EXCELing with Mathematical Modeling
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Week – 07
Lecture – 34 (Vegetation in a Desert Model)

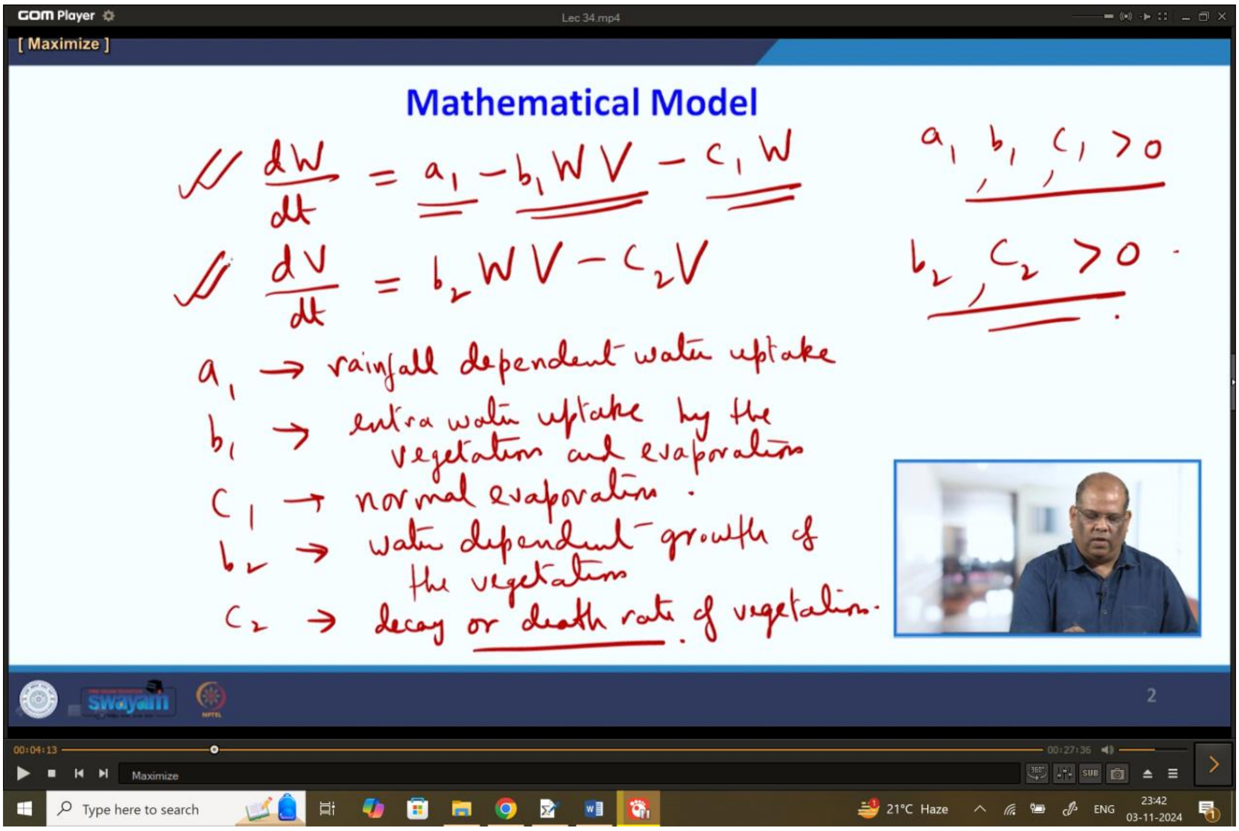
Hello, welcome to the course EXCELing with Mathematical Modelling.

Today we will be discussing about vegetation in a desert model.

So, as you know in a desert there is a very less vegetation because of the scarcity of water.

So, we will be formulating a very simple model where our variables will be the water and the vegetation.

So, let us go directly to the model.



So, if we say that W is our water that is available and V is the vegetation in the desert. So, let us say there is a constant water which is available in the desert and how that constant water comes? Say, the most common thing is the rainfall.

So, it is rainfall dependent water uptake and then there will be a vegetation which will require water.

So, there will be interaction between that vegetation and the water and the water there will be a loss of water.

$$\frac{dW}{dt} = a_1 - b_1WV - c_1W\,, \qquad \frac{dV}{dt} = b_2WV - c_2V$$

So, your b_1 is going to be the extra water uptake by the vegetation and some evaporation, and then there is going to be a normal evaporation of the water. So, we say c_1 is the normal evaporation.

So, this is the equation which represents the water the only source of vegetation in the desert. So, it says that there is a constant source of water which may be due to rainfall mostly due to rainfall and then that water intake is taken up by the vegetation for its growth which is happening at the rate of b_1 and this is the normal evaporation of water and since the water is decreasing in both the cases it is negative sign and obviously your a_1, b_1 and c_1 they are greater than zero.

The second equation the vegetation so when it gets the water obviously it will grow say it grows at a rate b_2, W and V and there is a decay or the death of the vegetation due to scarcity of water which is minus c_2V.

So, your b_2 is water dependent growth of the vegetation and your c_2 is decay or death rate of vegetation and obviously your c_2, b_2 and c_2 they are also positive.

So once we have the formulation of the model of the water and the vegetation, let us go for the analysis the very first thing which we look into is the equilibrium solution.

So, to find the equilibrium solution, we put

$$a_1 - b_1WV - c_1W = 0,$$

$$b_2WV - c_2V = 0.$$

Now from the second equation I get

$$V(b_2W - c_2) = 0$$

This gives me

$$V = 0 \qquad \text{or} \qquad W = \frac{c_2}{b_2}.$$

So when $V = 0$, I put this in (1), I get

$$a_1 - c_1W = 0 \Rightarrow W = \frac{a_1}{c_1}.$$

So one equilibrium solution is

$$\left(\frac{a_1}{c_1}, 0\right).$$

When $W = \frac{c_2}{b_2}$, I substitute again in (1) and I get

$$a_1 - b_1 \frac{c_2}{b_2} V - c_1 \frac{c_2}{b_2} = 0$$

and this will give me

$$V = \frac{a_1 b_2 - c_1 c_2}{b_1 c_2}$$

So, another equilibrium solution is

$$\left(\frac{c_2}{b_2}, \frac{a_1 b_2 - c_1 c_2}{b_1 c_2}\right).$$

Now, since this is the water and the vegetation they cannot be negative. So, for the existence of this solution

$$a_1 b_2 - c_1 c_2 > 0.$$

So, we got our equilibrium solution and next we go for the stability analysis.

So, to calculate the stability analysis let me rewrite the equation one more time.
So,

$$\frac{dW}{dt} = a_1 - b_1 WV - c_1 W = f_1$$

and

$$\frac{dV}{dt} = b_2 WV - c_2 V = f_2$$

So your Jacobian matrix is going to be

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial W} & \frac{\partial f_1}{\partial V} \\ \frac{\partial f_2}{\partial W} & \frac{\partial f_2}{\partial V} \end{pmatrix} = \begin{pmatrix} -b_1 V^* - c_1 & -b_1 W^* \\ b_2 V^* & b_2 W^* - c_2 \end{pmatrix}$$

So, at the equilibrium point $\left(\frac{a_1}{c_1}, 0\right)$, I put those values here and I get A. So, $V^* = 0$, so here it is

$$A = \begin{pmatrix} -c_1 & -b_1 \frac{a_1}{c_1} \\ 0 & b_2 \frac{a_1}{c_1} - c_2 \end{pmatrix}$$

So, if I want to calculate the eigenvalue which is straightforward

$$|A - \lambda I| = 0$$

So I get

$$\begin{vmatrix} -c_1-\lambda & -b_1\dfrac{a_1}{c_1} \\ 0 & b_2\dfrac{a_1}{c_1}-c_2-\lambda \end{vmatrix} = 0.$$

So if I just solve this, I will get $\lambda = -c_1 \quad , \quad b_2\frac{a_1}{c_1} - c_2 = \frac{a_1b_2 - c_1c_2}{c_1}$

So, clearly this is negative and for this to be negative, I will need

$$a_1b_2 - c_1c_2 < 0.$$

So, if you want this equilibrium point $\left(\frac{a_1}{c_1}, 0\right)$ to be stable, you need to have this particular quantity must be less than zero.

But this will mean that you will only have the water source and zero vegetation which obviously we do not want, we want the desert to have both the water and the vegetation, both has to coexist. So, this is not preferable, but this is mathematically just gives you the condition that this for which this particular equilibrium point is stable.

So, let us move to the next one, next equilibrium point and see that what is the condition for stability of that equilibrium point. And that equilibrium point is

$$\left(\frac{c_2}{b_2}, \frac{a_1b_2 - c_1c_2}{b_1c_2}\right).$$

And if you substitute here, you will get your Jacobian matrix to be

$$A = \begin{pmatrix} -c_1 - \dfrac{a_1b_2 - c_1c_2}{c_1} & -b_1\dfrac{c_2}{b_2} \\ \dfrac{b_2(a_1b_2 - c_1c_2)}{b_1c_2} & 0 \end{pmatrix}$$

Now for this, it is better to calculate the characteristic equation. So, it will be given by

$$|A - \lambda I| = 0$$

and we will be getting

$$\left(-c_1 - \frac{a_1b_2 - c_1c_2}{c_2} - \lambda\right)(-\lambda) + \frac{b_1c_2}{b_2}.\frac{b_2}{b_1c_2}(a_1b_2 - c_1c_2) = 0$$

$$\Rightarrow \lambda^2 + \left(c_1 + \frac{a_1b_2 - c_1c_2}{c_2}\right)\lambda + (a_1b_2 - c_1c_2) = 0$$

$$\Rightarrow \lambda^2 + \left(\frac{a_1b_2}{c_2}\right)\lambda + (a_1b_2 - c_1c_2) = 0$$

Now let us use the stability condition and if you recall that if your equation is the form

$$\lambda^2 + A_1\lambda + A_2 = 0,$$

if your characteristic equation is this, then condition that the system will be asymptotically stable is $A_1 > 0$ and $A_2 > 0$.

So, in this particular case, this is your A_1 and this is your A_2.

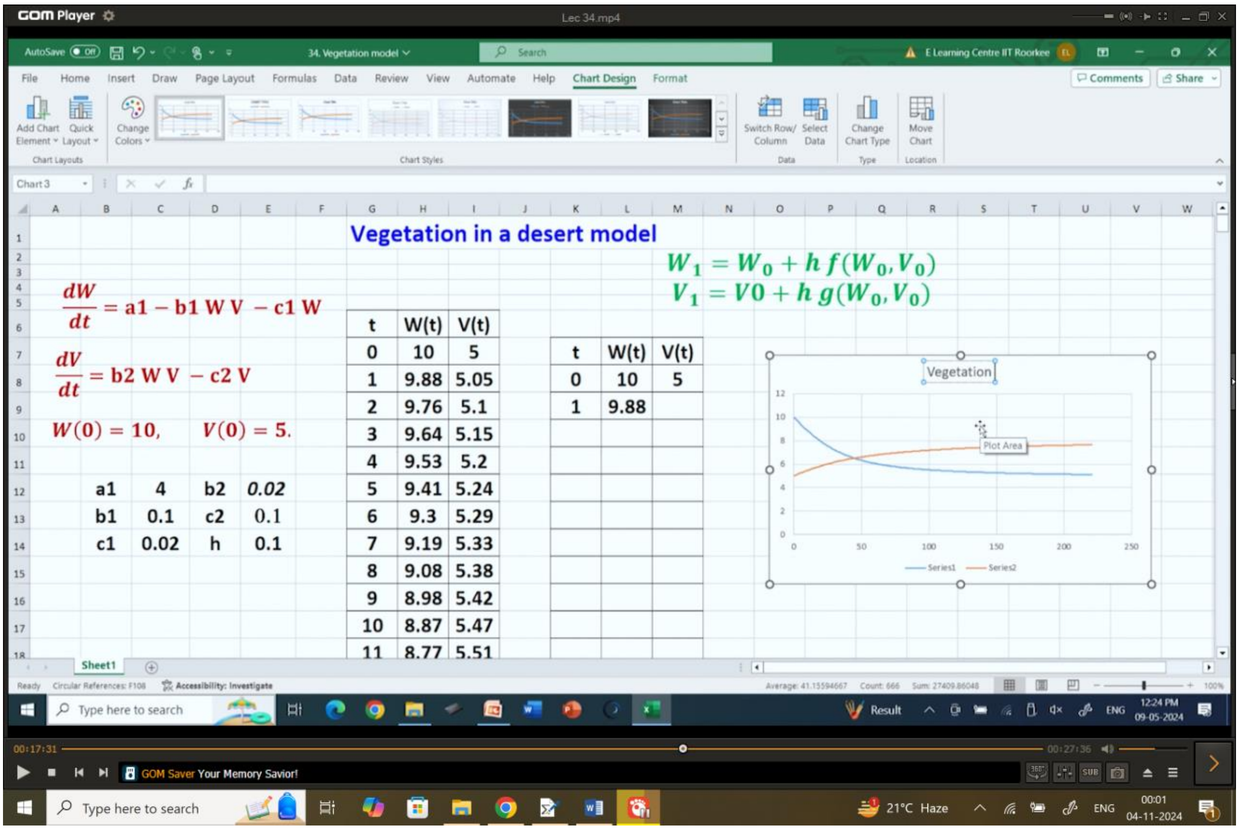
Now, for the existence of the solution, we have already taken that

$$a_1 b_2 - c_1 c_2 > 0.$$

So, $\frac{a_1 b_2}{c_2}$ is positive, $a_1 b_2 - c_1 c_2$ is also positive and this system is always asymptotically stable.

So, the conclusion is that whatever may be the values of this a_1, b_2, c_1, c_2, this conditions is always satisfied and your system is asymptotically stable.

Now, let us solve this model numerically for which we will be using this Microsoft Excel and let us see how this model behaves.



So, you can see that this is the equation, the initial condition will be the water source is some 10 units, the vegetation is 5 units and these are the a_1, b_1, a_2, b_2, c_1 and c_2 and the value of h.

So, to calculate this, I put this equal to this plus 1 and I drag this. Let us complete the rest of it. So, let me just increase the font size. Okay, a little bit bold here.

Okay, to calculate this, this is equal to, so we use this Euler's method as explained before, the initial value plus H which is the constant multiplied by a1 which is 4 again a constant minus b1 times W, which is this 10 multiplied by V minus c_1 which is again a constant and multiplied by W.

In the similar manner, you can calculate this V(t) and I already have this table here.

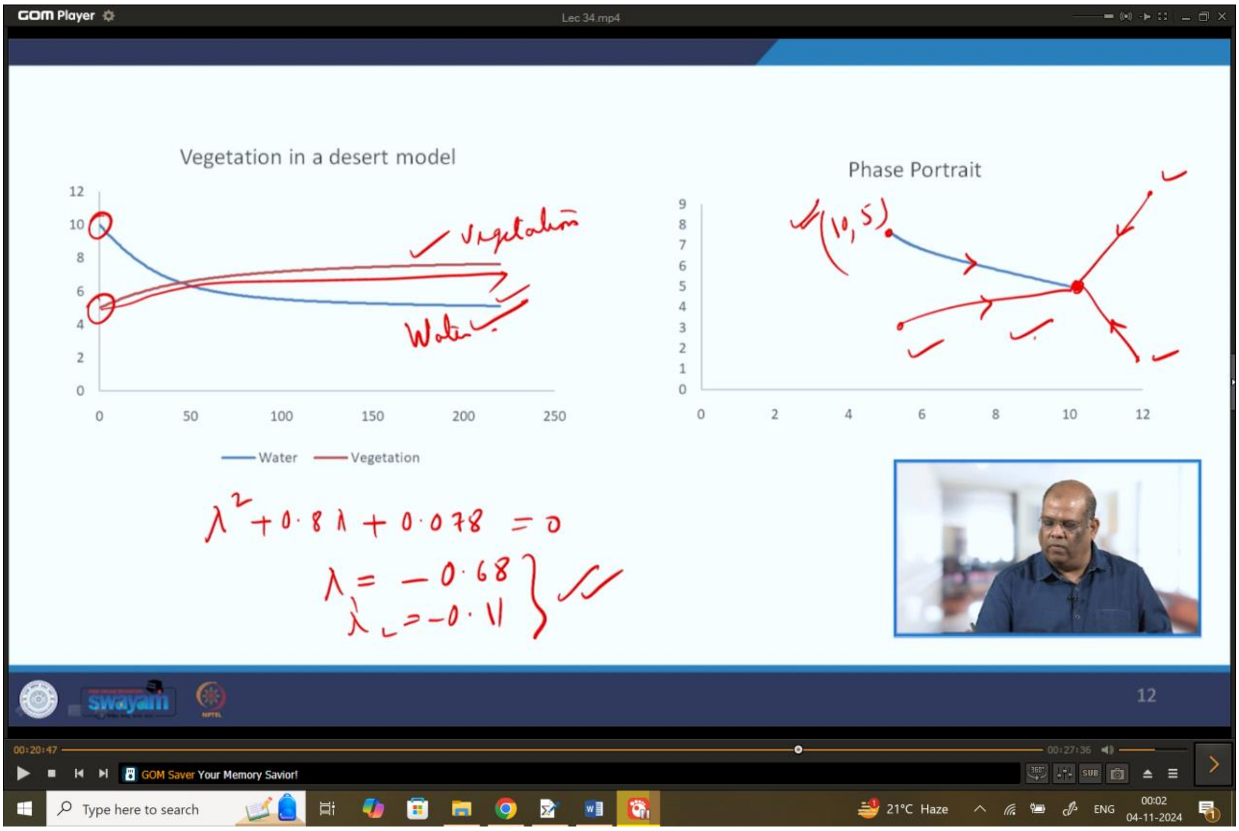
So, I am going to plot this. So, you highlight them till the calculated values, go to insert chart and this one. So, this is for the water and this is for the vegetation you can change the title vegetation in a desert model, this one go to select data highlight series edit and type water highlight this edit type vegetation and okay

And if you want to calculate the phase portrait, then you highlight these 2 values only and go calculate all the values.

Again go to insert, chart and draw and you get this. So, let me go back to the slides and let me explain these figures in detail.

So, once you plot them you get something like this. So, this blue one is the water, this is your water and this is the vegetation. So, what you see that initially the water has the high level but as it is used by the vegetation it is goes down but reaches a steady value.

And initially when the vegetation was not getting much water it was in the lower level but then once it starts getting water it is growing and it is growing and come to some steady value which is some sort of expected in a desert.



Now for the phase portrait if you calculate the eigenvalues numerically you will see that your characteristic equation will give

$$\lambda^2 + 0.8\lambda + 0.078 = 0$$

If you substitute all the values of a_1, b_1, c_1 and b_2, c_2 and if you solve this you will get your

$$\lambda = -0.68$$

and the second value is $\lambda = -0.11$

So, both are real and negative and you will be getting a stable node.

Now, if you know that the stable node which we have drawn it is something like this, like this, like this and all arrows are towards this point, but here we have taken only one initial condition which is 10 and 5.

So if you take many different initial conditions you are going to get this kind of curve which starts from different initial condition and always converts to this equilibrium point.

So basically you are getting from here it is concluded that you are getting a stable node which is again asymptotically stable and your figure will look something like this.

We now look into a modified model where I have added a competition term c_3V^2.

So, what we are going to look here into this vegetation model is when there is a competition among the vegetation and they will compete for water in this case.

So, this diagram is from the previous model when your c_3 is 0.

Now we will be plot another graph where we include the competition where c_3 is not equal to 0 but we put the value 0.05.

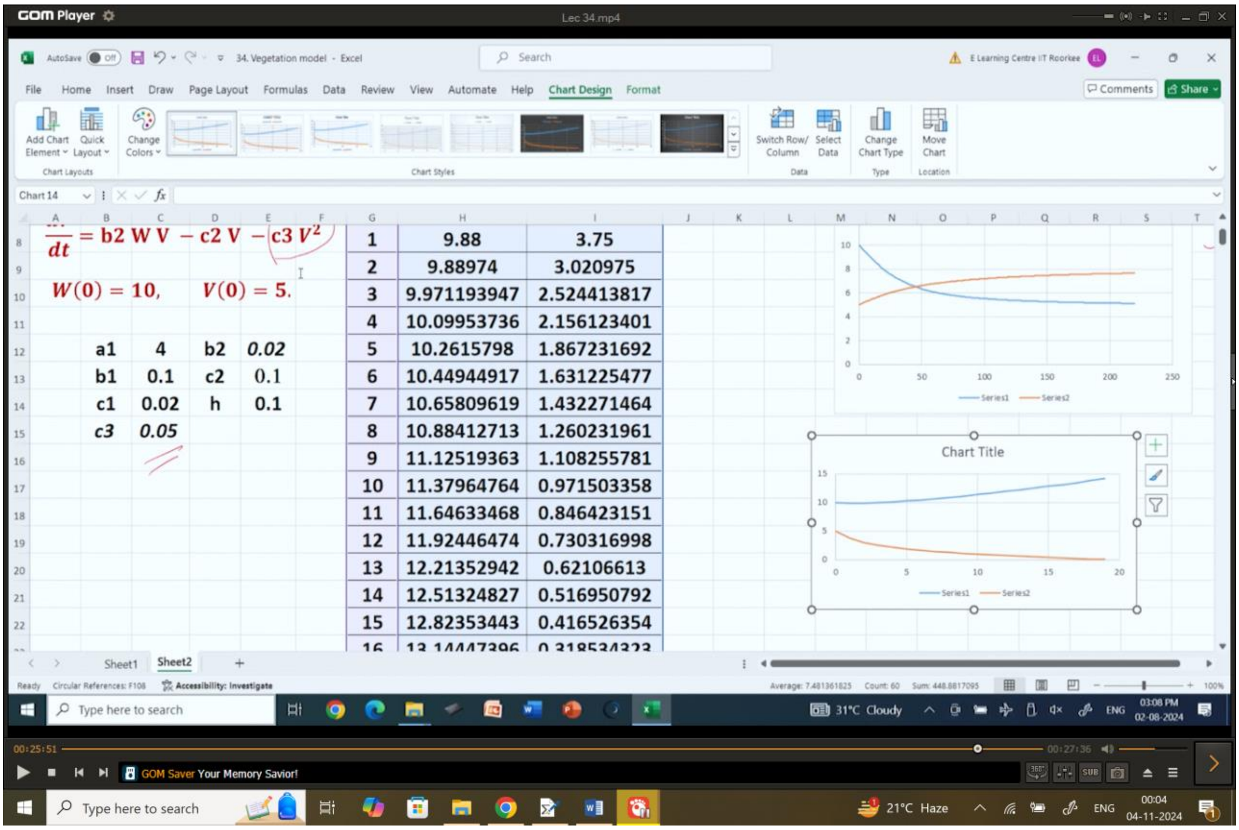
So as you can see I already have the equations with c_3V^2

So, this is denotes the competition term between the vegetation the initial condition I remain same the w0 is 10 v0 is 5 all this parameter values remain same only I have added c_3 which is 0.05.
Now, let us calculate these values and see what new dynamics you get.

So, this is equal to the initial value plus h which is a constant multiplied by a_1 again a constant minus b_1 again a constant, so I put the dollar multiplied by W multiplied by V minus c_1, again a constant times W.

And for the vegetation, this is equal to the initial value plus h which is a constant multiplied by b_2, again a constant so I put the dollar multiplied by W into V minus c_2 a constant minus c_3, another constant into V^2, the competition between the vegetation of same species and enter.
We calculate till 221.

So, let us first plot a bit positive values only.
Let us see what we get, go to insert the charts and the plots. So, this is the figure which we get.



So, if I compare with the previous case then what is happening in this particular case.

So what we see here due to this competition term the vegetation they do not get enough water and hence they die and hence the water level also increases.

But this is not what we really want but this is just a case where we take into consideration about the competition among the vegetation.

So the idea is that you if you plant the vegetation you keep a sufficient distance so that they do not compete against each other with the natural resources in this case the water which is very much needed for their survival.

So summing up we have taken a model for vegetation in a desert. In the initial case we do not have a competition term among the vegetation and we saw that the both water amount and the vegetation they coexist and there is a life of the vegetation in the desert.

But if you add a competition term we see that the vegetation they die and the water level goes up but ultimately they will also evaporate and then there is a no need.

So take home message is that if you want to plant this vegetation you have to be careful so that they do not compete among each other and hence they survive.

In my next lecture we will be talking about another important aspect of the social dynamics mainly the mathematical model of love affairs.

Till then bye-bye.