EXCELing with Mathematical Modeling

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Week – 06
Lecture – 29 (Epidemic Model 2)

Hello, welcome to the course EXCELing with Mathematical Modelling.

In my previous lecture, we have talked about this SI model.

So, today we will talk about the SIS model.

Now to start with a little recapitulation.

So, this endemic it is defined as usually a large short term disease outbreak and the outbreak will be called a pandemic if the disease growth is exponential and it will be called endemic if it is consistently present but limited to a particular region.

For pandemic you can recall COVID-19 and for endemic you can recall malaria, cholera etc.

So, this endemic models they are primarily designed to explain and predict the spread of the disease and the way how you can control it, and the goal of the epidemiologist is to first to understand the cause of the disease and then to predict its course and finally a way how to content it or how to control it including comparisons of different possible approaches.

So, today our lecture will be on Susceptible Infectious Susceptible Model in short SIS model.

Now we have seen in SI model that when a susceptible comes in contact with the infected person they get infected.

So, this is one class the susceptible class and this is the infected class.

So, what is new in SIS?

So, what we are assuming here that you can take the example of common cough and cold.

So, a person comes in contact with another person who is suffering from this cough and cold and immediately he catches it.

So, from the susceptible he moves to the infected class and since he is moving to this class, this is negative and the addition to this class, this is positive.

Now, by taking some medicine which is easily available, you get rid of this cough and cold but again you are susceptible.

So, that is why I put a negative sign that is you were infected but then you recovered and you become susceptible again.

So, from here (−α I), a part of it get recovered, but then again becomes vulnerable so (α I). So, here what we are doing is that a susceptible person first gets infected, but he took some medicine and gets recovered and again move to the susceptible class.

So, now you have this set of differential equation and we will see that how your solution changes compared to the SI model.

$$\frac{dS}{dt} = -\beta SI + \alpha I, \quad S(0) = n,$$

$$\frac{dI}{dt} = \beta SI - \alpha I, \quad I(0) = 1.$$

So, when you solve this differential equation again we use the same technique

$$\frac{dS}{dt} + \frac{dI}{dt} = -\beta SI + \alpha I + \beta SI - \alpha I = 0$$

$$\Rightarrow \frac{d}{dt}(S + I) = 0$$

$$\Rightarrow S(t) + I(t) = \text{ constant,}$$

and at t = 0,

$$S(0) = n, \qquad I(0) = 1 \Rightarrow \text{constant} = n + 1$$

$$S(t) + I(t) = n + 1.$$

So now you either eliminate S or you eliminate I from (1) and (2). Eliminating S we get

$$\frac{dI}{dt} = \beta SI - \alpha I = \beta I(n + 1 - I) - \alpha I$$

$$\Rightarrow \frac{dI}{dt} = \beta(n + 1)I - \beta I^2 - \alpha I = [\beta(n + 1) - \alpha]I - \beta I^2$$

$$\Rightarrow \frac{dI}{dt} = cI - \beta I^2, \text{ where } c = (n + 1)\beta - \alpha,$$

$$\Rightarrow \frac{dI}{dt} = I(c - \beta I) \quad \Rightarrow \frac{dI}{I(c - \beta I)} = dt$$

$$\Rightarrow \frac{dI}{\beta I\left(\frac{c}{\beta} - I\right)} = dt \quad \Rightarrow \frac{\frac{c}{\beta}\,dI}{I\left(\frac{c}{\beta} - I\right)} = c\,dt$$

$$\Rightarrow \left[\frac{dI}{I} + \frac{dI}{\frac{c}{\beta} - I}\right] = c\,dt \quad \Rightarrow \ln(I) - \ln\left(\frac{c}{\beta} - I\right) = ct + \text{Constant}$$

At time t = 0, $I(0) = 1$,

$$\Rightarrow \ln(I) - \ln\left(\frac{c}{\beta} - I\right) = ct + \text{Constant} \Rightarrow \text{Constant} = -\ln\left(\frac{c}{\beta} - I\right)$$

$$\Rightarrow \ln(I) - \ln\left(\frac{c}{\beta} - I\right) = ct - \ln\left(\frac{c}{\beta} - 1\right)$$

$$\Rightarrow \ln(I) - \ln\left(\frac{c}{\beta} - I\right) + \ln\left(\frac{c}{\beta} - 1\right) = c\,t \implies \ln\left[\frac{I\left(\frac{c}{\beta} - 1\right)}{\left(\frac{c}{\beta} - I\right)}\right] = c\,t$$

$$\implies \frac{I\left(\frac{c}{\beta} - 1\right)}{\left(\frac{c}{\beta} - I\right)} = e^{ct} \implies \frac{\left(\frac{c}{\beta} - I\right)}{I\left(\frac{c}{\beta} - 1\right)} = e^{-ct}$$

$$\implies \frac{c}{\beta\, I} - 1 = \left(\frac{c}{\beta} - 1\right) e^{-ct} \implies \frac{c}{\beta\, I} = 1 + \left(\frac{c}{\beta} - 1\right) e^{-ct}$$

$$\implies I = \frac{\frac{c}{\beta}}{1 + \left(\frac{c}{\beta} - 1\right) e^{-ct}}, \qquad c = (n+1)\beta - \alpha$$

$$\implies I(t) = \frac{(n+1) - \frac{\alpha}{\beta}}{1 + \left(n + 1 - \frac{\alpha}{\beta} - 1\right) e^{-[(n+1)\beta - \alpha]t}}$$

and $$S(t) = n + 1 - I(t) = \frac{(n+1)\left(n + 1 - \frac{\alpha}{\beta} - 1\right) e^{-[(n+1)\beta - \alpha]t} + \frac{\alpha}{\beta}}{1 + \left(n + 1 - \frac{\alpha}{\beta} - 1\right) e^{-[(n+1)\beta - \alpha]t}}$$

Now, let us see what happens as your t becomes large.

So, as $t \longrightarrow \infty$, $e^{-[(n+1)\beta - \alpha]t} \longrightarrow 0$, which implies

$$I(t) \longrightarrow n + 1 - \frac{\alpha}{\beta} \quad \text{and} \quad S(t) \longrightarrow \frac{\alpha}{\beta},$$

provided

$$(n+1)\beta - \alpha > 0.$$

So, this is the condition which need to be followed such that you get some finite value for this S(t) and I(t), as n becomes large.

So, what does this mean?

This means that a fraction $\frac{\alpha}{\beta}$ of the infected person recovers and goes to the susceptible class, after recovery. So, this is the conclusion for this SIS model.

Let us see how the model numerically behaves for which we will be using this Microsoft Excel.

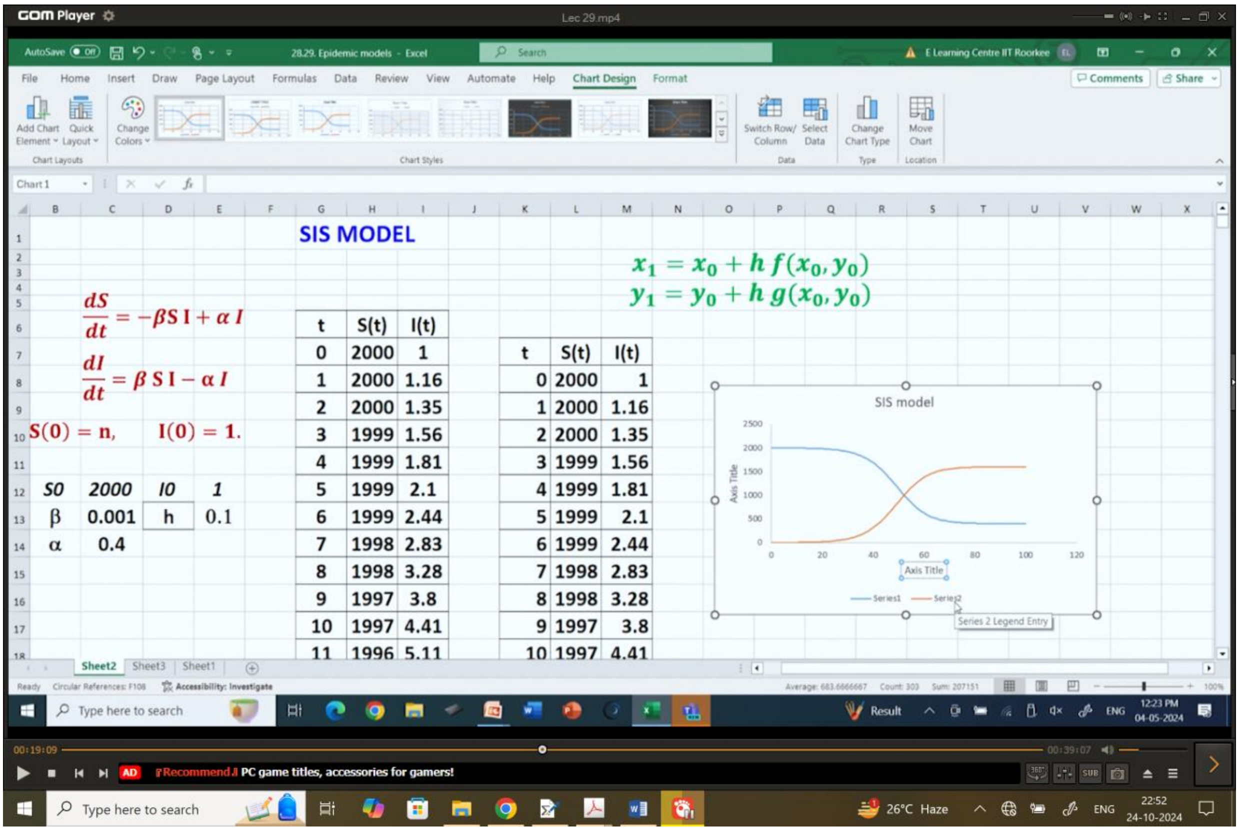
So, as you can see that I already have the equations here.

So, I take the initial value to be 2000 and the infected one is 1, your beta is 0.001 and this value of alpha is 0.4.

So, as usual this is your 0, this value is 2000 and this value is 1. So, let me first put this, this is equal to 0 plus 1 and let me drag this one.

Next, we calculate this S i, this is equal to S 0 plus, which is again a constant. So, I put a dollar multiplied by beta with a negative sign multiplied by S0 into I0 plus alpha which is again a constant multiplied by I, close and we get this value and your infected this is equal to I0 plus H which is a constant multiplied by this particular thing. So beta, which is again a constant multiplied by SI.

So, this is your S multiplied by I minus alpha I So, I drag these values and let us plot this. I highlight using the shift key and the cursor, go to insert the chart and this chart. This is SIS model.



I don't like grid lines, so I will remove unchecked grid lines. I want the axis title. So this is again time. This is the population. And I will change this series 1 and series 2, go to chart design, then select data, select series 1, edit, so this is susceptible and series 2, edit, infected.

So I get this chart for the SIS model.

Now let us now go back to the slides.

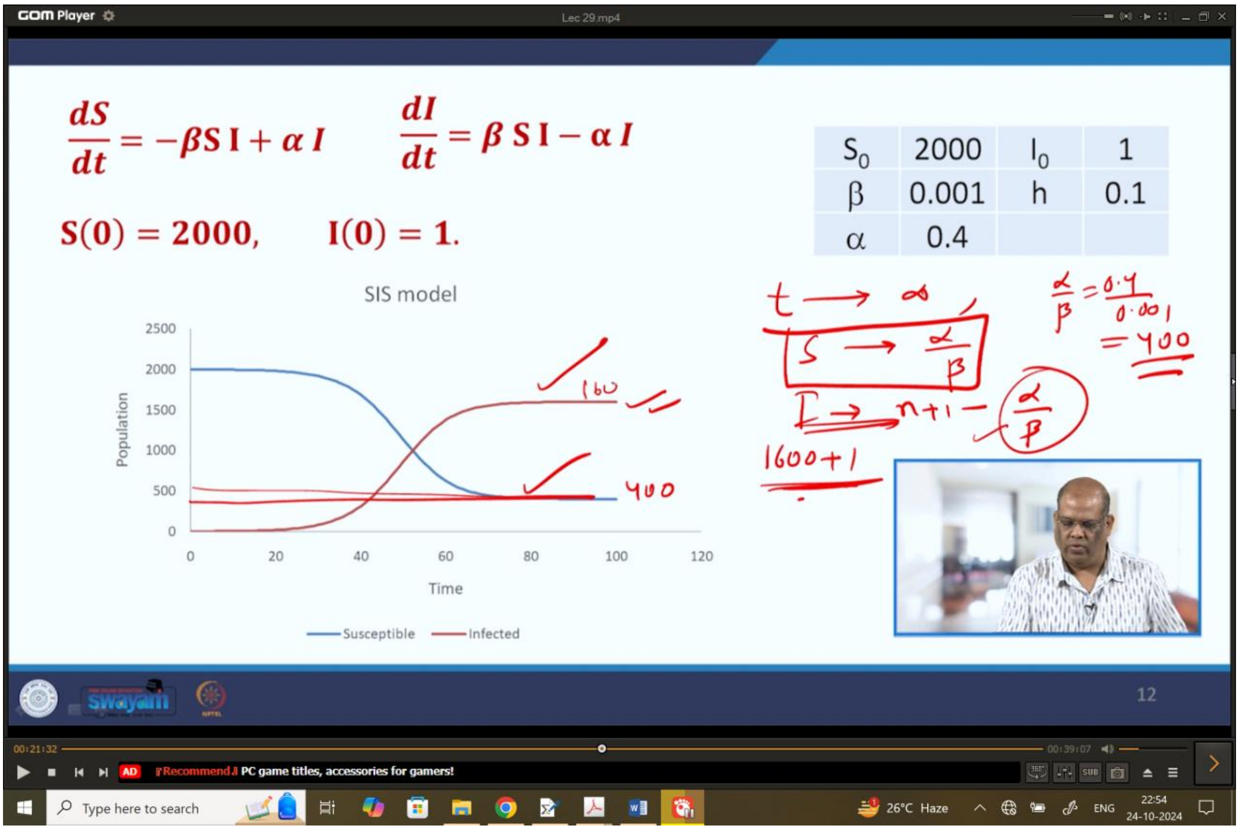
So our numerical solution to this SIS model shows that this value this is the infected one and this is the susceptible one.

So if you look into the solution it says that as your t becomes large your S is approaching alpha by beta and your I is approaching n plus 1 minus alpha by beta.

So, if I calculate my alpha by beta from here that is 0.4 by 0.001 which is 400, so you see that you are susceptible alpha by beta, so this is reaching the value 400 sorry 400 and your i that is n plus 1 alpha minus beta will reach the value 1600 plus 1.

So which is approximately this particular value.

So the conclusion is that as your time increases a fraction of the susceptible persons will be there and in this case which is 400, and of the fraction which has not been infected is n plus 1 minus this alpha by beta and due to which your infected person has only reached 1600 plus 1 which is this value.



So with this we go to the next model which is called the SIR model.

It is just an extension of the SIS model but only with another compartment.

So there you have your rate of change of susceptible.

This is the infected and this is called recovered.

These three classes are there now.

So when the person infected person comes in contact with a susceptible one they infect that person and hence this particular term comes here the infection rate is beta.

Now from the infected class some of the person gets recovered at a rate alpha and since they move from this infected class to the recovered class.

So from that is why here it is minus alpha i and it is added in the recovered class to plus alpha I. Now from the recovered class some people are also again susceptible. So minus gamma r is going back to the susceptible class.

So, we have now three class susceptible, infected and recovered and this is the dynamics between them.

$$\frac{dS}{dt} = -\beta SI + \gamma R$$
$$\frac{dI}{dt} = \beta SI - \alpha I$$
$$\frac{dR}{dt} = \alpha I - \gamma R$$

$$S(0) = n, \qquad I(0) = 1, \qquad R(0) = 0$$

We now look into the equilibrium solution of this SIR model and as we know we put

$$\frac{dS}{dt} = -\beta SI + \gamma R = 0,$$

$$\frac{dI}{dt} = \beta SI - \alpha I = 0,$$

$$\frac{dR}{dt} = \alpha I - \gamma R = 0,$$

to find the equilibrium solution.

But before that if I add

$$\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0 \implies S(t) + I(t) + R(t) = \text{constant}$$

Now, initial condition, initially at t = 0, you have

$$S(0) = n, \qquad I(0) = 1, \qquad R(0) = 0$$

So, if I substitute it here, I will get

$$S(t) + I(t) + R(t) = \text{constant} \implies n + 1 + 0 = \text{constant} \implies \text{constant} = \text{n} + 1$$

$$\implies S + I + R = n + 1.$$

Now,

$$\frac{dS}{dt} = -\beta SI + \gamma R = 0 \ \Rightarrow -\beta SI + \gamma R = 0$$

$$\frac{dI}{dt} = \beta SI - \ \alpha I = 0 \Rightarrow I(\beta S - \ \alpha) = 0 \Rightarrow I^* = 0, S^* = \frac{\alpha}{\beta}$$

$$\frac{dR}{dt} = \alpha I \ - \gamma R = 0 \ \Rightarrow \alpha I \ = \gamma R$$

Now if I = 0, it means there is no infections and none of the person is infectious.

So, there is no point of studying this model because if I put I = 0, then R = 0. So, we get some value $\frac{\alpha}{\beta}$ of the susceptible and the no infectious, no recovery, there is no point of studying this equilibrium point.

So, we will study where you have infectious people where you have recovered people and at the same time you have the susceptible one.

So you have this your $S^* = \frac{\alpha}{\beta}$ and

$$I^* + R^* = n + 1 - S^* = n + 1 - \frac{\alpha}{\beta} \ \Longrightarrow \ I^* + \frac{\alpha}{\gamma} I^* = n + 1 - \frac{\alpha}{\beta}$$

$$\Longrightarrow I^* = \frac{n + 1 - \frac{\alpha}{\beta}}{1 + \frac{\alpha}{\gamma}} \quad \text{and} \quad \Longrightarrow R^* = \frac{\alpha}{\gamma} I^* = \frac{\alpha}{\gamma}\left(\frac{n + 1 - \frac{\alpha}{\beta}}{1 + \frac{\alpha}{\gamma}}\right)$$

So, I get the value of (S^*, I^*, R^*) and this is the equilibrium point which we will study.

If you want to look into the stability analysis, that is, the linear stability analysis.

So you know you have to find the matrix A where you differentiate this with respect to S, I and R, and the matrix which you will get is

$$A = \begin{pmatrix} -\beta I^* & -\beta S^* & \gamma \\ \beta I^* & \beta S^* - \alpha & 0 \\ 0 & \alpha & -\gamma \end{pmatrix}$$

So, I leave the stability analysis to you, this is the Jacobian matrix which you will get, you have to find the eigenvalues, you have to form the characteristic equation and use Routh-Harvey's criteria to find the condition for stability.

Now, what we look into is the numerical solution of this SIR model.

So, I have this SIR model already typed, the initial condition is N 1 and 0, where the value of N I have taken to be 2000, the initial value of infected is 1 and there is no recovery is 0.

The value of beta it is 0.001, H is 0.1, alpha is 0.4 and gamma is 0.01. So, with this we start with t equal to 0 and this is equal to 0 plus 1 and I will drag them later.

So, this is equal to S0 plus H which is a constant, so I put a dollar multiplied by this whole thing minus beta SI plus gamma R. So, minus beta multiplied by S multiplied by I since beta is a constant I put a dollar plus gamma which is 0.01 multiplied by r which is 0.

So, this gamma is also a constant. So, I put the dollar sign and enter. so plus this minus this star.

Now, this is equal to I0 plus h which is a constant, so I put a dollar multiplied by this beta SI minus alpha I so beta multiplied by S multiplied by I minus alpha which is 0.4 multiplied by I.

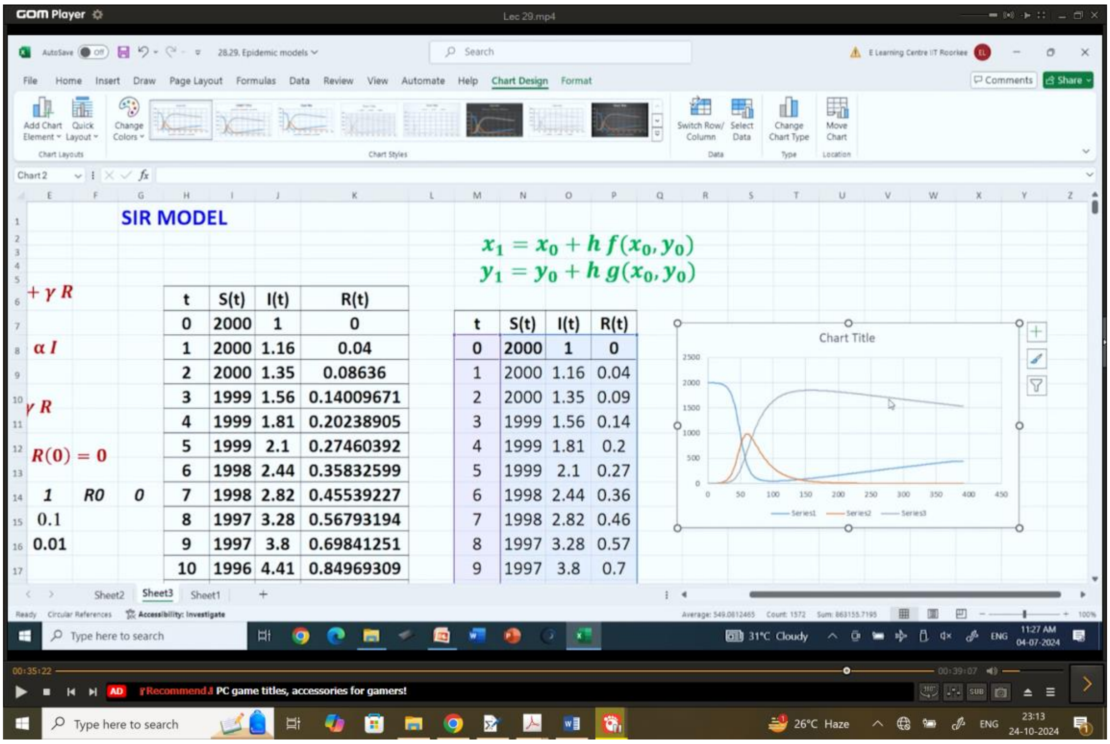
I have to put dollars in front of the constants. So, this is a dollar, this is another dollar.

This is beta SI minus alpha so which is again constant and this value finally this is equal to R0 plus H which is constant multiplied by so alpha i minus gamma r. So, this is alpha multiplied by i minus gamma multiplied by r.

So, this alpha is a constant I put a dollar and this gamma is a constant. So, the reason I already calculated is that here I just show the calculation and here the correct one is already there.

So, you can just compare. So, I now just drag to more than 300 values and let me plot them.

So, I change the font size to 20, I just align them and let us now plot the graphs.



So, in this case I have to take a little more values just to find the true dynamics of the model and that is the reason that we have taken so many values.

Okay now I go to insert, I choose the graph and I get the values.

So I just change the color of this, let us make it green. okay and if you want to change the title this is a SIR model if you want to change the series go to select data go to edit and first one is susceptible The second one is infected and the third one is recovered. So we get this dynamics.

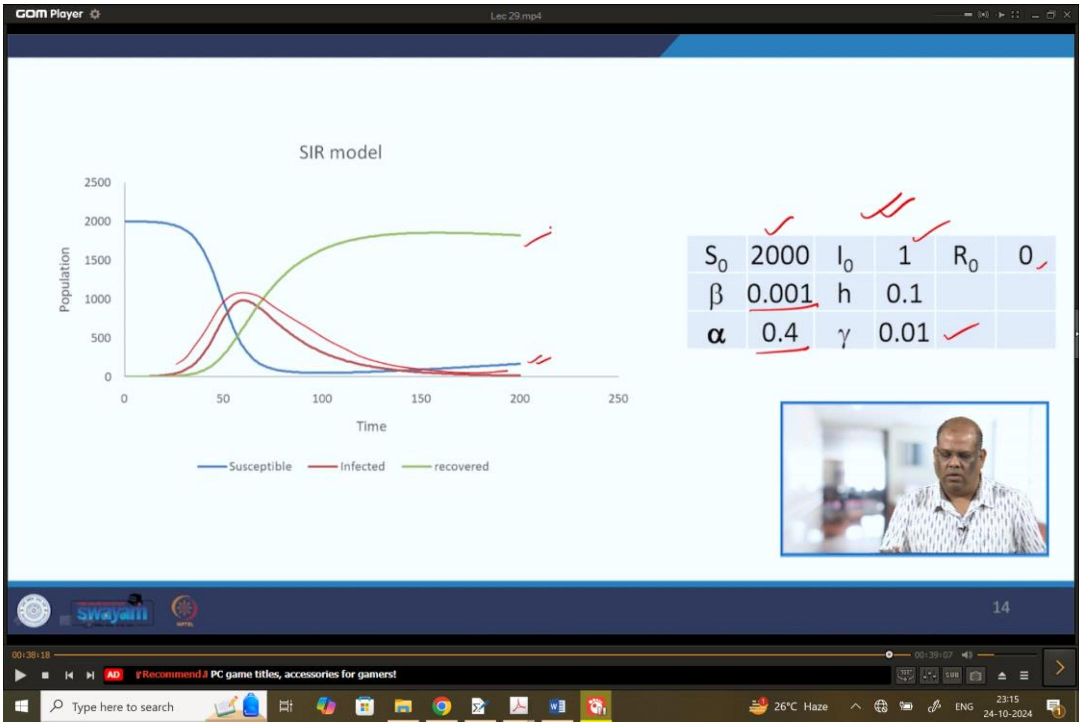
So one thing you need to know here that our equilibrium point does not show any zero value but you can see this curve has touched almost touched the x-axis that is where the value is zero.

But in reality that is not happening.

What is happening is that if I see the last value of the infected, you can see that this is becoming less and less and less and ultimately it is coming to 1 and slowly you will get a non-zero equilibrium value here. It is not actually going to 0, but the value is very very less and that is why this is infected one shows that this there is a increase at some point and slowly the value goes down.

It never goes to 0 it goes to some very small equilibrium value which is positive. The susceptible as from the equilibrium solution you will see reach certain equilibrium value non-zero and so is the recovered class. Now if you plot this figure by using this set of parameter values.

I have taken again an initial value as 2000, 1 infected, no recovered, beta remains the same, alpha remains the same, gamma is also 0.01. So, you get this figure.



So, your susceptible as you can see it is going to 0, but then a little increase because some of the recovered will move back to the susceptible class your infected there is an increase in the infection but then it goes down and then a part of the population they have also recovered. So, this is a modification of the previous model, which is known as the SIR model.

With this, we come to the end to the lecture of this epidemic models.

In our next lecture, we will be talking about more interesting models. Till then, bye-bye.