EXCELing with Mathematical Modeling
Prof. Sandip Banerjee
Department of Mathematics
Indian Institute of Technology Roorkee (IITR)
Week – 05
Lecture – 24 (Rectilinear motion under variable force)

Hello, welcome to the course EXCELing with Mathematical Modeling.

Today we will be discussing about the rectilinear motion under variable forces.

Now in physics you have already learned that this rectilinear motion which means motion in a straight line.

And we know from Newton's law that force is equal to mass × acceleration.

And for this acceleration which is rate of change of velocity, we have the expression

$$\frac{dv}{dt} = \frac{dv}{dx}.\frac{dx}{dt}$$

This is rate of change of distance which is again velocity.

So this becomes

$$\frac{dv}{dt} = \frac{dv}{dx}.\frac{dx}{dt} = v\frac{dv}{dx}$$

or again from here I can write

$$\frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$$

So we have three expressions for this acceleration.

One is this $\frac{dv}{dt}$, one is this $v\frac{dv}{dx}$ and another is $\frac{d^2x}{dt^2}$.

Now let us take that a particle moves in a straight line with a constant acceleration F. So, if it moves in a straight line with a constant acceleration F, we have force is equal to mass into acceleration.
So, we write mass × acceleration, that is, m × f. The mass cancels and you get

$$m\frac{dv}{dt} = mf \Rightarrow \int dv = \int f\, dt$$

$$\Rightarrow v = ft + \text{constant}$$

And if you take initial condition, initially at time t equal to zero, the initial velocity is u, this implies your constant is equal to u and you get the formula

$$v = u + ft$$

This you have already done in physics.

Now, the question is here it is a constant acceleration f. What will happen if your acceleration now varies?

So, it can be a function of f, it can be a function of v and then how will you solve that kind of problem with the help of mathematical modelling.

Well in physical problems, we have some sort of laws, which we have to follow in this case this Newton's law which says force equal to mass into acceleration.

So let us take the change in law now we assume that let

$$f \propto \frac{1}{x}$$

f is the acceleration now it is not constant it varies as 1/x , and then your force is going to be mass × acceleration.

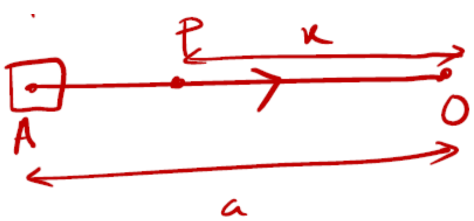
So, mass and acceleration varies 1/x. So,

$$F = m\frac{\mu}{x}$$

where μ is the constant of proportionality. Since this is a function of x, we use $v\frac{dv}{dx}$, this is the acceleration multiplied by mass and that is equal to

$$m\left(v\frac{dv}{dx}\right) = F = m\frac{\mu}{x}.$$

So, we state the problem like this that a particle moves in a straight line with an acceleration which varies 1 by x or with a force $m\frac{\mu}{x}$ in a straight line towards a point O. So, it starts from A and it is moving towards O. Let this distance OA, let it be A and P be the position of the particle at any time t such that OP is equal to X.



Now as you can see that the particle is moving towards a point which we called origin and this is an attractive force.

So, if it is an attractive force this total force will be negative which is moving towards the origin. If it is a repulsive force that is it is moving away from this fixed point origin, then this would be positive.

So, this part you have to remember attractive force we have to take this to be negative and if it is a repulsive force we have to take this to be positive. So, our equation is

$$mv\frac{dv}{dx} = -m\frac{\mu}{x}$$

So, this is our model and we say a particle moves in a straight line and be acted upon by a force

$$F = m\frac{\mu}{x},$$

which is always directed towards a fixed point. So, this is your fixed point O, this is from where it is starting, this distance is a, this is this point at any time t and this point is your variable point x and it is moving towards P. So, this is the situation.

So, now you have to solve this differential equation and this m cancels and we get

$$\frac{v^2}{2} = -\mu \ln x + \text{constant.}$$

So the initial condition, say, initially the particle is at rest at the point A such that OA equal to a. So if the particle is at rest then the initial velocity is zero.

So, at time t = 0, v = 0 and x = a because the particle is here. If I substitute it, I will get

$$0 = -\mu \ln a + \text{constant} \Rightarrow \text{constant} = \mu \ln a$$

So I substitute the value of constant and you get

$$\frac{v^2}{2} = -\mu \ln x + \mu \ln a \Rightarrow v^2 = 2\mu \ln\left(\frac{a}{x}\right).$$

So this is your step 1 where you have calculated the velocity in terms of the distance.

So it was given the acceleration in terms of the distance you have integrated use the initial condition and calculated the velocity in terms of the distance.

So let us now find a relation between the time and the distance. So I have

$$v^2 = 2\mu \ln\left(\frac{a}{x}\right).$$

Your figure was this was your O this is the point A this is P such that this distance is A and the particle is moving towards the origin. Now when you calculate the velocity, this is

$$\left(\frac{dx}{dt}\right)^2 = 2\mu \ln\left(\frac{a}{x}\right)$$

$$\Rightarrow \frac{dx}{dt} = -\sqrt{2\mu}\sqrt{\ln\left(\frac{a}{x}\right)}$$

Now when you take the square root obviously there will be plus minus sign but then you have to see what is happening.

So you see that with respect to time whether your distance is increasing or whether your distance is decreasing.

So, particular started from this position and it is moving towards the point O. So, with time your distance is decreasing and that is why you put a negative sign and you write the distance decreases with time. If it increases, you keep it as positive.

So, now you have to integrate this differential equation and find a relation between x and t. So, we just use the separation of variables and you will get

$$\sqrt{2\mu}\int dt = -\int \frac{dx}{\sqrt{\ln\left(\frac{a}{x}\right)}}$$

So, let us ask the question how much time the particle will take to reach the origin O? So if you ask this question, let us say, t be the time, so it starts from where t = 0 and it reaches here when t = T. So, from t = 0, it is going to t = T, that capital T, I have to find.

So, while doing the integration now you can put from t = 0 to t = T and when t = 0, x = a, because this OA is x = a, and when t is at the point O, the value of the distance x = 0.

So, here

$$\sqrt{2\mu}\int_{t=0}^{t=T} dt = -\int_{x=a}^{x=0} \frac{dx}{\sqrt{\ln\left(\frac{a}{x}\right)}}$$

$$\Rightarrow \sqrt{2\mu}\,T = -\int_{x=a}^{x=0} \frac{dx}{\sqrt{\ln\left(\frac{a}{x}\right)}}$$

Substitute

$$y^2 = \ln\left(\frac{a}{x}\right) = \ln a - \ln x \;\Rightarrow 2y\,dy = -\frac{1}{x}dx$$

Again,

$$\ln\left(\frac{a}{x}\right) = y^2 \Rightarrow \frac{a}{x} = e^{y^2} \Rightarrow x = ae^{-y^2}$$

And, also when

$$x = a, \qquad y = 0$$

and

$$x \to 0, \;\; y \to \infty$$

$$\Rightarrow \sqrt{2\mu}\, T = -\int_a^0 \frac{dx}{\sqrt{\ln\left(\frac{a}{x}\right)}} = \int_0^\infty \frac{2y\, ae^{-y^2} dy}{\sqrt{y^2}} = 2a \int_0^\infty e^{-y^2}\, dy$$

The easy way of doing this is using the beta gamma function. So, in this case it will be the gamma function and if you recall the formula is

$$\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1} dx$$

So, what we do is you substitute $z = y^2$, so, $dz = 2ydy$

and

$$2dy = \frac{dz}{y} = \frac{dz}{\sqrt{z}}$$

So, I have

$$\sqrt{2\mu}\, T = 2a \int_0^\infty e^{-y^2}\, dy = a \int_{z=0}^\infty e^{-z} \frac{dz}{z^{\frac{1}{2}}} = a \int_0^\infty e^{-z}\, z^{-\frac{1}{2}}\, dz = \Gamma\left(\frac{1}{2}\right)$$

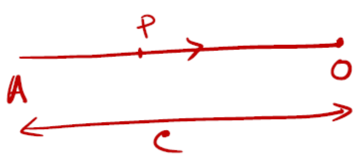
And if you are familiar with this gamma function then you know that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, so

$$\sqrt{2\mu}T = a\sqrt{\pi} \Rightarrow T = a\sqrt{\frac{\pi}{2\mu}}.$$

So, that is how you model this kind of situation, where the particle is moving in a straight line following some variable acceleration and you find the velocity and then you find the relation between the x and t and ultimately find the time that the particle will reach the origin.

Let us now take second example, only now I will change this law. So let me put the force to be $m\left(\frac{c^5}{x^2}\right)^{\frac{1}{3}}$. So the idea is taking such example is that though the force looks so complicated but as we go on with the problem and the solution, you will see that, the moment you follow the steps everything will fall into places and it will be quite easy to solve this kind of problem no matter what kind of complicated expressions are given in the force.

So, it is the same situation a particle is moving in a straight line moving towards the origin at a distance, say c, and let P be the position of the particle at any time t.



Then, our equation of motion will be

$$m\left(v\frac{dv}{dx}\right) = \text{Force} = -m\frac{c^{\frac{5}{3}}}{x^{\frac{2}{3}}}$$

$$\Rightarrow \int v\,dv = -\int \frac{c^{\frac{5}{3}}}{x^{\frac{2}{3}}}dx$$

$$\Rightarrow \frac{v^2}{2} = -c^{\frac{5}{3}}\int x^{-\frac{2}{3}}\,dx = -c^{\frac{5}{3}}\frac{x^{\frac{1}{3}}}{\frac{1}{3}} + \text{constant}$$

Initially the particle starts from rest and therefore initial condition is

$$v = 0, x = c \Rightarrow \text{constant} = 3c^{\frac{5}{3}}.c^{\frac{1}{3}} = 3c^2$$

$$\Rightarrow \frac{v^2}{2} = 3c^2 - 3c^{\frac{5}{3}}x^{\frac{1}{3}}$$

Now if you ask the question that what will be the velocity of the particle at the origin?

This distance is c, this distance is x. So at the origin, the value of x is zero.

So if you put x = 0, in this particular expression, you get

$$\frac{v^2}{2} = 3c^2 \Rightarrow v = -\sqrt{6}\,c$$

So your velocity is $\sqrt{6}\,c$ and you take this to be negative because your distance is decreasing with time.

Now you want a relation between the x and t and you integrate this. So you write

$$\left(\frac{dx}{dt}\right)^2 = 6c^{\frac{5}{3}}\left(c^{\frac{1}{3}} - x^{\frac{1}{3}}\right)$$

So, your $\frac{dx}{dt}$ is negative because your distance decreases with time. So this is going to be

$$\frac{dx}{dt} = -\sqrt{6}\left(c^{\frac{5}{3}}\right)^{\frac{1}{2}}\sqrt{c^{\frac{1}{3}} - x^{\frac{1}{3}}}$$

$$\Rightarrow \sqrt{6}c^{\frac{5}{6}}\,dT = \frac{dx}{\sqrt{c^{\frac{1}{3}} - x^{\frac{1}{3}}}}$$

So, I need to find the time from this point till it reaches the origin and let that time be capital T. So, from t = 0 to t = T and when t = 0, x = c, and when it reaches here, x = 0. So, it is from c to 0. So, we have to integrate this,

$$\sqrt{6}c^{\frac{5}{6}}\int_{t=0}^{T} dt = -\int_{c}^{0}\frac{dx}{\sqrt{c^{\frac{1}{3}}-x^{\frac{1}{3}}}} \implies \sqrt{6}c^{\frac{5}{3}}\,T = -\int_{c}^{0}\frac{dx}{\sqrt{c^{\frac{1}{3}}-x^{\frac{1}{3}}}}$$

Substitute

$$x^{\frac{1}{3}} = c^{\frac{1}{3}}\sin^2\theta \implies \frac{1}{3}x^{-\frac{2}{3}}dx = c^{\frac{1}{3}}\,2\sin\theta\cos\theta\ d\theta$$

$$\implies dx = 6c^{\frac{1}{3}}c^{\frac{2}{3}}\sin^4\theta\sin\theta\cos\theta\ d\theta \implies dx = 6\,c\sin^5\theta\cos\theta\ d\theta$$

When $x = 0,\ \theta = 0$ and $x = c,\ \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$. Therefore,

$$\sqrt{6}c^{\frac{5}{3}}\,T = -\int_{c}^{0}\frac{dx}{\sqrt{c^{\frac{1}{3}}-x^{\frac{1}{3}}}} = -\int_{\frac{\pi}{2}}^{0}\frac{6\,c\sin^5\theta\cos\theta\ d\theta}{\sqrt{c^{\frac{1}{3}}(1-\sin^2\theta)}}$$

Interchange the limit and adjust the plus sign,

$$\sqrt{6}c^{\frac{5}{3}}\,T = +\int_{0}^{\frac{\pi}{2}}\frac{6\,c\sin^5\theta\cos\theta\ d\theta}{c^{\frac{1}{6}}\cos\theta}$$

$$\implies \sqrt{6}c^{\frac{5}{3}}\,T = \int_{0}^{\frac{\pi}{2}} 6\,c^{\frac{5}{6}}\sin^5\theta\ d\theta$$

$$\implies \sqrt{6}\,T = 6\int_{0}^{\frac{\pi}{2}}\sin^5\theta\ d\theta = 6\int_{0}^{\frac{\pi}{2}}\sin^4\theta\,(\sin\theta)d\theta = 6\int_{0}^{\frac{\pi}{2}}(1-\cos^2\theta)\sin\theta\,d\theta.$$

Substitute

$$z = \cos\theta, \qquad dz = \sin\theta\,d\theta$$

When $\theta = 0,\ z = 1$, and $\theta = \frac{\pi}{2},\ z = 0$

$$\Rightarrow \sqrt{6}\,T = 6\int_{1}^{0}(1-z^2)^2 d\theta \Rightarrow T = \frac{8}{15}\sqrt{6}$$

So, the time taken by the particle from A to reach the point O is $\frac{8}{15}\sqrt{6}$.

So, with this example, we come to the conclusion to how to deal with the motion in a straight line or rectilinear motion where your acceleration is a variable quantity.

So, in our next lecture, we will be doing about this dynamics of rowing, which is again an interesting area of modeling.

Till then, bye-bye.