EXCELing with Mathematical Modeling
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Week – 05
Lecture – 21 (Carbon Dating)

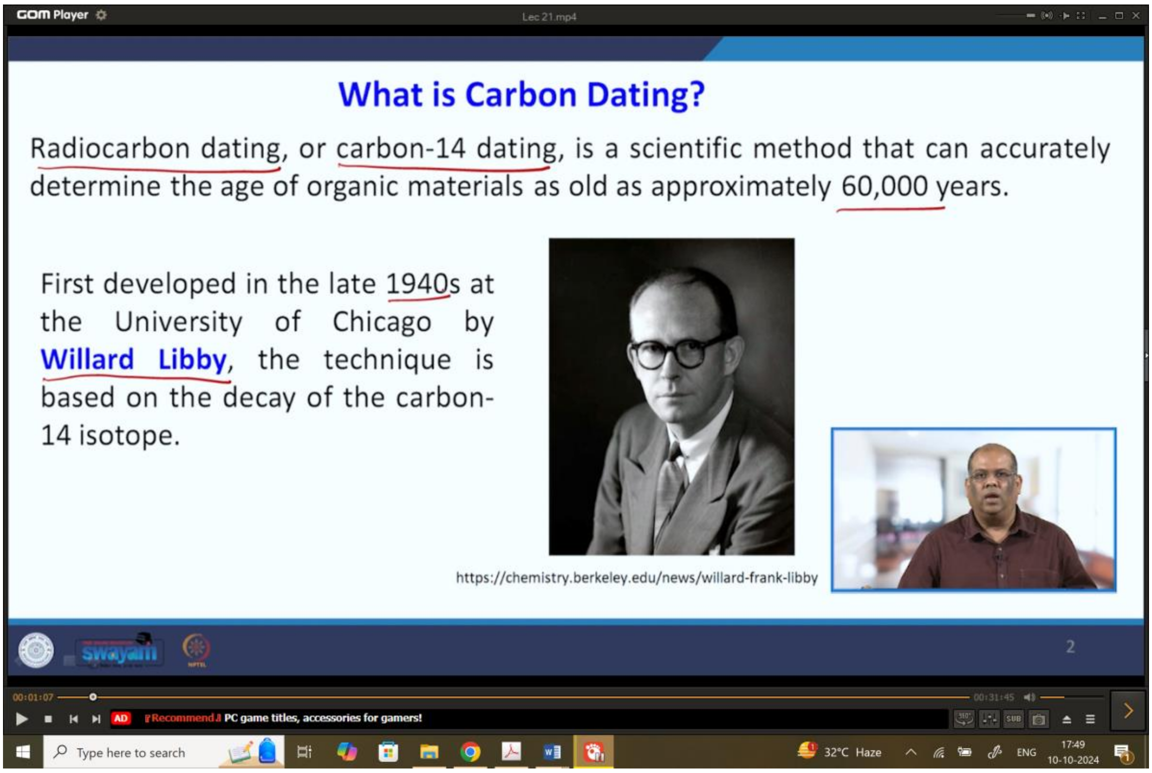
Hello, welcome to the course EXCELing with Mathematical Modelling.

Today, we will be discussing about carbon dating.

So, what is this carbon dating?

Radioactive dating or carbon dating or carbon-14 dating, it is a scientific method that can actually determine the age of an organic material or any fossil as old as approximately 60,000 years.

Now, it was first developed by Willard Libby in late 1940s at the University of Chicago and the technique is based on the decay of carbon-14 isotope and he also won Nobel Prize later on because of this discovery.



So, what is the technique?

So, what happens is that every living being or thing is made of carbon.

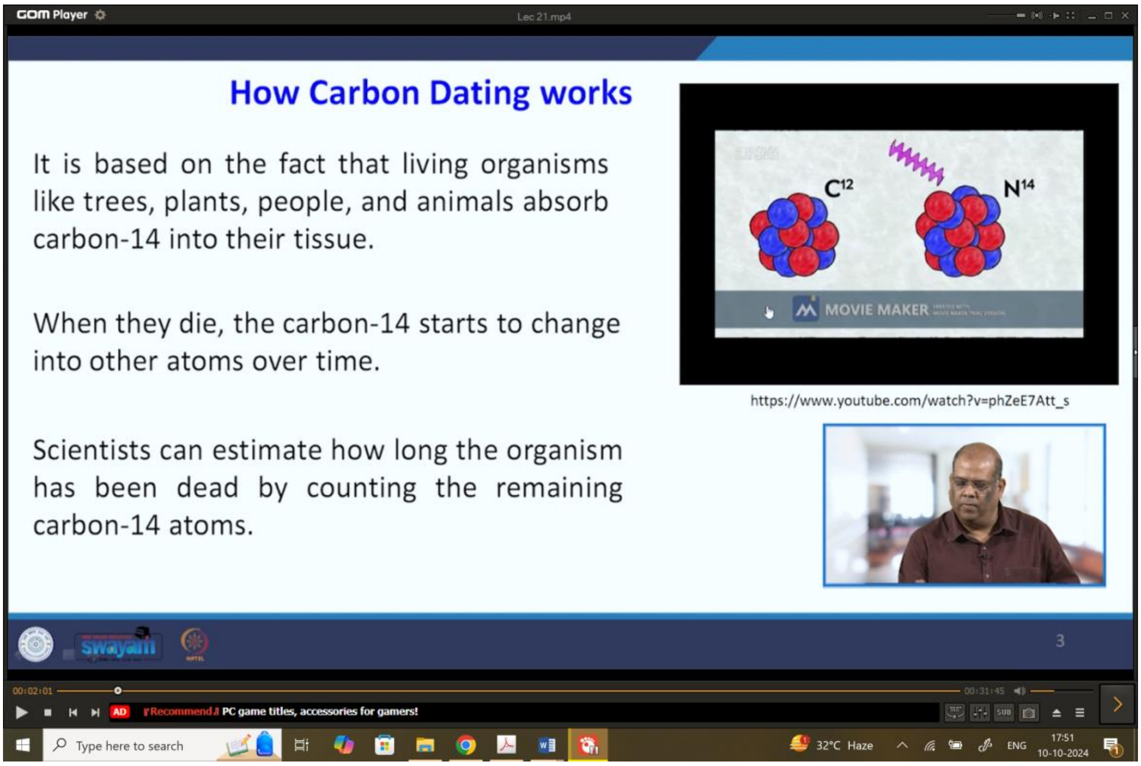
So, you look at the video, so this plant, they grab this CO2 from the outside and use it to form some complex molecules. The animals eat the plants and then they get the carbon from there. But, there is more than one form of carbon.

So, a typical carbon atom will consist of 6 protons and 6 neutrons, we call them carbon 12.

Now, high above the atmosphere this cosmic rays, they hit this nitrogen atom and create a carbon which consists of 6 protons but 8 neutrons, we call this carbon 14.

Now, both carbon 12 and carbon 14 they behave alike but there is one attribute that separates this carbon 14 from carbon 12.

Carbon 14 is unstable and when person dies or when an animal dies this carbon 14 it disintegrates and it takes approximately 5730 years to reduce this carbon-14 to half of its size and we called it the half-life of carbon-14.



So, after 5730 it reduced to half and again another 5730 it becomes one-fourth and then it becomes one-eighth.

So, scientists use this information to calculate this approximate age of the fossil or of animal, plant, trees that absorbs this carbon 14.

So, let us see that how we can use this mathematical modelling to find the approximate age of a fossil or of a relic that suddenly the archaeologists discover.

So, to start with let us say an archaeological sample has been found and whose age needs to be determined.

So we consider let A(t) is the amount of carbon-14 present in the sample at time t. So, please remember, this carbon-14 when the thing is living there is no problem but when that thing is dead then this carbon 14 starts disintegrating or leave the body.

So, the differential equation which will follow this, will be given by

$$\frac{dA}{dt} = -\lambda A,$$

and we call it a radioactive decay law. Now, this is quite easy to solve.

So, you separate the variables, you integrate both sides,

$$\int \frac{dA}{A} = -\int \lambda \ dt \ \Rightarrow \ln A = -\lambda \, t + \ln A_0$$

$$\Rightarrow \ln A - \ln A_0 = -\lambda \, t \Rightarrow \ln \frac{A}{A_0} = -\lambda \, t \Rightarrow A(t) = A_0 e^{-\lambda t}$$

So, A(t) gives you the amount of carbon 14 present in the sample at any time t and this A_0 is the initial value.
So, we can say, A_0 is A(0), which is the amount of C-14 present in the sample, when it was discovered.

Now, say if we want to calculate what is the rate of disintegration. So, we have to find this

$$M(t) = -\frac{dA}{dt} = A_0 \lambda e^{-\lambda t}, \qquad M(0) = A_0 \lambda$$

$$\frac{M(t)}{M(0)} = e^{-\lambda t} \Rightarrow t = \frac{1}{\lambda} \ln \left(\frac{M(0)}{M(t)} \right)$$

So, M(0) is the original rate of disintegration at time t equal to zeri and t gives the age of the sample provided, you can measure this M(0) and this M(t).

The same can be calculated from

$$A(t) = A_0 e^{-\lambda t} \Rightarrow \frac{A(t)}{A_0} = e^{-\lambda t} \Rightarrow t = \frac{1}{\lambda} \ln \left(\frac{A_0}{A(t)} \right)$$

So, you can calculate the approximate age of the given sample either using this or using this provided which information is available to you.

Now, let us quickly define what is half-life.

So, half-life is the amount of time required by the disintegrating or decreasing substance to reduce to half.

So, we have the equation

$$A(t) = A_0 e^{-\lambda t},$$

where your A_0 is the amount of carbon 14 that was present in the sample, when it was discovered.

So, what happens when this reduced to half? That is,

$$A(t) = \frac{A_0}{2}$$

at time, say, $t = \tau$.

So, this becomes

$$\frac{A_0}{2} = A_0 e^{-\lambda\tau}$$

where τ is your half life.

So, this will give

$$\lambda\tau = \ln(2) \Rightarrow \tau = \frac{1}{\lambda}\ln(2)$$

So, this gives the formula for half-life.

Now, let us take an example that how we calculate the age of a fossil or a sample or an archaeological sample.

So, an example, say, a fossil is found that has 20% C^{14} compared to the living sample. So, how old is the fossil and it is given half-life of carbon 14 is 5730 years.

So, you have the formula which is

$$A(t) = A_0 e^{-\lambda t} \Rightarrow \frac{A(t)}{A_0} = e^{-\lambda t}$$

and it is told that you have 20% of the carbon 14 present compared to the living sample. So,

$$A(t) = A_0 e^{-\lambda t} \Rightarrow \frac{A(t)}{A_0} = e^{-\lambda t} \Rightarrow \frac{20}{100} = 0.2 = e^{-\lambda t}$$

$$\Rightarrow e^{\lambda t} = 5 \Rightarrow \lambda t = \ln 5 \Rightarrow t = \frac{\ln(5)}{\lambda} \qquad (1)$$

Also the half-life tau is given by

$$\tau = \frac{1}{\lambda}\ln(2) \qquad (2)$$

So, from (1) and (2), we eliminate this λ and you get

$$t = \tau\frac{\ln(5)}{\ln(2)} = 5730 \times \frac{1.6094}{0.6931} = 13305 \text{ years,}$$

so that the age of the fossil is approximately found to be 13305 years old. So, that is how your mathematical model for this carbon dating works.

We now look into another example, say an archaeologist digs up a human skull at a dig site. The age of the skull is 11460 years. So, we have to find, what is the concentration of this carbon-14 of its initial concentration?

So, if the concentration of carbon-14 of its initial concentration be x, then what is the value of x?

So, our governing equation is

$$A(t) = A_0 e^{-\lambda t},$$

where A(0) is the amount of C^{14} present in the skull, when it was discovered.

So, you will get

$$e^{-\lambda t} = \frac{A(t)}{A(0)} \Rightarrow -\lambda t = \ln\left(\frac{A(t)}{A(0)}\right) \Rightarrow t = -\frac{1}{\lambda}\ln\left(\frac{A(t)}{A_0}\right)$$

Now, I have to get an expression for this lambda in terms of the half-life.

So what is this half-life? It is the amount which reduces to half of its size and we have the time τ. So, τ is the half-life in which this initial concentration reduces to half of its size and this will give me

$$\Rightarrow e^{\lambda\tau} = 2 \Rightarrow \lambda\tau = \ln(2) \Rightarrow \lambda = \frac{1}{\tau}\ln(2).$$

So, I will put this value of λ here. So, we have

$$e^{-\lambda\tau} = \frac{A(t)}{A(0)} = \frac{x}{100} \Rightarrow x = 100 \times e^{-\frac{\ln(2)}{\tau}\times t}$$

$$\Rightarrow x = 100 \times e^{-\frac{\ln(2)}{5730}\times 11460}$$

$$\Rightarrow x = 100 \times e^{-1.386} = 100 \times 0.2500 = 25\%$$

And which gives your answer to be 25%. So, the concentration of the carbon-14 to its initial concentration is 25%.

We next move to another example; a museum is testing the authenticity of Leonardo da Vinci (1452-1519) manuscript which is supposed to exist between this period.

So, this the museum sends a paper sample to a lab in 2022 and learned that it has 97.4% of its initial carbon-14 concentration.

So the question is, is the manuscript authentic?

So the museum got a manuscript claimed to be the by Leonardo da Vinci and they spent the paper for this testing and learned that 97.4% of its initial carbon 14 concentration and they want to check that whether the authenticity of this manuscript.

So, we have the governing equation

$$A(t) = A_0 e^{-t}, \qquad A(0) = A_0$$

where your A_0 and sometimes denoted by A(0), it is the amount of carbon 14 present in the sample when it was discovered, and A(t) is the amount of C^{14} present in the sample at time t. Now, here this λ is the rate of the decay.

So we calculate an expression for half-life, which is the amount of time required by the decreasing substance to reduce to half. So,

$$A(t) = A_0 e^{-\lambda t} \Rightarrow \frac{A_0}{2} = A_0 e^{-\lambda \tau}$$

$$\Rightarrow e^{\lambda \tau} = 2 \Rightarrow \lambda \tau = \ln(2) \Rightarrow \lambda = \frac{1}{\tau} \ln(2)$$

and half-life of C^{14} is 5730 years, so we can get the value of λ. Now, from the governing equation,

$$A(t) = A_0 e^{-\lambda t}$$

$$\Rightarrow e^{-\lambda t} = \frac{A(t)}{A(0)} = \frac{97.4}{100} = 0.974$$

$$\Rightarrow -\lambda t = \ln(0.974)$$

$$\Rightarrow t = -\frac{1}{\lambda} \ln(0.974) = \frac{\tau}{\ln(2)} \ln(0.974) = -\frac{5730}{0.693} \times (-0.02634) = 218 \text{ years},$$

and that is the age of the paper.

So, this paper is made, so that it went for the lab test in 2022 and if the age is 218 years, I subtract, so I get this is equal to the year 1804. So, this does not fall in this period (1452-1519) and hence the conclusion is that the manuscript, that is claimed to be of Leonardo da Vinci, is not authentic.

So, that is how you test a sample and check the authenticity of a relic or a fossil or a bone like that.

Let us take another example, say this Shroud of Turin.

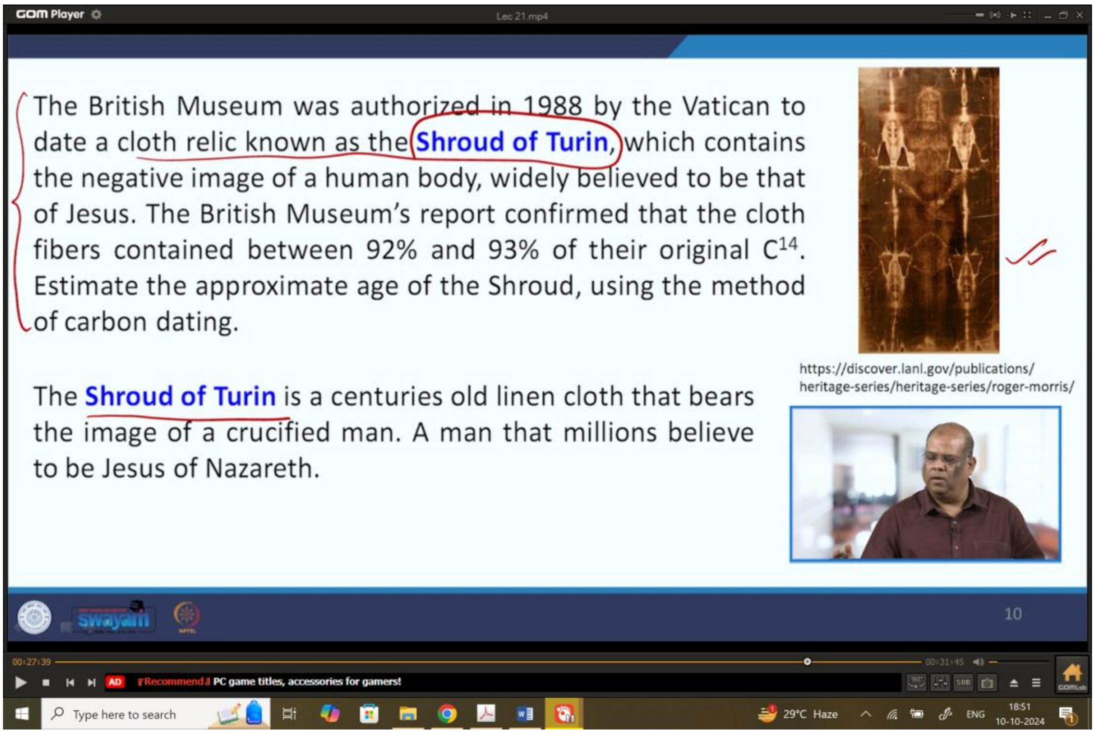
So, the problem is that the British Museum was authorized in 1988 by the Vatican to find the approximate date of this relic known as Shroud of Turin.

So, if you can see this figure, so it is believed that when Jesus Christ was crucified, this was the only cloth that covers his body and there is a mysterious image that has appeared on that cloth.

So that is why it is quite important and the Vatican wants to find what is the approximate age of that cloth.

And the information they have given that the British Museum, the reports confirm that the cloth fibre contains 92% and 93% of their original carbon 14.

So, what will be the approximate age of this shroud of touring?



So, that was the challenge that was placed before them and they use this method of carbon dating.

So, if we do that, so we have this half-life

$$\lambda = \frac{1}{\tau}\ln(2)$$

So, if I put the value of half-life of carbon-14, it is 5730 and this is approximately will be equal to 0.000121.

So, I get

$$A(t) = A_0 e^{-0.000121t} \Rightarrow t = -\frac{1}{0.000121}\ln\left(\frac{A(t)}{A_0}\right)$$

So, this is now my formula.

So, in that formula, I have the information that contains 92 and 93 percent.

So, they give you an approximate value.

So, what is my t_1?

So,

$$t_1 = -\frac{1}{0.000121}\ln(0.92) \approx 689 \text{ years}$$

$$t_2 = -\frac{1}{0.000121}\ln(0.93) \approx 600 \text{ years}$$

So, they were authorized in 1988.

So, if I write, 1988 – 689 = 1299 years and 1988 – 600 = 1380 years.

So, they give that, okay the shroud of Turin, it is between 1299 AD to 1388 AD.

So, that is how we calculate using mathematical model and this technique of carbon dating to approximately find the age of a particular fossil or relic, in this case, it was Shroud of Turin.

So, with that we come to the end of this lecture about the mathematical model of this carbon dating.

In the next lecture, we will take up some very interesting models.

Till then bye-bye.