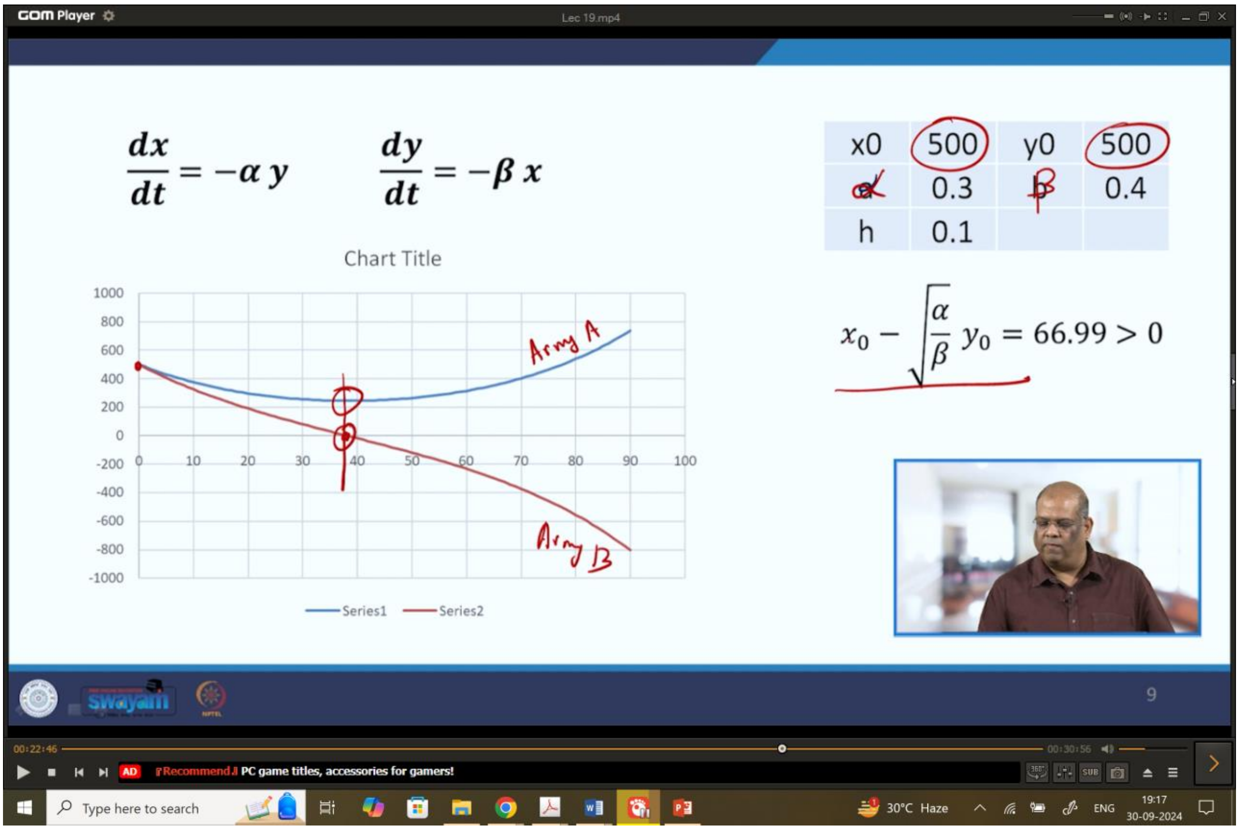
So, $y(t)$ means the troops of army B becomes zero and hence army A wins the battle.
So, that is the case if your $x_0 - \sqrt{\frac{\alpha}{\beta}}y_0 > 0$.

Now, if you want to see the numerical values, this is what you will get.

So, this is your α and that is your β. So, α is 0.3 and β is 0.4.

We have taken both the initial values of the army to be 500.

So, it had starts from this. So, I have prolonged this curve, but actually it will stop somewhere here.



This is your army A, this is your army B.

So, as we have shown that the number of troops for army B will be zero at some finite time.

So, this is that finite time for which your army B becomes zero and your army A still have this positive number and you can see it is growing.

So, the numerical value confirms that this particular value $x_0 - \sqrt{\frac{\alpha}{\beta}} y_0$ is 66.99, which is positive and hence the numerical result confirms the analytical results.

Let us take the second case. So, in the second case, we take that when

$$x_0 - \sqrt{\frac{\alpha}{\beta}}y_0 < 0.$$

So, then what happens?

So, if this is less than 0, then from here you see that this is less than 0, and with this negative sign this whole thing becomes positive.

So, this whole expression along with this negative sign becomes a positive quantity, this is also a positive quantity, this is again a positive quantity, this is again a positive quantity.

$$y(t) = \frac{1}{2}\sqrt{\frac{\beta}{\alpha}}\left[-\left(x_0 - \sqrt{\frac{\alpha}{\beta}}y_0\right)e^{\sqrt{\alpha\beta}t} + \frac{1}{2}\left(x_0 + \sqrt{\frac{\alpha}{\beta}}y_0\right)e^{-\sqrt{\alpha\beta}t}\right]. \quad (6)$$

So, in this particular case, if we write this as (5) and this as (6), so from (6),

$$y(t) > 0, \forall\, t \geq 0.$$

Again, you can check, if you put t = 0, you are getting finite value of this quantity, which will be y(0).

So, now let us with the same logic, let us come to $x(t)$.

So, we say that this is now a negative quantity, this is a positive.

So, I do not know whether $x(t)$ will be totally positive or it may go to zero.

So, let us assume that there exist a time T_1, a finite time T_1 for which this $x(T_1)$ vanishes and I will put

$$\frac{1}{2}\left(x_0 - \sqrt{\frac{\alpha}{\beta}}y_0\right)e^{\sqrt{\alpha\beta}T_1} + \frac{1}{2}\left(x_0 + \sqrt{\frac{\alpha}{\beta}}y_0\right)e^{-\sqrt{\alpha\beta}T_1} = 0.$$

So, please note that this is a negative quantity.

So, when I calculate this T_1, in the similar manner, I will get

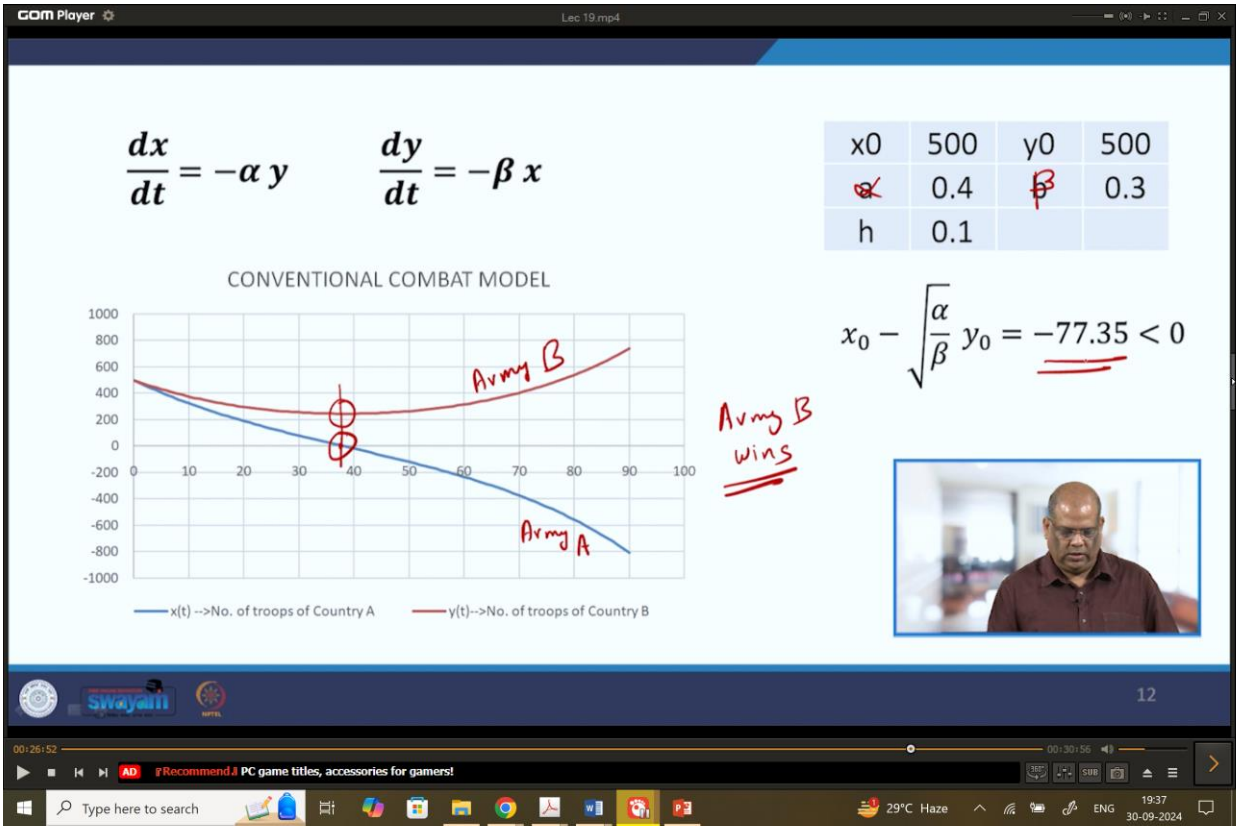
$$T_1 = \frac{1}{2\sqrt{\alpha\beta}}\ln\left[\frac{x_0 + \sqrt{\frac{\alpha}{\beta}}y_0}{\underbrace{-\left(x_0 - \sqrt{\frac{\alpha}{\beta}}y_0\right)}_{>0}}\right].$$

Now, this quantity is negative and along with this negative sign this whole quantity is positive.

So, inside the log nothing is negative, this is also positive, this is also positive.

So, you get a finite time T_1 for which your $x(T_1) = 0$ and hence in this case you have army B wins.

If you see the numerical result, so, in this particular case, again it reaches zero somewhere at this point.



So, this is for army A and this is for army B. So, army A becomes zero, the number of troops $x(t)$ at some finite time and your army B is still positive and hence your army B wins.

And in this particular case $x_0 - \sqrt{\frac{\alpha}{\beta}}\, y_0$ is negative quantity, and again the numerical results supports the analytical calculations or findings.

And let us consider the third case, where your $x_0 - \sqrt{\frac{\alpha}{\beta}}\, y_0 = 0$.

So if it equals to zero, then from here, what happens?

Then this one vanishes, this one vanishes.

So what do you get? You get your

$$x(t) = \frac{1}{2}\left(x_0 + \sqrt{\frac{\alpha}{\beta}}\, y_0\right) e^{-\sqrt{\alpha\beta}t}$$

and

$$y(t) = \frac{1}{2}\sqrt{\frac{\beta}{\alpha}}\left(x_0 + \sqrt{\frac{\alpha}{\beta}}y_0\right)e^{-\sqrt{\alpha\beta}t}$$

Here you can see that if your t becomes large, then x(t) goes to zero, y(t) also goes to zero.

So, when your t becomes large, both your $x(t)$ and $y(t)$ equals to zero. But, if I divide

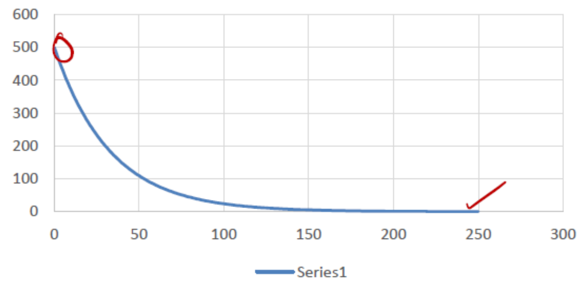
$$\frac{x(t)}{y(t)} = \sqrt{\frac{\alpha}{\beta}} = \text{constant.}$$

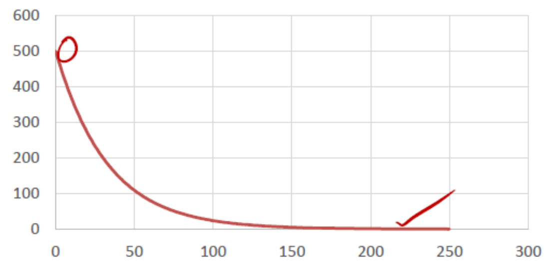
So, conclusion is, the ratio between the number of troops of the two armies in battle remains constant in time.

So, basically the kind of damage that one army produces to the other, they remain some sort of constant, if you take the ratio of both the troops, they give you a constant value, and if that is the case then and over time, it becomes zero.

So, if that is the case, it is called that this is a case of tie, this is the case of tie or draw.

So, none of the army wins and at the end, in infinite time both the armies, the number of troops dies they become zero.





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If you see the numerical it is also giving the same thing that over time it starts with 500 over time this army A goes to zero, over time this army B also goes to zero.

So this is about this conventional combat model.

In my next lecture, we will be talking about another type of combat model, which is a guerrilla type combat model.

Till then bye-bye.