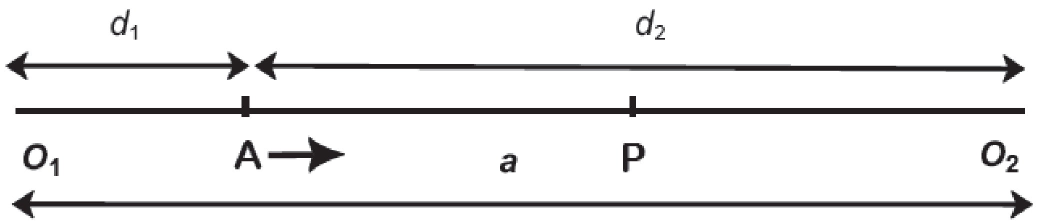
EXCELing with Mathematical Modeling
Prof. Sandip Banerjee
Department of Mathematics
Indian Institute of Technology Roorkee (IITR)
Week – 06
Lecture – 26 (Horizontal Oscillations)

Hello welcome to the course EXCELing with Mathematical Modeling.

Today we will be talking about horizontal oscillations and how to model those kind of oscillations.

So, if you consider this string and the mass attached to it, as you can see that the surface is we assume to be frictionless and there is an oscillation of this particular object moving to and fro.

So, to model this kind of phenomena, so we say that a particle of mass m say it rests at this point A and there are two forces which is acting on it.



One is this force O_1 origin, another is force O_2, the origin is O_2 and the law of force is

$$m\,\mu_1{}^n (distance)^n$$

towards this force O_1 and

$$m\,\mu_2{}^n (distance)^n$$

towards the fixed centre O_2.

So, at the point A both the forces are acting and the particle is in equilibrium. So, the forces are equal and opposite. So, as you can see that this O_1, A and this O_2 they are in a straight line and hence they are called collinear.

Now a small push is given to the right hand side. So, it disturbs the equilibrium we assume that it is a stable equilibrium. So, if it is given a small push to the right, the tendency will be to maintain the equilibrium the stable equilibrium that it will go back to its original position.

So, we assume that let after the push let P be the position of the particle at any time t and the tendency of this particle is to move back towards the point A. So, in such a scenario, let us see what happens.

So, we assume that this $O_1O_2 = a, O_1A = d_1, AO_2 = d_2$.

So, basically if I add $d_1 + d_2 = a$ and since this force the at the point A, forces are equal and opposite because the particle is in equilibrium.

So, both the forces are keeping this particle at this point A in equilibrium and therefore the law

$$\mu_1{}^n (d_1)^n.$$

So the distance from here is d_1 and from here is d_2. So that must be equal to

$$\mu_2{}^n (d_2)^n$$

If I simplify this a bit, this will be

$$\mu_1{}^n (d_1)^n = \mu_2{}^n (d_2)^n \quad \Rightarrow \mu_1 d_1 = \mu_2 d_2$$

$$\Rightarrow \frac{\mu_1}{d_2} = \frac{\mu_2}{d_1} = \frac{\mu_1 + \mu_2}{d_1 + d_2} = \frac{\mu_1 + \mu_2}{a}$$

$$\Rightarrow d_2 = \frac{a\mu_1}{\mu_1 + \mu_2} \text{ and } d_1 = \frac{a\mu_2}{\mu_1 + \mu_2}$$

So, I get the distance in terms of $a, \ \mu_1, \ and \ \mu_2$.

Now let us consider the particle is given a small push towards O_2 and let P be the position at any time t. So, the particle is slightly displaced from the equilibrium position A to the position P and it will be it will have a tendency to move towards the point A and let us take this AP to be some x. So your equation of motion, the left hand side, is given by $m \frac{d^2x}{d\,t^2}$.

Now there are two forces, one is acting at the point O_1, another is acting at the point O_2.

So if the particle is at the point P, its tendency will be to move towards this O_1 so that we assume that the equilibrium is stable.

So if it moves towards this O_1 then this O_1 becomes an attractive force and this O_2 becomes a repulsive force.

So here is what you have to understand that, okay here is your point A and you are giving a push in this side. This is your O_2 and this is your O_1.

If the particle is not in equilibrium, then it is going towards this direction and then your O_2 would have been the point of attraction and O_1 the point of repulsion.

But in this case at this point A, the particle is in equilibrium and we assume it is a stable equilibrium.

So, if you now give a small push and by the differentiation of stable equilibrium this will tend to come back to its original position A and in doing so, it is its actually motion will be like this and if it is actual motion is like this after the push, then O_1 becomes the point of attraction and O_2 becomes the point of repulse.

Now if O_1 is the point of attraction, so let us see what is the force acting on this. So, by the definition it is given that it is

$$m\,{\mu_1}^n (O_1P)^n.$$

And for the O_2 it is

$$m\,{\mu_2}^n (O_2P)^n.$$

Now which one will be positive and which one will be negative.

So as I have explained that it is moving towards O_1 after the push to maintain the equilibrium O_1 becomes the point of attraction and if the force is attractive, you have a negative sign and O_2 becomes the pulse of repulsion and if the force is repulsive, we have a positive sign. So that is how your sign is determined.

So, equation of motion is

$$m\frac{d^2x}{d\,t^2} = -\,m\,{\mu_1}^n (O_1P)^n + m\,{\mu_2}^n (O_2P)^n$$

$$\Rightarrow\ m\frac{d^2x}{d\,t^2} = -\,m\,{\mu_1}^n (d_1 + x)^n + m\,{\mu_2}^n (d_2 - x)^n$$

$$\Rightarrow\ \frac{d^2x}{d\,t^2} = -\ {\mu_1}^n (d_1 + x)^n +\ {\mu_2}^n (d_2 - x)^n$$

$$\Rightarrow\ \frac{d^2x}{d\,t^2} = -\ {\mu_1}^n {d_1}^n \left(1 + \frac{x}{d_1}\right)^n +\ {\mu_2}^n {d_2}^n \left(1 - \frac{x}{d_2}\right)^n$$

The next step is we expand them binomially so

$$\frac{d^2x}{d\,t^2} = -\ {\mu_1}^n {d_1}^n \left(1 + \frac{nx}{d_1} + \frac{n(n-1)}{2!}\frac{x^2}{{d_1}^2} + \cdots\right) +\ {\mu_2}^n {d_2}^n \left(1 - \frac{nx}{d_2} + \frac{n(n-1)}{2!}\frac{x^2}{{d_2}^2} + \cdots\right)$$

And, since x is small, this will be approximately equal to, so we neglect the square and the higher terms, and you will be getting

$$\frac{d^2x}{d\,t^2} = -\ {\mu_1}^n {d_1}^n \left(1 + \frac{nx}{d_1}\right) +\ {\mu_2}^n {d_2}^n \left(1 - \frac{nx}{d_2}\right),$$

x^2 and higher powers of x are neglected because x is small. So, if we simplify this

$$\frac{d^2x}{d\,t^2} = -\ {\mu_1}^n {d_1}^n - nx\ {\mu_1}^n {d_1}^{n-1} +\ {\mu_2}^n {d_2}^n - nx{\mu_2}^n {d_2}^{n-1}$$

Now in the equilibrium position ${\mu_1}^n {d_1}^n = {\mu_2}^n {d_2}^n$ and we are left with

$$\frac{d^2x}{d\,t^2} = -\,{\mu_1}^n \left(\frac{a\mu_1}{\mu_1 + \mu_2}\right)^{n-1} nx - {\mu_2}^n \left(\frac{a\mu_1}{\mu_1 + \mu_2}\right)^{n-1} nx.$$

So, what I did is, I substitute the value of d_1 and I substitute the value of d_2, which we have calculated here.

And if I simplify this, I will be getting

$$\frac{d^2x}{d\,t^2} = -\left(\frac{a\mu_1\mu_2}{\mu_1 + \mu_2}\right)^{n-1} x$$

So, this equation is of the form

$$\frac{d^2x}{d\,t^2} = -\lambda^2\, x$$

and this is a known form equation which gives you a simple harmonic motion.

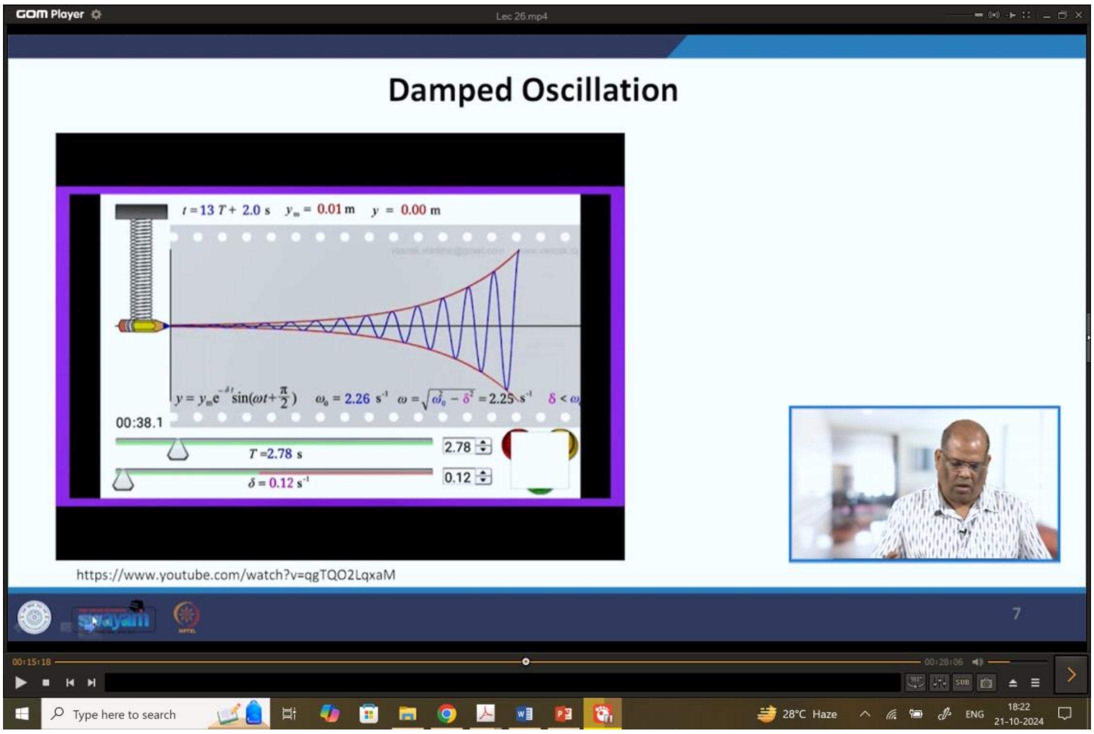
So, the motion of the particle in this case is simple harmonic and the period of oscillation will be

$$\frac{2\pi}{\sqrt{\lambda^2}} = \frac{2\pi}{\lambda}, \text{where} \quad \lambda = \sqrt{\left(\frac{a\mu_1\mu_2}{\mu_1 + \mu_2}\right)^{n-1}}.$$

So in case of this horizontal oscillation, we see that it gives to a simple harmonic motion with period $\frac{2\pi}{\lambda}$, where λ is given by square root of this quantity.

Now let us consider a case where there is a damped oscillation. By damped oscillation means there will be some force which will be opposing this motion. So, let us see what happens.

So, if you consider a damped oscillation in this particular video you will see that this spring is having an oscillation though in this case it is a vertical one, but some force is acting on it and you can see that the curve, it is though giving a cycle but that cycle is slowly decreasing with respect to time because there is a force which is working against this motion it is not frictionless at all and slowly that curve decreases and ultimately it will go to zero.



So this is what happens during this damped oscillation.

So that damped oscillation law can be anything so you can take that your original equation was

$$\frac{d^2x}{d\,t^2} = -\,\mu^2 x$$

anyway it was λ but let us take μ, really does not matter, and let us there be a force which varies as the velocity.

So, if there is a force which is acting against it and varies as the velocity, so we can put it some

$$\frac{d^2x}{d\,t^2} = -\,\mu^2 x - k\frac{dx}{dt}$$

To make it a bit convenient for our calculation, I can put 2k instead of k and we get

$$\frac{d^2x}{d\,t^2} = -\,\mu^2 x - 2\,k\frac{dx}{dt}$$

If I want an additional force, other than this damping effect, I add up a periodic acceleration of the form F $\cos(bt)$.

So, in this particular case we are adding two things, one is this damping force, another is this additional periodic force.

So, basically if your string or your spring is like this and it is moving to and fro.

So, what is happening is as it moves this side there will be a force which will oppose and it varies as the velocity. So, this is 2kv, plus there is an additional disturbing force which is of the form a periodic one F $\cos(bt)$.

So, what will happen if we consider such case?

So, in that case your equation of motion will be some

$$\frac{d^2x}{d\,t^2} = -\,\mu^2 x - 2k\frac{dx}{dt} + \mathrm{F}\cos(bt)$$

$$\Longrightarrow (D^2 + 2kD + \mu^2)x = \mathrm{F}\cos(bt)$$

So, this is a second order differential equation where your operator

$$D \equiv \frac{d}{dt}$$

So, if you are familiar with this kind of equation which you should be, then you know that we take the solution of the form

$$x = Ae^{mt}$$

and we have two parts of the solution as a general solution.

First is the complementary function and another is the particular integral.

So for the complementary function, we first take

$$x = Ae^{mt}$$

be the trial solution and then your

$$D \equiv \frac{dx}{dt} = Ae^{mt}, \frac{d^2x}{d\,t^2} = A\,m^2 e^{mt}$$

You substitute both of them here, you get

$$m^2 + 2km + \mu^2 = 0$$

as your auxiliary equation.

So you solve for m which will give you

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -k \pm i\sqrt{\mu^2 - k^2}$$

I assume that $\mu > k$

So, your complementary function will be

$$C.F. = e^{-kt} A_1 \cos\left(\sqrt{\mu^2 - k^2}\, t\right) + \epsilon_1$$

If you are not familiar with the second order differential equation, you just have to go through it, this is quite I mean a simple solution. So, you need to know how to solve this kind of differential equation.

So, after this we look for the so this is the complementary function and then you have to look for the particular integral and to do that you have to write x_p so for the particular integral

This is

$$x_p = \frac{1}{(D^2 + 2kD + \mu^2} \mathrm{F}\cos(bt)$$

This I write it as

$$\frac{D^2 + \mu^2 - 2kD}{(D^2 + \mu^2)^2 - 4\,k^2 D^2} \mathrm{F}\cos(bt)$$

So, what I did is I have taken this as $D^2 + \mu^2 + 2kD$ and multiply both numerator and denominator by $D^2 + \mu^2 - 2kD$

So, the denominator becomes $\mathrm{A}^2 - \mathrm{B}^2$, this is the part with which we have multiplied and here it is $\mathrm{F}\cos(bt)$ and by the rule it says that you replace this D^2 by $-b^2$, and you will get

$$x_p = \frac{D^2 + \mu^2 - 2kD}{(-b^2 + \ \mu^2)^2 + 4\,k^2 b^2}\,\mathrm{F}\cos(bt)$$

So, you have replaced this D^2 by $-b^2$ and hence this becomes plus.

So this part is the constant

$$x_p = \frac{D^2 + \mu^2 - 2kD}{(\,\mu^2 - b^2)^2 + 4\,k^2 b^2}\,\mathrm{F}\cos(bt)$$

So if you multiply this F $\cos(bt)$ with each of this, what you are going to get is

$$x_p = \frac{D^2\,\mathrm{F}\cos(bt) + \mu^2\,\mathrm{F}\cos(bt) - 2kD\,\mathrm{F}\cos(bt)}{(\,\mu^2 - b^2)^2 + 4\,k^2 b^2}$$

Now D of F $\cos(bt)$ is nothing but that you have to differentiate this particular function, so this will be $-\mathrm{F}\sin(bt)$.

So, if you do that, you will get it in the form

$$x_p = \frac{(\mu^2 - b^2)\ \cos(bt) + 2kb\ \sin(bt)}{(\,\mu^2 - b^2)^2 + 4\,k^2 b^2}$$

The next thing what you have to do is, so if I write the general solution

$$x = e^{-kt}\,A_1\cos\left(\sqrt{\mu^2 - k^2}\;t\right) + \frac{(\mu^2 - b^2)\ \cos(bt)}{(\,\mu^2 - b^2)^2 + 4\,k^2 b^2} + + \frac{2kb\ \sin(bt)}{(\,\mu^2 - b^2)^2 + 4\,k^2 b^2}$$

As such this is the solution but we simplify a bit for the better understanding of the problem.

So, what you do is you put

$$\frac{(\mu^2 - b^2)}{(\,\mu^2 - b^2)^2 + 4\,k^2 b^2} = \cos\epsilon_2 \quad \text{and} \quad \frac{2kb}{(\,\mu^2 - b^2)^2 + 4\,k^2 b^2} = \sin\epsilon_2.$$

So, they come in a formula and you will get this as

$$x = e^{-kt}\,A_1\cos\left(\sqrt{\mu^2 - k^2}\;t\right) + \cos\epsilon_2\ \cos(bt)\ +\ \sin(bt)\ \sin\epsilon_2$$

$$\Rightarrow\ x = e^{-kt}\,A_1\cos\left(\sqrt{\mu^2 - k^2}\;t\right) +\ \cos(bt - \epsilon_2)$$

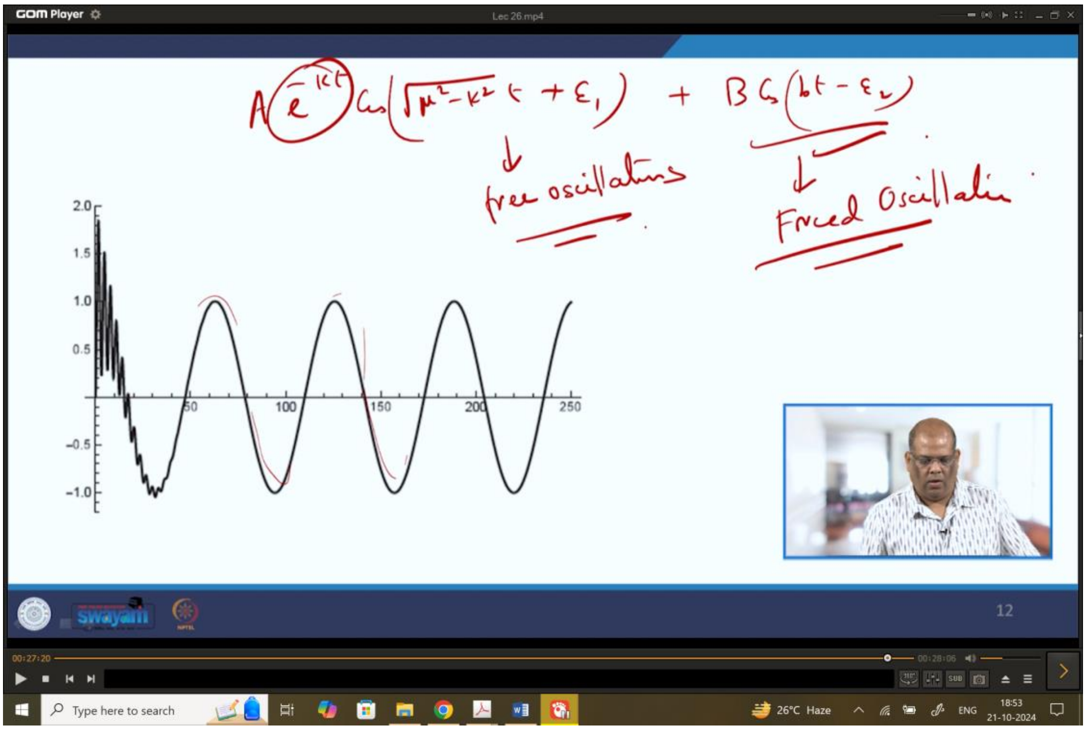
where

$$\tan \epsilon_2 = \frac{2kb}{\mu^2 - b^2} \implies \epsilon_2 = \tan^{-1} \frac{2kb}{\mu^2 - b^2}$$

So, now what can you tell about the solution?

So, we have a damping force and an additional periodic force disturbing the motion.

So, from here you can see that this will give to a damping oscillation, that is, it will start with a normal oscillation and slowly it will die out, and then this is going to give another oscillation.



So, if you plot this solution you will get it like this that it has started with an oscillation and because of that damping factor

$$e^{-kt} A_1 \cos\left(\sqrt{\mu^2 - k^2}\, t + \epsilon_1\right)$$

because of this e^{-kt} this is going to die out. But there is a factor

$$e^{-kt} A_1 \cos\left(\sqrt{\mu^2 - k^2}\, t + \epsilon_1\right) + \mathrm{B} \cos(bt - \epsilon_2)$$

and due to this factor again it is going to give you this periodic solution.

So, this is called the free oscillation but a damped one and this is called the forced oscillation.

So with this, we come to the end of this lecture about this horizontal oscillation and how you can model some particular situations.

In our next lecture, we will be taking up the vertical oscillations. Till then, bye bye.