EXCELing with Mathematical Modeling
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Week – 05
Lecture – 25 (Dynamics of Rowing)

Hello, welcome to the course EXCELing with Mathematical Modelling.

Today we will be talking about the dynamics of rowing.

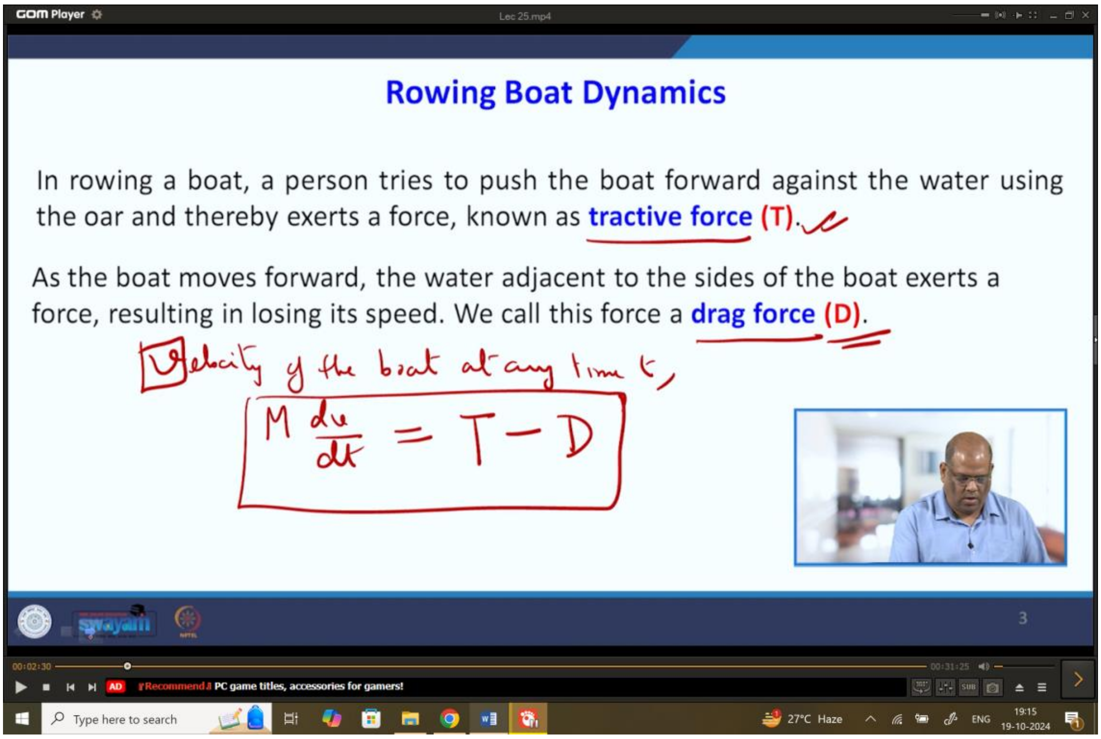
Now, if you see the video, you can see that in rowing this boat, a person has to push the boat forward against water using this oar and hereby he exerts a force which is known as tractive force. Let us denote it by T.

As the boat moves forward you will see that the water adjacent to the sides of the boat it exerts a force and resulting it in losing its speed. So, basically it will be in this direction in the opposite direction and we call that force as drag force. Let us denote it with D.

So, while forming the equation of motion, we have to take care that what is a tractive force and what is a drag force?

So, if v is the velocity of the boat at any time t, then you have mass into acceleration, this is the velocity and since we want the boat to move forward, so T must be greater than D, I mean if T is less than D, obviously it will not move forward.

So, this will be T because the boat is moving forward, its force has to be greater and there will be a drag force, -D. So, this is the equation of motion for a person rowing a boat.



Now, if we elaborate this equation, so we get

$$M\frac{dv}{dt} = T - D.$$

Let us now see on what other forces this tractive force and this drag force, they depend.

So, if we say P to be the effective power that a person can sustain for the entire length he has to row. That is, we assume that the person has entered a race and then he has to keep rowing till he reach a particular length which can be say 500 meter or 700 meter and let us the effective power that the person can sustain is denoted by P. So, you will have effective power which is equal to the tractive force multiplied by the velocity.

So, you have

$$P = T \times v$$

And from fluid dynamics, we know that this drag force it is proportional to the square of the velocity and to the surface area in contact with the water, that is, the surface area below the boat that is in contact with the water.

So, you can say that this drag force

$$D = kv^2S$$

and this S is that weighted surface area or you can just say surface area since it is in touch with the water we use the word wet.

So now this tractive force and this drag force they will be substituted here.

So we will be getting our equation of motion

$$M\frac{dv}{dt} = T - D \implies M\frac{dv}{dt} = \frac{P}{v} - kv^2S$$

$$\implies \int \frac{vdv}{P - kv^3S} = \int \frac{dt}{M} \implies \frac{1}{kS}\int \frac{vdv}{\frac{P}{kS} - v^3} = \int \frac{dt}{M}$$

$$\implies \int \frac{vdv}{a^3 - v^3} = \frac{kS}{M}\int dt$$

where

$$a^3 = \frac{P}{kS}.$$

So basically this constant is written in the form of a cube because we have a v^3 here and it just look a bit symmetric, when $a^3 = \frac{P}{kS}$.

Now this is a bit known form and we will integrate this you use partial fraction this will be

$$\int \frac{vdv}{(a-v)(a^2+av+v^2)} = \frac{kS}{M}t + \text{constant}$$

This can be written as, so you have to use this partial fraction and you can easily show that this is

$$\Rightarrow \int -\frac{dv}{3a(v-a)} + \int \frac{1}{3a}\frac{v-a}{v^2+av+a^2}dv = \frac{kS}{M}t + \text{constant}$$

If you integrate this, you are going to get this

$$\Rightarrow -\frac{1}{3a}\ln(v-a) + \frac{1}{6a}\int \frac{(2v+a)dv}{v^2+av+a^2} - \frac{3}{6}\int \frac{dv}{v^2+av+a^2} = \frac{kS}{M}t + \text{constant}$$

$$\Rightarrow -\frac{1}{3a}\ln(v-a) + \frac{1}{6a}\ln(v^2+av+a^2) - \frac{1}{2}\int \frac{dv}{v^2+2*v*\frac{a}{2}+\frac{a^2}{4}-\frac{a^2}{4}+a^2}$$
$$= \frac{kS}{M}t + \text{constant}$$

$$\Rightarrow -\frac{1}{3a}\ln(v-a) + \frac{1}{6a}\ln(v^2+av+a^2) - \frac{1}{2}\int \frac{dv}{\left(v+\frac{a}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{kS}{M}t + \text{constant}$$

$$\Rightarrow -2\ln(v-a) + \ln(v^2+av+a^2) - 3a\frac{2}{\sqrt{3}a}\tan^{-1}\frac{v+\frac{a}{2}}{\frac{\sqrt{3}}{2}a} = \frac{6kS}{M}t + \text{constant}$$

$$\Rightarrow \ln\frac{(v^2+av+a^2)}{(v-a)^2} - 2\sqrt{3}\tan^{-1}\left(\frac{2v+a}{\sqrt{3}a}\right) = \frac{6kS}{M}t + \text{constant}$$

Then, you assume that at time t = 0, the boat was at rest. and which will imply that the velocity is zero.

So, this is a relation between the velocity v and the time t and if you substitute t = 0 and v = 0, your constant is

$$\text{constant} = \ln\left(\frac{a^2}{a^2}\right) - 2\sqrt{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = -\frac{2\sqrt{3}\pi}{6} = -\frac{\pi}{\sqrt{3}}$$

So you substitute this constant here and you get

$$\Rightarrow \ln\frac{(v^2+av+a^2)}{(v-a)^2} - 2\sqrt{3}\tan^{-1}\left(\frac{2v+a}{\sqrt{3}a}\right) + \frac{\pi}{\sqrt{3}} = \frac{6kS}{M}t$$

So, this gives you a relation between the velocity and the time in the dynamics of rowing a boat.

We just now let us change the equation of motion and let us see that what alternate solution we get here. So, we start with the same thing

But now our acceleration is instead of $m\frac{dv}{dt}$, we just put $mv\frac{dv}{dx}$

$$\Longrightarrow\ M\frac{dv}{dt} = M\frac{dv}{dx} = T - D = \frac{P}{v} - kv^2S = \frac{P-kSv^3}{v}$$

$$\Longrightarrow\ v^2\frac{dv}{dx} = \frac{P-kSv^3}{v}$$

$$\Longrightarrow\ \int M\frac{v^2\,dv}{P-kSv^3} = \int dx$$

So I want a relation between the velocity and the distance.

So now the integration becomes a bit easier because now you can substitute this

$$z = P - kSv^3, \qquad dz = -3kSv^2dv$$

$$\Rightarrow -\frac{M}{3kS}\int\frac{dz}{z} = x + \text{constant}$$

$$\Rightarrow -\frac{M}{3kS}\ln z = x + \text{constant}$$

$$\Rightarrow -\frac{M}{3kS}\ln(P - kv^3S) = x + \text{constant}$$

Now at time t = 0, the boat was at rest, the velocity is zro and there is no movement, so x is also zero and this gives you the constant

$$\text{constant} = -\frac{M}{3kS}\ln P$$

So, you substitute it there and you get

$$\Rightarrow -\frac{M}{3kS}\ln(P - kv^3S) = x - \frac{M}{3kS}\ln P$$

$$\Rightarrow \frac{M}{3kS}\left[\ln(P - kv^3S) - \ln P\right] = -x$$

And this can be written as

$$\Rightarrow \ln(P - kv^3S) - \ln P = -\frac{3kS}{M}x$$

$$\Rightarrow \frac{P - kv^3S}{P} = e^{-\frac{3kS}{M}x} \quad \Rightarrow \quad 1 - \frac{kv^3S}{P} = e^{-\frac{3kS}{M}x}$$

$$\Rightarrow \frac{kv^3S}{P} = 1 - e^{-\frac{3kS}{M}x} \quad \Rightarrow \quad v^3 = \frac{P}{kS}\left(1 - e^{-\frac{3kS}{M}x}\right)$$

So, you can say

$$\Rightarrow v = \left(\frac{P}{kS}\right)^{\frac{1}{3}}\left(1 - e^{-\frac{3kS}{M}x}\right)^{\frac{1}{3}}.$$

So, this gives you a relation between the velocity and the distance.

We now look into the numerical solution of this dynamics of rowing for which I have two equations, one is

$$\frac{dv}{dt} = \frac{P - kv^3S}{Mv}$$

and the other one was

$$\frac{dv}{dx} = \frac{P - kv^3S}{Mv^2}$$

So, I will solve these two equations with initial condition v0 equal to 5

And let us see the dynamics, so this will be the dynamics between the velocity and time and this will be the dynamics between the velocity and distance.

So, I have the time and the velocity, I have the distance and the velocity.

So, I change both of them to 20, the time is 0 and this will be equal to 0 plus 1.

So, I give an increment 1 because h is 1 here and I drag them to say not much, say, 50 values.

So, at time t equal to 0, I have taken v equal to 5. Let me quickly change their format also.

So, I change them to 20. So, I am first solving this first equation.

So, this is equal to V0 plus h which is a constant multiplied by this whole expression.

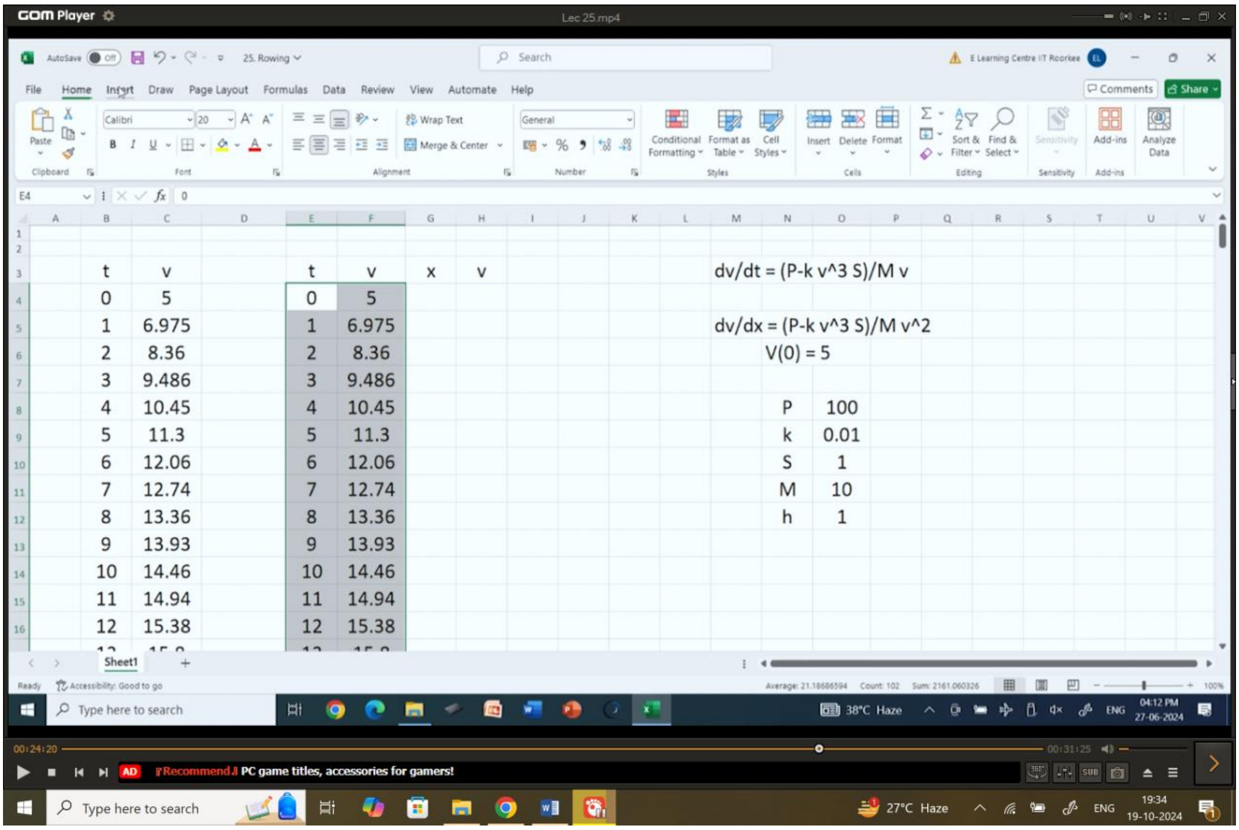
So, P which is 100 and is a minus k which is 0.01

Again a constant multiplied by v cube which is v cube bracket closed divided by m, which is 10 again a constant so I put a dollar multiplied by p bracket closed and I put the whole thing in the bracket again so this could be some value

I just drag them up to this much.

So, if I want to plot this to between the time and the velocity, I go to insert chart and this one.

So it is reaching some steady state value.



If I want to increase so up to say 100, so let me calculate say till 150 and let us see where this is going.

So, I select these two again up to 150 and then we will plot this.

So, you go to insert charts and this smooth diagram and I get this one.
So, if I want to remove the grid lines I remove them want to put the axis title so this is your time and this is your velocity

And the charge title is dynamics of rowing.

So, let us look into the other part, see the distance I can put it to 0.

So, at time t equal to 0, the distance is also 0, the velocity is initial 5.

So, this is equal to the 0 value plus 1 and I again drag it up to 150 values.

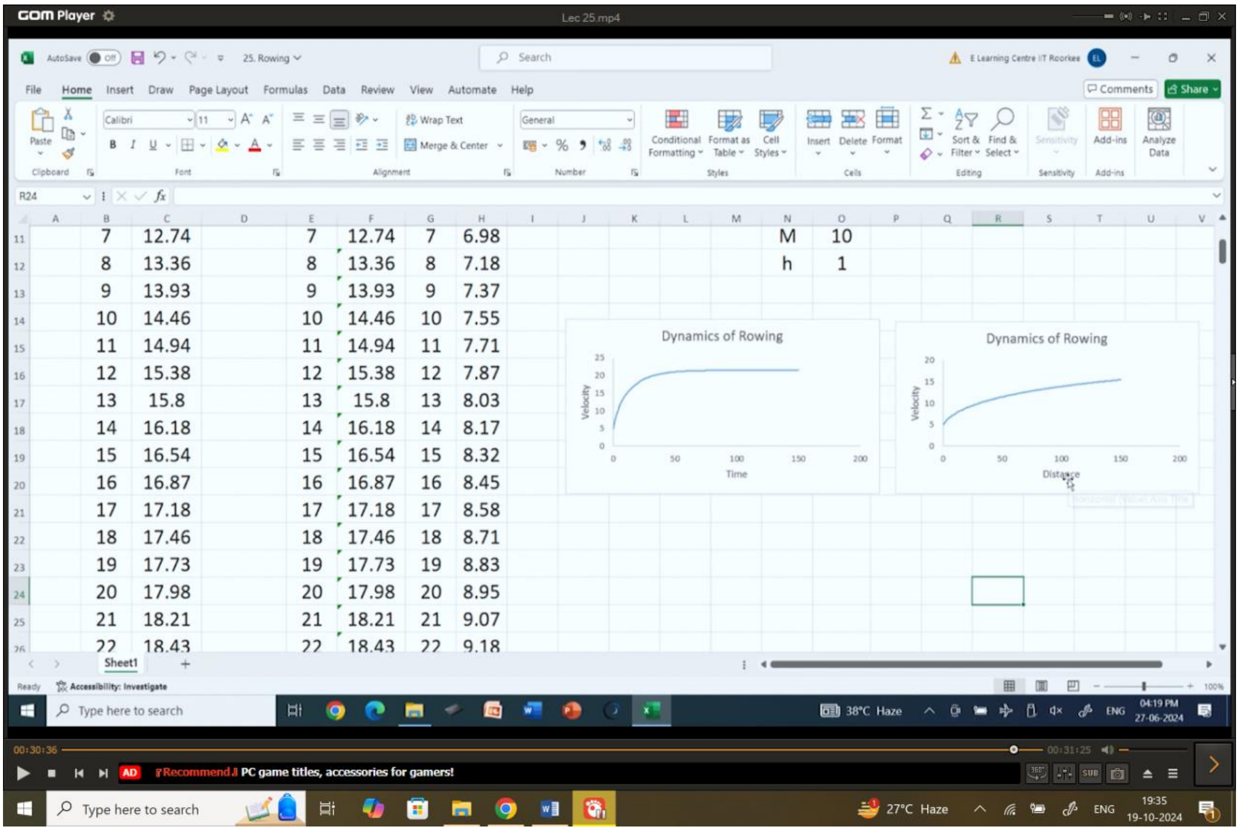
This is now equal to V0 plus h which is a constant.

So, put a dollar multiplied by P which is again a constant minus K which is again a constant multiplied by V cube multiplied by S which is again a constant divided by M which is again a constant multiplied by V square and I put a bracket here and a bracket here and I get a value so I drag it again to 150 values and if I plot them

So I go to insert, choose the chart and this one.

So I remove the grid lines, I want to add axis title, so this is your distance and this is the velocity and this remains the same dynamics of rowing.

If I make the graph a little smaller, I can bring them side by side and I can see the dynamics this is the velocity of the rowing with respect to time and this is the velocity with respect to the distance.



So, this gives you the dynamics of rowing a boat in our next lecture we will be talking oscillations mainly the vertical and the horizontal oscillations.

Till then bye-bye.