EXCELing with Mathematical Modeling
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Week – 07
Lecture – 31 (Varying Gravity model)

Hello, welcome to the course EXCELing with Mathematical Modelling.

Today we will be talking about models with varying gravity.

So, to start with we look into Newton's law of universal gravitation which says that every particle in our universe attracts every other particle with a force which is directly proportional to the product of their masses and is inversely proportional to the square of the distance between them.

If you want to write them mathematically then that force varies as the product of the masses directly and inversely to the square of the distance between them and we get the formula

$$F \propto \frac{Mm}{r^2}$$

$$\Rightarrow F = G\frac{Mm}{r^2}$$

where this capital G is called the universal gravitational constant.

We defined another term which is called E, it is called gravitational field strength, which is defined by

$$E = \frac{F}{m}$$

but this m is the mass of the smaller object.

If I want to see the unit of this, so the unit of force is Newton and mass is kg. Also, Newton is nothing but kg meter per second square and this is divided by kg, so

$$\frac{kg\ m/s^2}{kg},$$

this and this cancels, and you get m/s^2.

So, if you note carefully this is just the unit for acceleration, in this case acceleration due to gravity. So, basically your gravitational field strength has a unit of the acceleration due to gravity.

Now, near the earth's surface the gravitational force

$$F = mg \implies g = \frac{F}{m} = E = \frac{GM}{r^2}.$$

So, your gravitational force (g) near the earth surface varies as $\frac{1}{r^2}$, that is,

$$g \propto \frac{1}{r^2}.$$

The point of proving this is that in the earth's surface if you if a body is attracted then the gravity varies as 1 by the distance square which we call the inverse square law.

However, when you are inside the earth's surface let us see what happens. So, inside the earth's surface, suppose, somebody is here, then this is the attracting mass. So, suppose you are moving from here towards the surface of the earth. So, as you rise from earth's surface the effective mass of earth attraction also increases.

So, it means that from here if you now take a new g, that is,

$$g' = \frac{G}{r^2}M,$$

where this M now increases as you move towards the surface of the earth. This r is now the distance from the centre to this current location.

So, if your M increases, then I know that mass equal to density into volume. So,

$$g' = \frac{G}{r^2}M = \frac{G}{r^2}\left(\frac{4}{3}\pi r^3 \rho\right) = \frac{4}{3}\pi\rho G r.$$

So, inside the earth's surface $g' \propto r$.

So, the take home message is that inside the earth's surface if a body moves away from the centre, it varies linearly as the distance and on the surface and above, it follows the inverse square law which varies as $\frac{1}{r^2}$.

With this information, let us take an example.

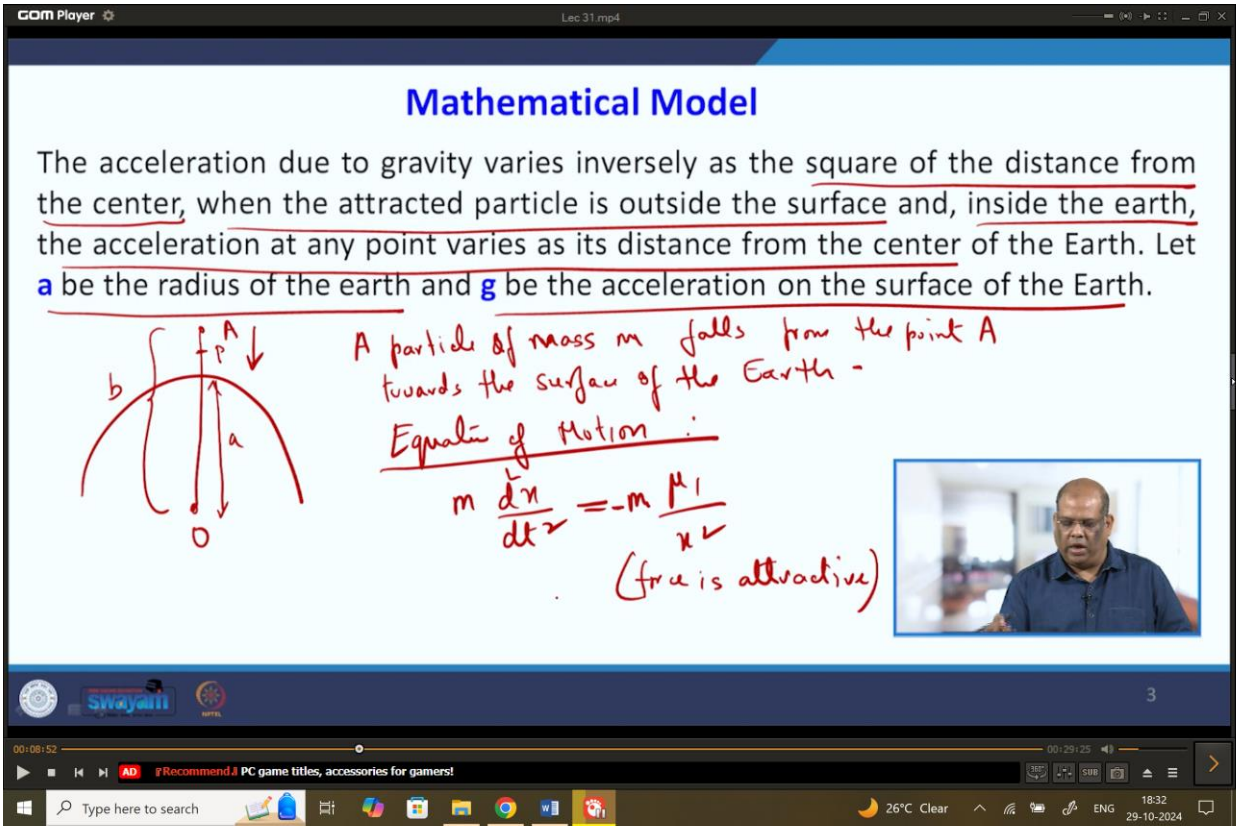
So, it says that the acceleration due to gravity varies inversely as the square of the distance from the centre when the attracted particle is outside the surface and inside the earth the acceleration at any point varies at its distance from the centre of the earth, the one which we just proved before.

Let a be the radius of the earth and g be the acceleration on the surface of the earth.

So, if we draw a figure. So, let us see, that this is the earth and its surface. Suppose A is a position of the particle and it is falling. So, this is the centre of the earth, this distance is b and the radius of the earth is a. Let P be the position of the particle at any time t. So, a particle of mass m falls from the point A towards the surface of the earth. So, now the equation of motion will be

$$m\frac{d^2x}{dt^2} = -m\frac{\mu_1}{x^2},$$

so it is falling in that direction it is attracted towards the centre O so the force is attractive.



So, this is our equation of motion and now we have to solve this say to find the velocity on the surface.

So, what we are going to do is we want to find the velocity from the point A on the surface and at the centre and let us see how this varying gravity affect this model.

So you cancel this m from both sides and you get your equation

$$\frac{d^2x}{dt^2} = -\frac{\mu_1}{x^2} \text{ (negative sign, sign force is attractive)}$$

$$\Rightarrow 2\frac{dx}{dt}\frac{d^2x}{dt^2} = -\frac{\mu_1}{x^2}\left(2\frac{dx}{dt}\right)$$

$$\Rightarrow \frac{d}{dt}\left(\frac{dx}{dt}\right)^2 = -2\frac{\mu_1}{x^2}\frac{dx}{dt}$$

$$\Rightarrow \int d\left(\frac{dx}{dt}\right)^2 = -\int 2\frac{\mu_1}{x^2}dx$$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 = \frac{2\mu_1}{x} + \text{constant.}$$

Now you have to find the value of this constant.

So if you notice that initially the particle is at the point A and it is falling freely from that point.

So the initial velocity is zero, and the distance is b from the centre of the earth.

So at the point A, the velocity is zero and the distance is b. So this information, we put here that at time t = 0,

$$\frac{dx}{dt} = 0, \qquad x = b,$$

$$\Rightarrow\ 0 = \frac{2\mu_1}{b} + \text{constant} \implies \text{constant} = -\frac{2\mu_1}{x}$$

$$\implies \left(\frac{dx}{dt}\right)^2 = 2\mu_1\left(\frac{1}{x} - \frac{1}{b}\right).$$

Now, on the surface, the acceleration due to gravity is g and i follows the law

$$g = \frac{\mu_1}{x^2} = \frac{\mu_1}{a^2}\ \text{(on the surface x = a)}$$

So, what we are doing here is we are trying to find out the value of this constant of proportionality μ_1 in terms of the gravitational constant. So, we say that on this surface of the earth, the gravitational constant is g and also it follows the inverse square law and this will give us

$$\mu_1 = ga^2.$$

So, you substitute here and you get

$$\left(\frac{dx}{dt}\right)^2 = 2\mu_1\left(\frac{1}{x} - \frac{1}{b}\right) = 2ga^2\left(\frac{1}{x} - \frac{1}{b}\right).$$

Now, if I want to find what is the velocity of this particle on this surface, then I say that if v_1 be the velocity of the particle on reaching the surface, then I get

$$\frac{dx}{dt} = v_1, \qquad x = a$$

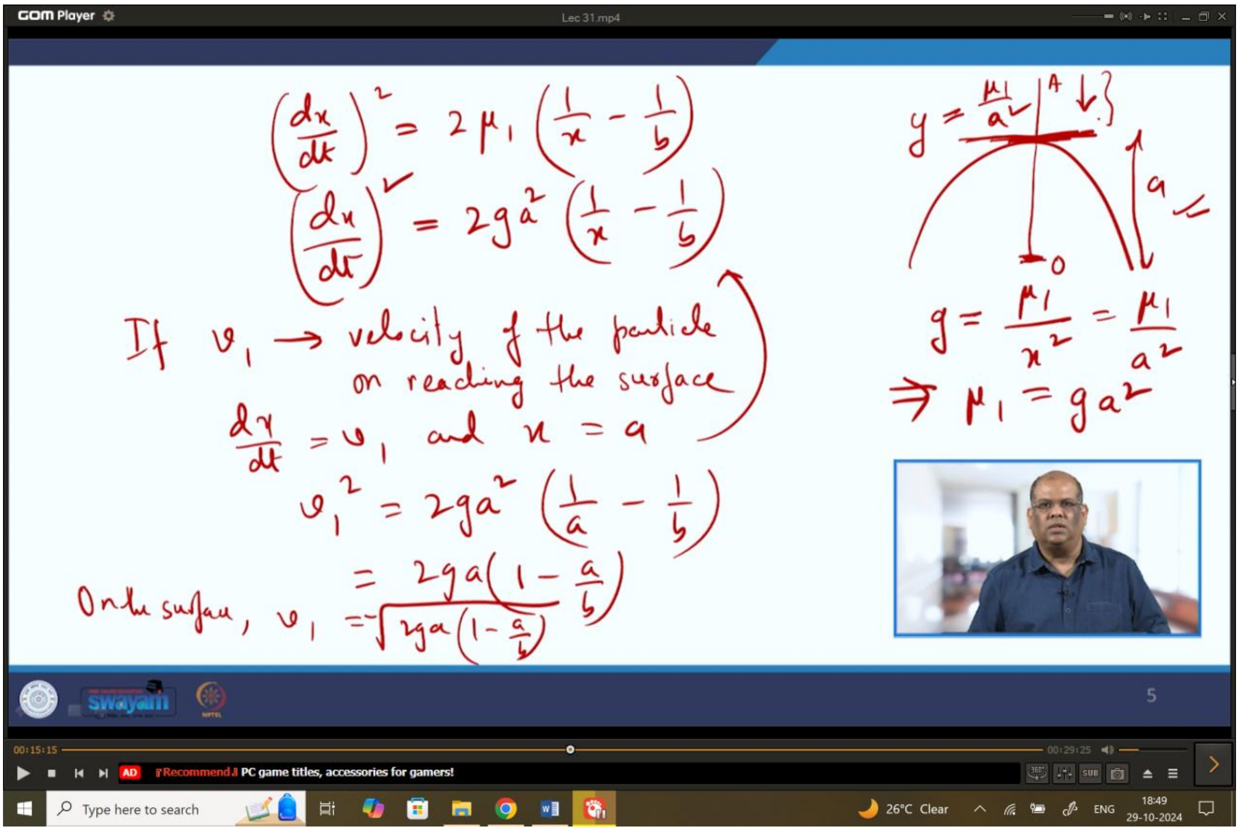
$$\Rightarrow\ {v_1}^2 = 2ga^2\left(\frac{1}{a} - \frac{1}{b}\right) = 2ga\left(1 - \frac{a}{b}\right)$$

So I can say that the velocity v_1 on the surface is given by

$$v_1 = -\sqrt{2ga\left(1 - \frac{a}{b}\right)}.$$

Since, the value of x decreases with time, the velocity should be taken as a negative quantity.

Now, if somebody asks, okay, what will be the velocity at the of the particle when it reaches the centre O? Let us see how your equation of motion changes.



So there is two parts of the problem. The first part is that it is falling from the point A till it reaches the surface, and then from A again it goes to O.

The reason for doing this is that in this particular space you have an inverse square law and in this the force varies linearly as the distance.

So, if we consider the second part, that is, inside the earth, your equation of motion inside the earth, this is going to be now

$$m\frac{d^2x}{dt^2} = -m\mu_2 x \text{ (negative sign, sign force is attractive)}$$

$$\Rightarrow\ 2\frac{dx}{dt}\frac{d^2x}{dt^2} = -\mu_2 x\left(2\frac{dx}{dt}\right) \ \Rightarrow\ \frac{d}{dt}\left(\frac{dx}{dt}\right)^2 = -2\mu_2 x\frac{dx}{dt}$$

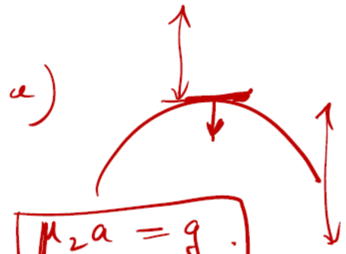
$$\Rightarrow\ \int \frac{d}{dt}\left(\frac{dx}{dt}\right)^2 = -2\mu_2\int x\frac{dx}{dt} \ \Rightarrow\ \left(\frac{dx}{dt}\right)^2 = \mu_2 x^2 + \text{constant.}$$

Now before finding the constant, let me find the value of μ_2, so with the same logic on this particular surface, if we consider now the motion in the downward direction, that is, inside the earth it follows linearly.

So, basically, if I follow this on the surface, $\mu_2 a = g$.

So, this one was considered for this lower part and the previous one was considered for this upper part but on the surface the value of g remains constant. Therefore,

$$\mu_2 = \frac{g}{a}$$



So, this you can substitute here and you get

$$\left(\frac{dx}{dt}\right)^2 = -\frac{g}{a}x^2 + \text{constant.}$$

$$\boxed{\mu_2 a = g.}$$

$$\mu_2 = \frac{g}{a}.$$

Now, I have to use the initial condition to find this value of the constant and the initial condition is

$$\text{when } x = a, \quad \frac{dx}{dt} = -\sqrt{2ga\left(1 - \frac{a}{b}\right)}.$$

So, we substitute it here and we get

$$\text{constant} = 2ga - 2g\frac{a^2}{b} + ga$$

$$\Rightarrow \text{constant} = \; 3ga \; - \frac{2ga^2}{b} = ag\left(3 - \frac{2a}{b}\right)$$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 = -\frac{g}{a}x^2 + ag\left(3 - \frac{2a}{b}\right)$$

Now the question is, what is the velocity of this particle, once it reaches the centre O? So, when it reaches the centre O, your x becomes zero because x is calculated from here (centre O), this distance was b, this distance was the radius of the Earth a and P be the position of the particle at any time t and such that this OP is equal to x.

So, you just substitute x = 0, you get

$$\left(\frac{dx}{dt}\right)^2 = {v_2}^2 = ag\left(3 - \frac{2a}{b}\right) \Rightarrow \; v_2 = \sqrt{ga\left(3 - \frac{2a}{b}\right)}.$$

We now look into the numerical solution of this varying gravity model.

So, as you can see this is the actual equation

$$\frac{d^2x}{dt^2} = -\frac{ga^2}{x^2}$$

and we write this second order differential equation as a system of two first order differential equations,

$$\frac{dx}{dt} = y \;\text{ and }\; \frac{dy}{dt} = \; -\frac{ga^2}{x^2}$$

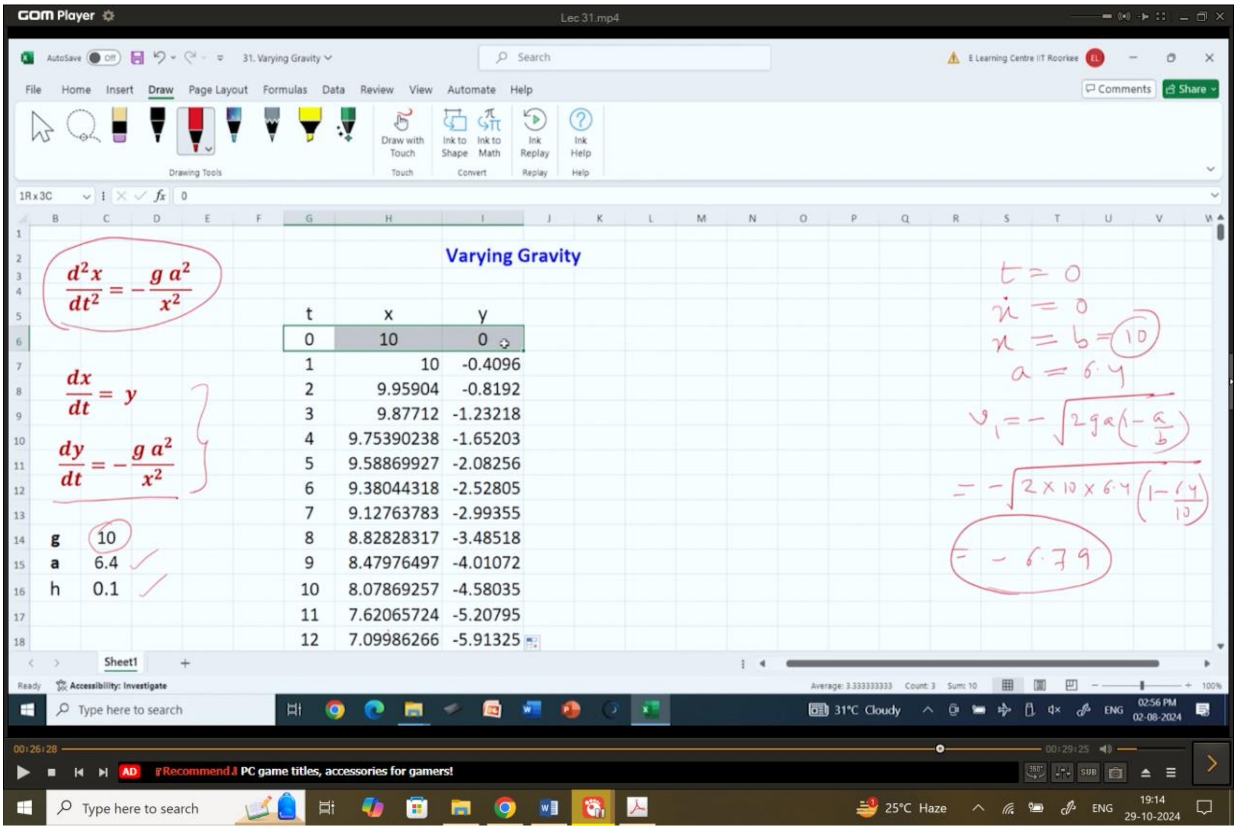
Now, g is the acceleration due to gravity. So, we take the value of g to be 10. The radius of the earth is approximately 6400 kilometer. We scaled it and we take the value 6.4 and the value of h is 0.1.

So, what we are going to do here is to check using Microsoft Excel that our numerical solution also matches with the analytical solution.

Now, we know that at time t equal to 0, we have taken that the value of the velocity x dot is equal to 0 and x is equal to some b, the value of which we take as 10 and the value of the radius is A which is 6.4. So, if we calculate the velocity this will be given by

$$v_1 = -\sqrt{2ga\left(1 - \frac{a}{b}\right)} = -\sqrt{2 \times 10 \times 6.4\left(1 - \frac{6.4}{10}\right)} = -6.79.$$

Now we solve this set of differential equation and we will see that it is approximately matching with this value.



So, let us use Microsoft Excel.

So, as you can see I already have time from 0 to 15, the value of the acceleration due to gravity is 10, the radius is 6.4, the value of B is 10. So, that is why initial value is 10 at time t equal to 0 and dy dt is nothing but x dot that value is also 0.

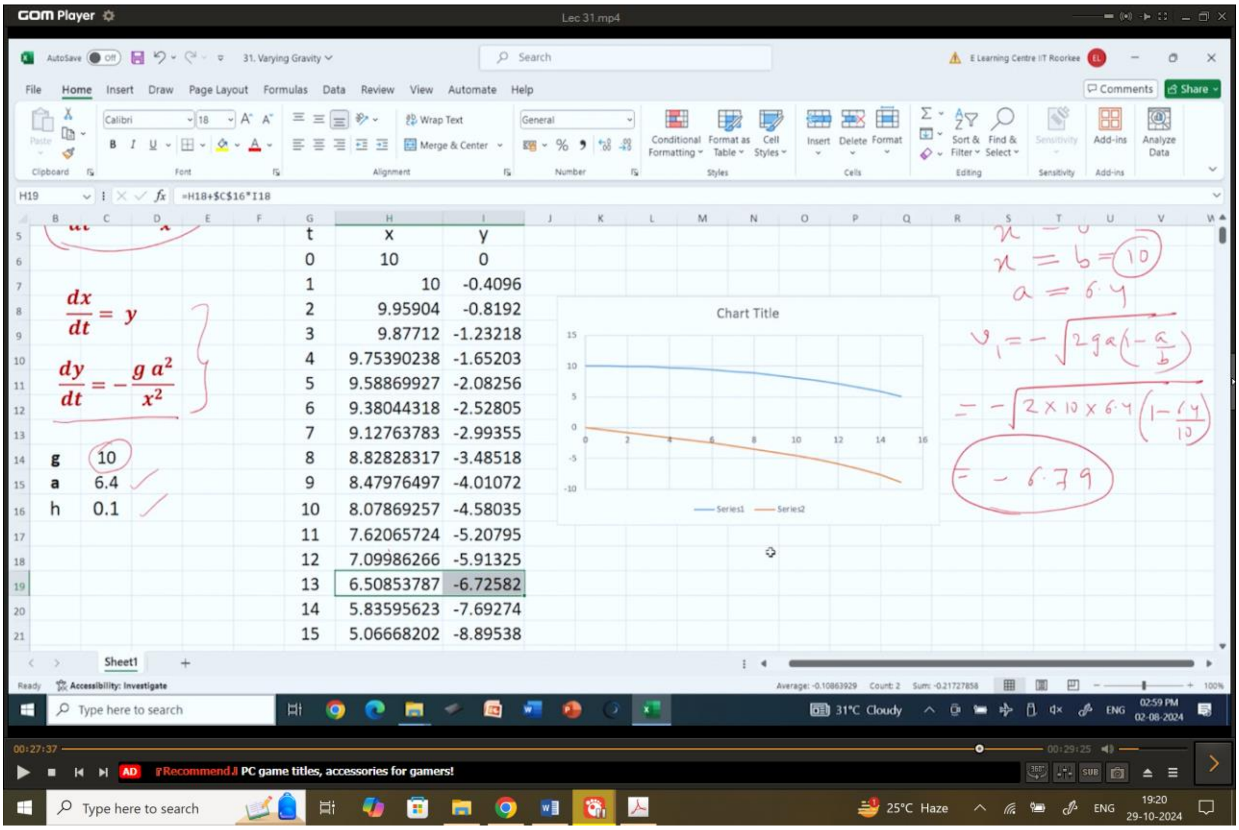
So, let us now calculate this.

So, this is equal to this x plus h which is a constant. So, I put a dollar multiplied by the y which is this value, and the value of y is equal to y0 plus h which is a constant multiplied by the value of g which is 10, again a constant, multiplied by a square which is again a constant, so I put dollar divided by x square, so it is 10 square, and this is in a bracket, so h and this is with a negative sign.

So, let us now drag and calculate the values. So, if we now plot them. It is what we get. Go to insert, go to the chart and click this. So, this will give the curve. So, the series 1 is the x values and the series 2 is the y values. So, y gives you the velocity and x gives you the distance.

Now if you look into the velocity value you can see that in this step it has attained the velocity as minus 6.72 which is approximately equal to this and the distance is 6.5.

So basically when the radius is or when the x value is approximately equal to 6.4 here it is 6.5 and your velocity is approaching 6.79 here it is 6.72 some sort of approximate value which is approaching as the particle reaches the surface of the earth.



So, thus we verify that our analytical solution is same as when we put the numerical values.

So, summing up, we took a model with varying acceleration, we solved the equation analytically and we got a relation between the velocity and the distance and then we use the numerical solution using Microsoft Excel and we have shown that that matches with our analytical calculation.

So, in our next lecture, I will be talking about various kind of tumor growth models, namely, say the linear growth, the exponential growth, the logistic growth, the Gompertzian growth and many more.

So, the reason we are studying so many growth model is that this tumor vary from person to person and we really do not know which model will actually fit to a particular kind of tumor. So, we will be going through all these growth models and learn their dynamics.

Till then, bye-bye.