EXCELing with Mathematical Modeling
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Week – 06
Lecture – 28 (Epidemic Model 1)

Hello, welcome to the course EXCELing with Mathematical Modeling.

Today we will be discussing about epidemic models.

So what do you mean by epidemic?

So an epidemic is defined as an unexpected increase in the number of disease cases in a specific geographical area.

That is an unusually large short term disease outbreak.

For example, you have seen this COVID-19. So that is an epidemic.

However, we generally use this word pandemic.

So, the difference between this pandemic is that the outbreak will be called a pandemic when the disease growth is exponential.

That means it is very, very high, very, very fast.

So as we have seen in the case of COVID-19, the infectious rate is very, very high.

It was just catching up exponentially.

So that's why we use the word pandemic before this COVID-19.

There is another word which we use, which is called endemic.

So if this disease is consistently present, but limited to a particular region.

For example, we can take this malaria, we can take cholera, we can take sleeping sickness.

So, these are endemic disease where in many parts of the world.

So, the difference between this pandemic and endemic is that this pandemic the growth will be exponential and in endemic it is present consistently, but limited to a particular region.

How do you model this kind of endemic diseases?

So, and what is the purpose of this endemic models?

So, basically endemic models they are primarily designed to explain and predict the spread of this infectious disease whatever the disease may be and the little history is that the first model not the first the first compartmental model it was proposed by W.H. Hamer in 1906 and he proposed the model where the spread of infection it depends on the number of susceptible individuals.

So, what do you mean by susceptible it is that the persons who are vulnerable to the disease. So, those are called susceptible individuals.

And the number of infective individuals, that is, the person who have already got the disease and they are called the infective individuals.

So, by compartmental model we mean that there are classes, like this class is called the susceptible class, this class is called the infected class and he suggested this law of mass action for the rate of new infections and this idea is the basic of this compartmental models.

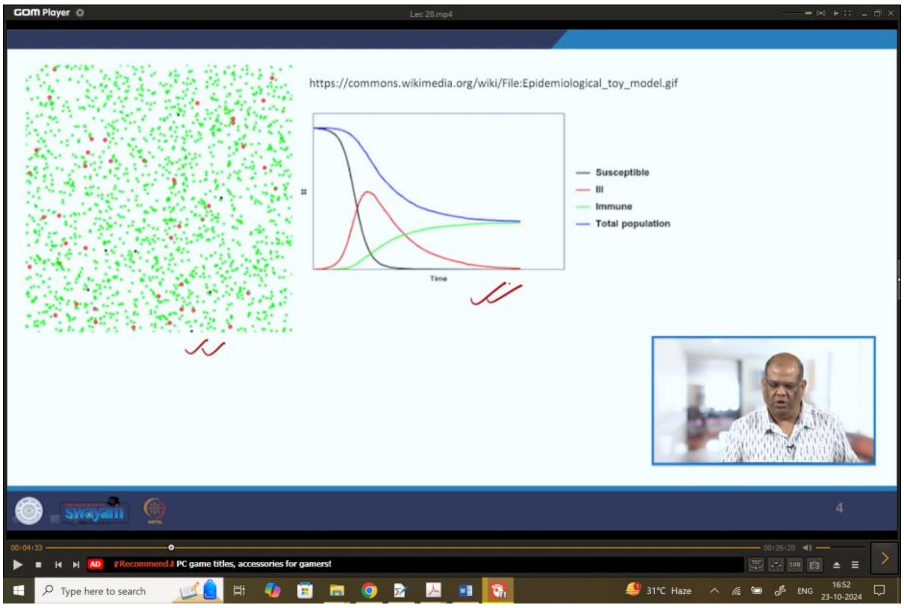
So, we will be looking up into some compartmental models.

The origin of such models is in the early 20th century.

The names famous for this model are Ross in 1916, Ross and Hudson in 1917, Kermack and McKendrick in 1927, Kendal in 1956.

So, let us start with a very simple kind of model that is the susceptible and infected model.

But before that, let us have a look at this particular graphics.



The left hand panel which you see here, the population is getting infected. And the right hand side, it gives you the graph that how this population is behaving. So it will start with some black spots like this. So these are the susceptible one and they are getting infected very fast, which is the red one. And here you can see that the susceptible is coming down. The infectious is growing up, and then this green one is the recovered one and this is the total population that has been infected.

So, this is what the mathematical model gives you and the goal of the epidemiologists, that is, the persons who study this epidemics, is to first understand this cause of the disease, then to predict its course and finally to develop a way of controlling those diseases.

So, we start with our first model that is the susceptible and infectious model.

So, the susceptible ones are the ones who are vulnerable to the disease and the infected one the one who has been infected.

If I connect these two compartments by some β so it means that when the susceptible is coming in contact with the when the infectious infected person is coming in contact with the susceptible they get infected.

So if S be the number of susceptible and I number of infected and β is the rate at which this is happening, a person gets infected, then the model will look very simple this is

$$\frac{dS}{dt} = -\beta SI, \quad --- \ (1)$$

$$\frac{dI}{dt} = +\beta SI \quad --- (2)$$

When the susceptible comes in contact with the infected person, the person gets infected at a rate β and from the susceptible class, it moves to the infected class.

So one more time, this is the rate of change of the susceptible, this is the rate of change of the infected person.

So when the infected person is coming in contact with the susceptible person who are vulnerable to the disease, they are getting infected at a rate β and then moving to the next class which is the infected class.

And since they are moving, this is the negative sign and the whole thing comes here, this is the positive sign.

Now, you have some initial condition, say initial values, say at time t = 0, let there be n susceptible and one infected, that is,

$$S(0) = n, \qquad I(0) = 1.$$

So, total population becomes

$$S(0) + I(0) = n + 1.$$

We want to see what happens to the whole population over large time.

(1) + (2) gives

$$\frac{dS}{dt} + \frac{dI}{dt} = 0 \quad \Longrightarrow \quad \frac{d(S+I)}{dt} = 0$$

$$\Longrightarrow S(t) + I(t) = \text{constant}.$$

Your initial condition $S(0) = n\,, I(0) = 1$, you substitute it here, you get

$$S(0) + I(0) = \text{constant} = n + 1 \Rightarrow \text{ constant} = n + 1$$

$$\Rightarrow S(t) + I(t) = n + 1$$

Now you have the equation

$$\frac{dS}{dt} = -\beta SI \quad \text{and} \quad \frac{dI}{dt} = \beta SI$$

Now, I will be solving this differential equation using the relation

$$S(t) + I(t) = n + 1 \text{ or } I(t) = (n + 1) - S(t),$$

So, I write

$$\frac{dS}{dt} = -\beta SI = -\beta S(n + 1 - S)$$

Separation of variables will give me

$$\frac{dS}{S(n + 1 - S)} = -\beta dt$$

$$\Rightarrow \frac{1}{n+1}\left(\frac{1}{n + 1 - S} + \frac{1}{S}\right) = = -\beta dt$$

$$\Rightarrow \left(\frac{1}{n + 1 - S} + \frac{1}{S}\right) dS = -(n + 1)\beta dt$$

And you integrate both sides to get

$$\int \left(\frac{1}{n + 1 - S} + \frac{1}{S}\right) dS = \int -(n + 1)\,\beta dt$$

$$\Rightarrow -\ln(n + 1 - S) + \ln(S) = -(n + 1)\beta t + \text{constant}$$

So, at time t=0, $S(0) = n$ and this gives

$$-\ln(n + 1 - n) + \ln(n) = -(n + 1)\beta \times 0 + \text{constant}$$

$$\Rightarrow \text{constant} = \ln(n) \quad (\text{since}, \ln(1) = 0)$$

$$\Rightarrow \; -\ln(n + 1 - S) + \ln(S) = -(n + 1)\beta t + \ln(n)$$

$$\Rightarrow \; -\ln(n + 1 - S) + \ln(S) - \ln(n) = -(n + 1)\beta t$$

$$\Rightarrow \ln\left(\frac{(n + 1 - S)n}{S}\right) = (n + 1)\,\beta t.$$

So,

$$\frac{(n+1-S)n}{S} = e^{(n+1)\beta t}$$

$$\Rightarrow \quad \frac{n(n+1)}{S} - n = e^{(n+1)\beta t} \quad \Rightarrow \quad \frac{n(n+1)}{S} = n + e^{(n+1)\beta t}$$

$$\Rightarrow S(t) = \frac{n(n+1)}{n + e^{(n+1)\beta t}}$$

And since we have this relation I(t),

$$S(t) + I(t) = n + 1$$

I do not have to solve $I(t)$ again from the other differential equation. All I have to do

$$I(t) = n + 1 - S(t)$$

$$\Rightarrow \quad I(t) = n + 1 - \frac{n(n+1)}{n + e^{(n+1)\beta t}}$$

$$\Rightarrow \quad I(t) = (n+1)\left(1 - \frac{n}{n + e^{(n+1)\beta t}}\right)$$

$$\Rightarrow \quad I(t) = (n+1)\left(\frac{n + e^{(n+1)\beta t} - n}{n + e^{(n+1)\beta t}}\right) = (n+1)\left(\frac{e^{(n+1)\beta t}}{n + e^{(n+1)\beta t}}\right)$$

So just divide by this particular quantity($e^{(n+1)\beta t}$), both numerator and denominator and you will get

$$I(t) = \frac{n+1}{ne^{-(n+1)\beta t} + 1},$$

and

$$S(t) = \frac{n(n+1)}{n + e^{(n+1)\beta t}}.$$

Let us see what happens as your t becomes large.

So as $t \longrightarrow \infty$, $e^{-(n+1)\beta t} \longrightarrow 0$ and $I(t) \longrightarrow (n+1)$

And, as $t \longrightarrow \infty$, $e^{(n+1)\beta t} \longrightarrow \infty$ and hence

$$S(t) = \frac{n(n+1)}{n + e^{(n+1)\beta t}} \longrightarrow 0.$$

So, what is happening is, as time increases, all the susceptible becomes infected, initially there was n susceptible and only one infected but as time becomes large, all the persons of the population gets infected because the total population is n+1 and your susceptible goes to zero.

So as time increases all the susceptible persons will become infected. So, this is just a basic model of the susceptible and the infected.

Let us see the numerical solution as this particular model where I will be using this Microsoft Excel.

So, I already have it here. So, this is the equation

$$\frac{dS}{dt} = -\beta SI, \quad \frac{dI}{dt} = \beta SI$$

So, the initially I am choosing the value of n=2000 and only one infected and the value of $\beta \neq 0$ and β =0.001.

This is your initial value for the susceptible, the initial value for infected.

So let us put the time at t=0, this value is S(0)=2000 and this value is I(0)=1.

So this will be equal to this plus 1 and let me drag it, say, to 83 of the values and this we use as usual the Euler's method to solve this system of differential equation, the formulas are given here.

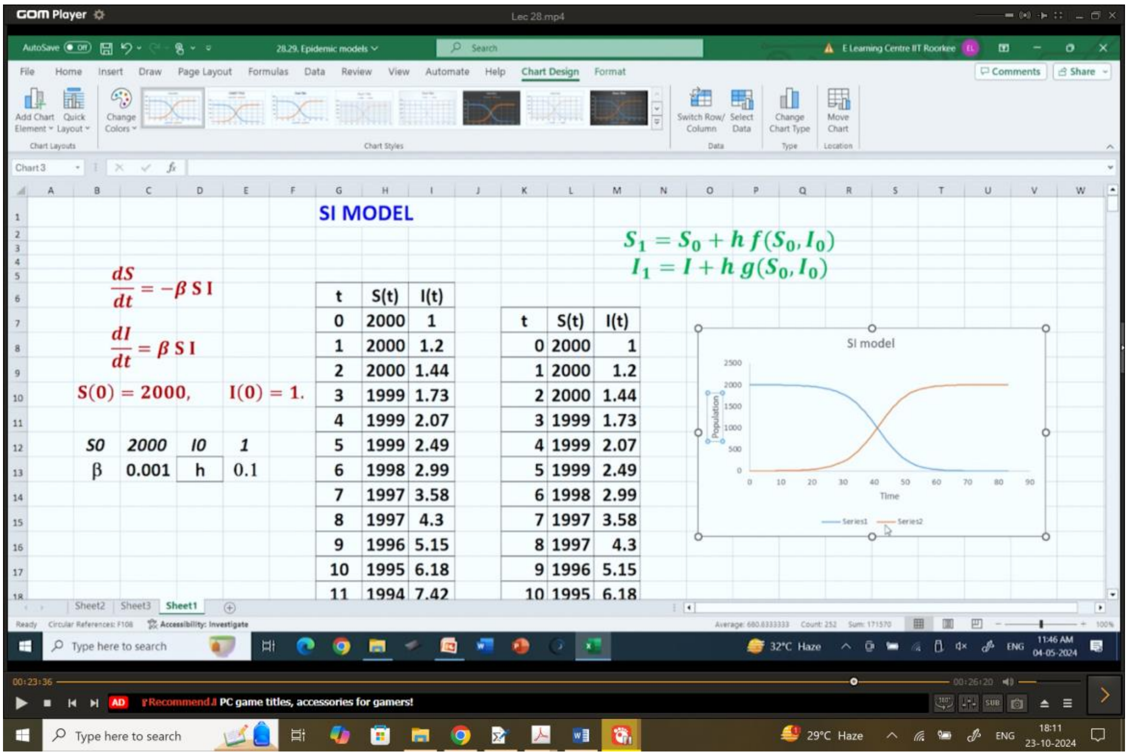
So, this is equal to this S(0)+h times which is again a constant.

So, I put $* β which is this again.

So, you get this value and this is equal to I(0) plus again the value of h, which is constant, again a constant multiplied by S×I. So now let us drag these two values.

Okay, now we plot this, you highlight by pressing the shift button and to the how many values we have up to 83, go to insert, choose this scattered diagram and this graph.

So, you get this curve.



I don't like the grid so you may keep the give if you want to remove click the plus sign and uncheck this grid line so you can see the grid lines are gone I want the axis so this is the axis title and the chart type so the chart title is SI model the axis title is time and to this.

So, here you do not have to bother the way it is written if you just click the cursor it will automatically will be in this in this vertical thing and you just press backspace it will remove.

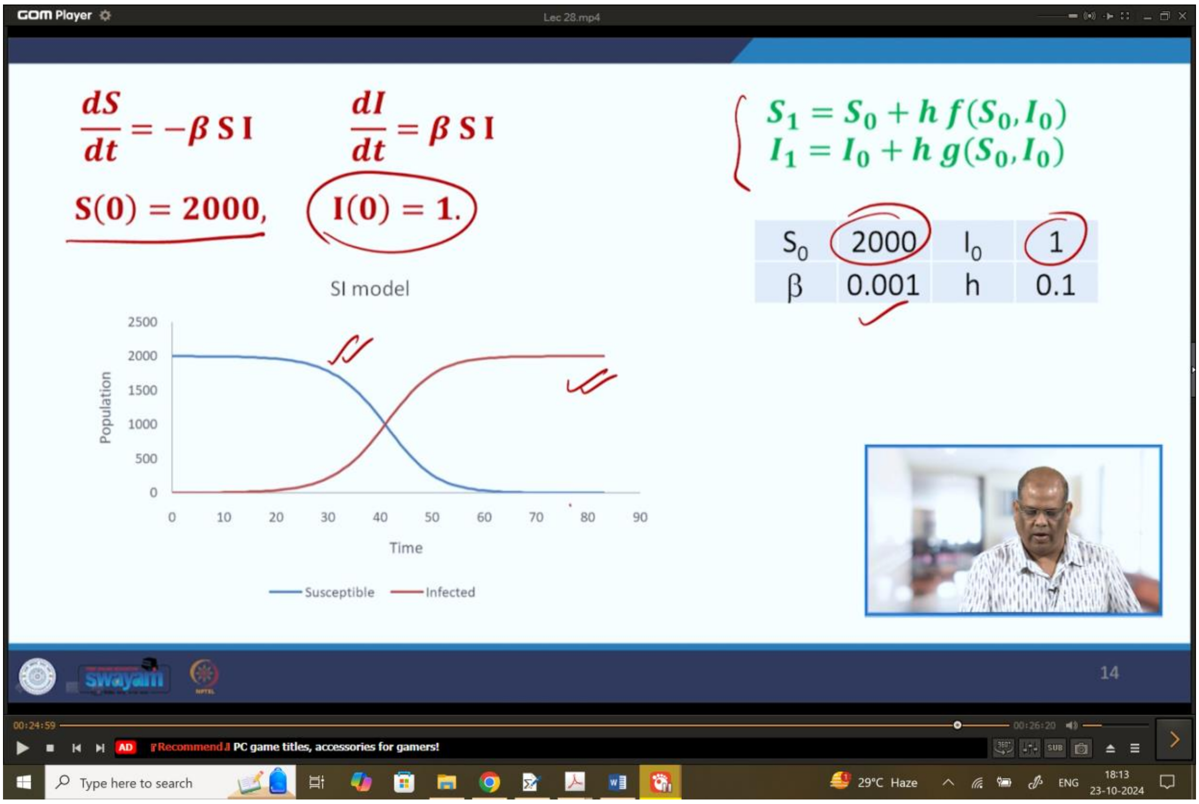
So, this is population.

And ultimately this series you go to this chart design select data here it will come series 1 edit, so series 1 is susceptible, susceptible and series 2 edit is the infect.

So, this is my curve there from 2000 all the susceptible goes to 0 and this reaches the value 2000.

Now, let us go back to our lectures.

Let us quickly see the numerical curve.



So, we plotted this curve using this Microsoft Excel where I have taken the initial value as 2000. for the susceptible, only one infected these are the parameter values where your β=0.001 initial values the Euler's method for solving and you get this curve, which is the susceptible and this is the infected, which shows that as time goes becomes large so with time all the susceptible will be infected.

This susceptible comes to zero and the infected reaches the value 2001.

So, summing up we have started with a SI model which is the susceptible and infected model.

We form the model with the help of differential equations, we solve them and we see the dynamics of the model as t becomes large.

In my next lecture we will be starting with a SIS model which is again a two compartmental model, but here we have a susceptible, we have an infected and from the infected class some is going again to the susceptible class.

So, we will be seeing that how the dynamics changes in an SIS model and how it differs from the SI model.

Till then bye-bye.