EXCELing with Mathematical Modeling
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Week – 04
Lecture – 20 (Combat model-II)

Hello, welcome to the course EXCELing with Mathematical Modelling.

In this lecture, we will be talking about a different combat model, which is a guerrilla combat model or guerrilla warfare.

Just a bit recall that this combat model was introduced by F. W. Lanchester, a British engineer. He used a simple formula, simple mathematical problem, to predict the outcomes of the number of soldiers surviving in a battle during First World War.

So, his mathematical models were built on the equation that the rate of change is equal to (rate-in – rate-out), where your rate in is the number of troops that is supplied and rate out is the number of troops that cannot take part in the war, either they are dead or they are injured or suffering from something so that they cannot take part.

So, what is a guerrilla warfare or guerrilla combat?

So, unlike the conventional combat, in guerrilla combat, there is no one-to-one confrontation. The troops were deployed in small groups and they are hidden. We use the word covert.

So, often you have heard it covert operation that means the small groups deployed and they are hidden.

So, if the troops are large they are more likely to get caught because then the enemy will always notice and also if the opposing army is quite large there is possibility they may find this guerrilla troops. So, they have to be very careful, being hidden they have to kill the enemy.

So, we take $x(t)$ and $y(t)$ to be the number of troops, $x(t)$ for army A and $y(t)$ for army B.

The assumptions are the operational losses are neglected.

There is no new reinforcement initially whatever the troops present, that will be present.

And the final assumption that the combat loss rate of the guerrilla troop army is proportional to the product of the sizes of both the armies A and B.

So, unlike the conventional combat model, in this case,

$$\frac{dx}{dt} \propto xy,$$

and why it is that because both the armies are not facing each other they are in a groups and they are quite far away. So, their loss will depend on both the parties, so that is why it is depending on both the x and y. So, your differential equation in this case will be

$$\frac{dx}{dt} = -\alpha xy,$$

and

$$\frac{dy}{dt} = -\beta xy,$$

your initial values are x_0 and y_0 and your α and β they are the fighting effectiveness coefficients or the army B and army A. So this is basically for army B and this is basically for army A.

Now we have to solve this.

In this particular case, we use the technique like this. You start with

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = -\beta xy \, . \frac{1}{-\alpha xy} = \frac{\beta}{\alpha}$$

So, if I cross multiply, I can write this as

$$\alpha \, dy = \beta \, dx \implies \beta \, dx - \alpha \, dy = 0,$$

and then I integrate, this will imply

$$\beta \, x(t) - \alpha \, y(t) = \text{constant}$$

And at $t = 0$, you have

$$\beta \, x(0) - \alpha \, y(0) = \text{constant},$$

which implies

$$\text{constant} = \beta \, x(0) - \alpha \, y(0) = \beta x_0 - \alpha y_0.$$

So, you will substitute this here and you get a relation between x and y. So, if I do that I will get

$$\beta x(t) - \alpha y(t) = \beta x_0 - \alpha y_0$$

and this will give me the value of

$$y = \frac{\beta x - (\beta x_0 - \alpha y_0)}{\alpha}.$$

Now you take any one of the equation either

$$\frac{dx}{dt} = -\alpha xy$$

and substitute this y, which will be

$$\frac{dx}{dt} = -\alpha x \frac{\beta x - (\beta x_0 - \alpha y_0)}{\alpha} = -\beta x^2 + (\beta x_0 - \alpha y_0)x$$

Now you have to remember this technique you divide both side by x^2. So this will be

$$\frac{1}{x^2}\frac{dx}{dt} = -\beta + \frac{(\beta x_0 - \alpha y_0)}{x}$$

And here you substitute $z = \frac{1}{x}$, then you get

$$\frac{dz}{dt} = -\frac{1}{x^2}\frac{dx}{dt}.$$

So basically this whole thing is replaced by $\frac{dz}{dt}$, so you will get

$$-\frac{dz}{dt} = -\beta + (\beta x_0 - \alpha y_0)\, z \implies \frac{dz}{dt} + (\beta x_0 - \alpha y_0)\, z = \beta.$$

So, this is a first order linear differential equation and I have to use the technique for finding the integrating factor, which is

$$I.F. = e^{\int (\beta x_0 - \alpha y_0) dt} = e^{(\beta x_0 - \alpha y_0)t}.$$

So, you multiply this equation by the integrating factor and you get

$$\frac{d}{dt}\left[z e^{(\beta x_0 - \alpha y_0)t}\right] = \beta e^{(\beta x_0 - \alpha y_0)t}$$

So, you have to be familiar with solving a first order linear differential equation, which I am sure you do and hence I think this particular step will be understandable to you. So,

If you integrate, you will get

$$z\, e^{(\beta x_0 - \alpha y_0)t} = \frac{\beta}{(\beta x_0 - \alpha y_0)} e^{(\beta x_0 - \alpha y_0)t} + \text{constant}.$$

Now, at $t = 0$, you will have $z(0) = \frac{1}{x(0)} = \frac{1}{x_0}$

So, if you substitute that you will get

$$\frac{1}{x_0} = \frac{\beta}{(\beta x_0 - \alpha y_0)} + \text{constant}$$

So, from here you find the value of the constant and substitute it here, and if you simplify you will get your

$$x(t) = \frac{x_0(\beta x_0 - \alpha y_0)}{\beta x_0 - \alpha y_0 e^{-(\beta x_0 - \alpha y_0)t}}.$$

So, in the similar manner you can calculate this $y(t)$ also and your solution will be like this

$$y(t) = \frac{(\beta x_0 - \alpha y_0) y_0}{-\alpha y_0 + \beta x_0 e^{(\beta x_0 - \alpha y_0)t}}.$$

Now, we come to three cases. So, first thing is this particular quantity $\beta x_0 - \alpha y_0$ when it is positive.

So, if $\beta x_0 - \alpha y_0 > 0$, what you see from here is

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{x_0(\beta x_0 - \alpha y_0)}{\beta x_0 - \alpha y_0 e^{-(\beta x_0 - \alpha y_0)t}} = \frac{x_0(\beta x_0 - \alpha y_0)}{\beta x_0} = \frac{(\beta x_0 - \alpha y_0)}{\beta}, \quad t \geq 0$$

And

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{(\beta x_0 - \alpha y_0) y_0}{-\alpha y_0 + \beta x_0 \, e^{(\beta x_0 - \alpha y_0)t}} = 0,$$

because $\beta x_0 - \alpha y_0$ is positive, so, $e^{(\beta x_0 - \alpha y_0)t}$, this positive quantity becomes large, this goes to infinite and if this goes to infinite, then the whole denominator becomes large or goes to infinity and hence your $y(t)$ tends to zero. So, as t becomes large your $y(t)$ becomes zero.

So, in this guerrilla warfare army A wins.

So, all you have to do is when this is positive you find the limit for large time and you see that this gives a constant value because this is positive, this is always greater than zero.

So, you have a positive value for $x(t)$ whereas as time becomes large $y(t)$ becomes zero and hence army A wins.

Let us now look into the numerical solution for which I will be using this Microsoft Excel.
So, I already have the page open where you have this equation

$$\frac{dx}{dt} = -\alpha xy$$

and

$$\frac{dy}{dt} = -\beta xy$$

So, the numerical value which I have chosen is for x_0, that is initial value of army A is just 50 and y_0 is 200.

This is the rates and this is the basically the fighting coefficients and h is 0.1.

Now so what you will be doing is, so if I just take t here, if I take x here and if I take y here.

So once again this Excel, they work on the cells. So, I will calculate this formula for one of the cell and this will work for all the cells.

So, let us say you put this is equal to 0, this is equal to 50, initial value and this is equal to 200, initial value.

Just increase the font size to 20 and make it bold.

So, this will be equal to, so first is 0 and I add 1 to each of them. So, the increment will be 1.

So, let us calculate say around 90 of them.

So, I make this font again to some bigger size bold.

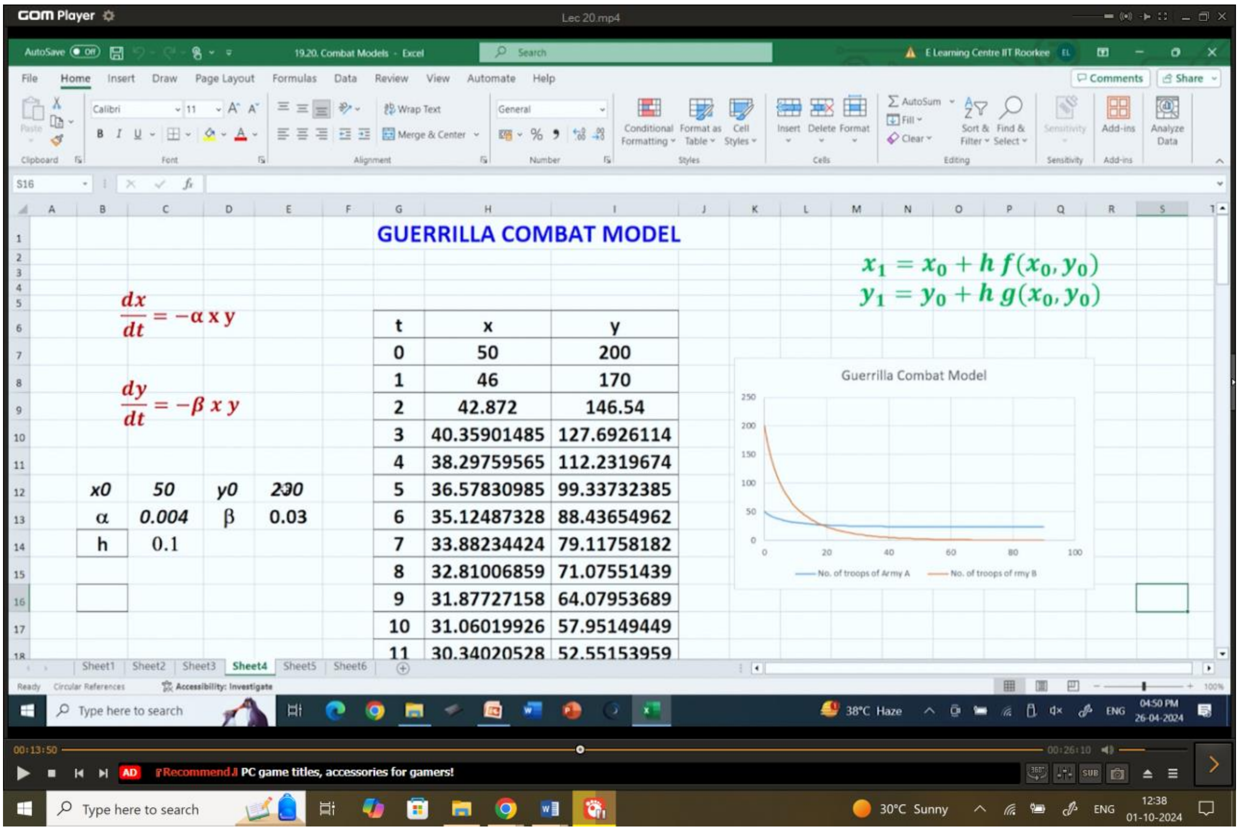
Now, if I want to calculate this one. So, I will calculate the values for one cell and drag to the next 80 or 90 values.

So, this is equal to I will be using this formula and this is my $f(x, y)$, this is my $g(x, y)$.

So, this is equal to x_0 like you can see x_0 here. So, $x_0 + h$ is 0.1, but this is a constant. So, I will put a dollar sign here to make it constant.

This is multiplied by which is a star. So, multiplied by $-\alpha$. The value of α is this, again which is a constant. This is again a constant times x value, which is 50 and y value, which is 200 bracket closed and end. so I get this value.

In the similar manner, I will calculate this is equal to y_0, this value plus h, again this is an instant multiplied by $g(x, y)$, so $(-\beta\ x\ y)$, and do not forget to make beta as a constant value. So, I put a dollar here and dollar here, enter and this gives some value.



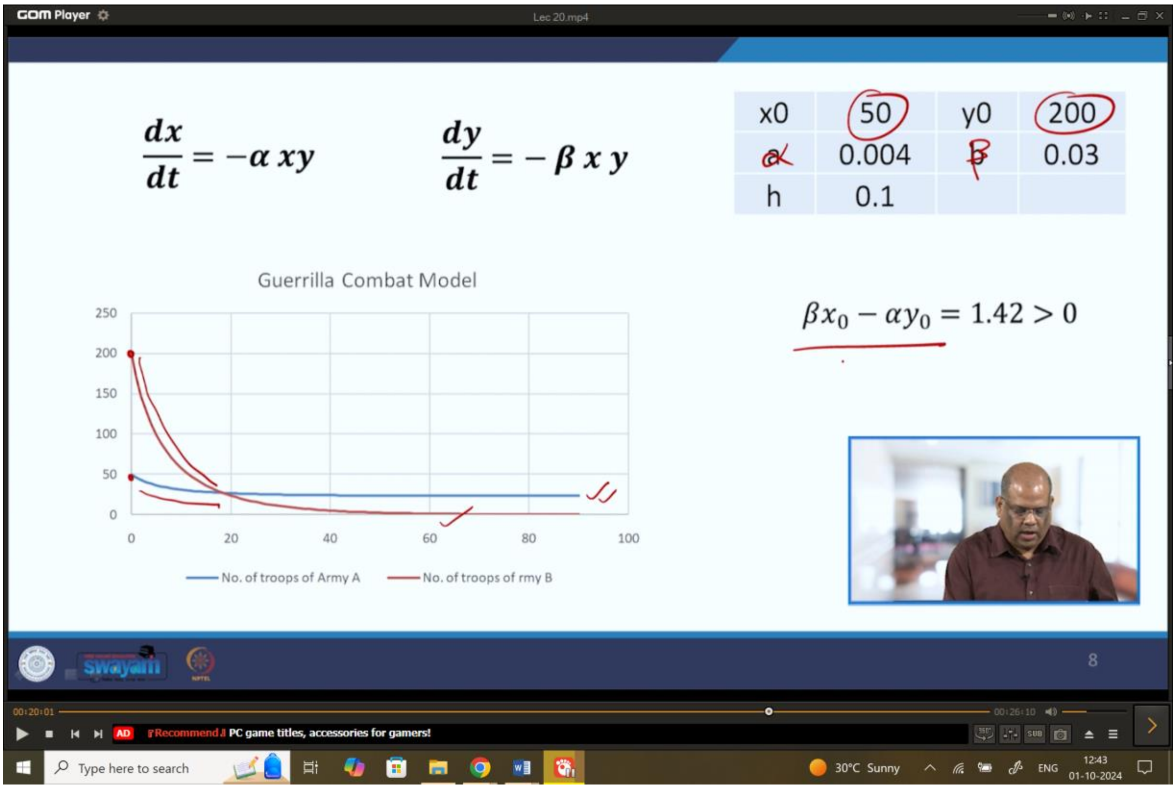
So now I just highlight this two and drag it till 80. So, I make them large.

So, if I now want to plot it, I just highlight the values and I go to insert, I go to this scattered chart and I click this one. So, you get the figure.

So, let me now go back to the slides and start explaining the numerical results.

So, I have this numerical figure here. So, again some reason this α, β is changing. So, this is your α and this is your β

So, I have my initial value for army A to be 50 which is here and for army B to be 200 which is here. I have already checked that $\beta x_0 - \alpha y_0 = 1.42 > 0$.



And you can see that initially though army B has an advantage of quite a large number of troops, but their decline is quite sharp and they are decreasing, and at one point, their number start becoming less than the army B and ultimately all of them are killed or dies.

Whereas this army B, their operation is successful and they maintain, there is a loss obviously, but they maintain some steady value.

So that is the advantage of this gorilla combat model.

So if you are efficient enough and your strategy and fighting coefficients are quite good, you can kill a large number of troops on your enemy side.

Let us take the case when your $\beta x_0 - \alpha y_0 < 0$

We have the same equations,

$$x(t) = \frac{x_0(\beta x_0 - \alpha y_0)}{\beta x_0 - \alpha y_0 e^{-(\beta x_0 - \alpha y_0)t}}, \qquad y(t) = \frac{(\beta x_0 - \alpha y_0)y_0}{-\alpha y_0 + \beta x_0 e^{(\beta x_0 - \alpha y_0)t}}.$$

So we start with

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{x_0(\beta x_0 - \alpha y_0)}{\beta x_0 - \alpha y_0 e^{-(\beta x_0 - \alpha y_0)t}} = 0.$$

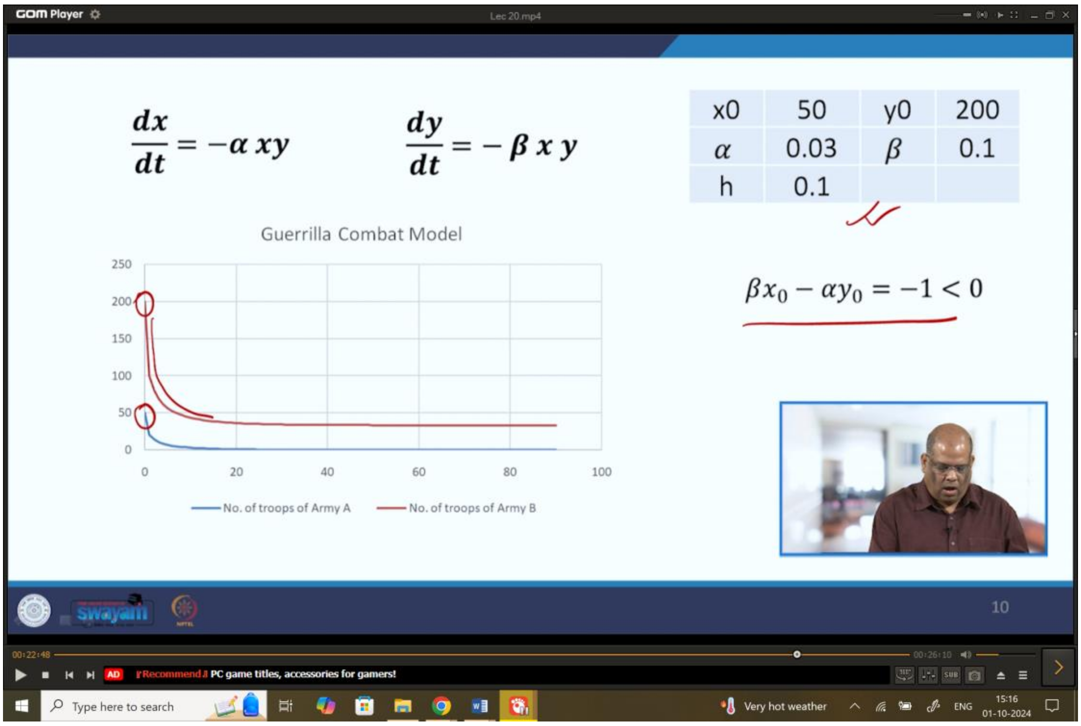
You see that if $\beta x_0 - \alpha y_0 < 0$, with the negative sign, this whole thing becomes positive. So, if this whole thing becomes positive, then $e^{-(\beta x_0 - \alpha y_0)t}$, it becomes infinite as t is infinite. Hence, this whole thing becomes infinite and your $x(t)$ is equal to 0 because this whole thing becomes large.

Whereas, when you come to the $\lim_{t\to\infty} y(t)$, in this particular case, $\beta x_0 - \alpha y_0 < 0$ and hence $e^{(\beta x_0 - \alpha y_0)t}$ goes to 0 and hence you are left with

$$\lim_{t\to\infty} y(t) = \frac{(\beta x_0 - \alpha y_0)y_0}{-\alpha y_0} = \frac{\alpha y_0 - \beta x_0}{\alpha} > 0.$$

As $\beta x_0 < \alpha y_0$ and $\alpha y_0 - \beta x_0 > 0$, so I take it to this side. So, if this quantity is positive, then this quantity is the same positive quantity and hence as a whole this is positive because $\alpha, \beta > 0$.

So, in this particular case, your $x(t) = 0$, your $y(t) > 0$, for all $t \geq 0$. So, your army B wins.



If you look into the numerical, you can see that in this particular case, the strategy which the army with less number of troops they have taken, but they have failed, they are easily located by the army of large size and then all of them get killed.

Though there is a huge loss also for army B, but at the end, they managed to win and kill all the enemies.

So, that is how you have to interpret the results by looking at the graphs and this condition has been satisfied for this set of numerical values.

Our third case is $\beta x_0 - \alpha y_0 = 0$.

So, in this particular case you do not get the actual answer when you substitute it here.

So, what you have to do is if you recall the equation

$$\frac{dx}{dt} = -\beta x^2 + (\beta x_0 - \alpha y_0)x$$

Since $\beta x_0 - \alpha y_0 = 0$, you are left with

$$\frac{dx}{dt} = -\beta x^2$$

and if you integrate this

$$\int \frac{1}{x^2} dx = \int \beta \, dt.$$

And, this implies

$$\frac{1}{x} = \beta t + \text{constant}.$$

At time $t = 0$, $x = x_0$ and your constant $= \frac{1}{x_0}$. So, if you substitute here, you get

$$\frac{1}{x} = \beta t + \frac{1}{x_0} \implies x(t) = \frac{x_0}{1 + \beta x_0 t},$$

and we have your

$$y(t) = \frac{\beta x - (\beta x_0 - \alpha y_0)}{\alpha} = \frac{\beta}{\alpha} x = \frac{\beta}{\alpha}\left(\frac{x_0}{1 + \beta x_0 t}\right)$$

So, again

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{x_0}{1 + \beta x_0 t} = 0 \quad \text{and} \quad \lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{\beta}{\alpha}\left(\frac{x_0}{1 + \beta x_0 t}\right) = 0.$$

So, both in the large time your $x(t)$ and $y(t)$ becomes zero.

If you take the ratio

$$\frac{x(t)}{y(t)} = \frac{\alpha}{\beta} = \text{constant}.$$

So, they always maintain a constant ratio, and this is the case for tie or a draw. just like the previous discussion on conventional combat model.

So, with this, we come to the end of this guerrilla war tactics.

In our next lecture, we will be looking into some more interesting model.

Till then, bye-bye.