

EXCELing with Mathematical Modeling
Prof. Sandip Banerjee
Department of Mathematics
Indian Institute of Technology Roorkee (IITR)
Week – 07

Lecture – 35 (Mathematical Model of Love affairs: Layla-Majnun model)

Hello, welcome to the course EXCELing with Mathematical Modelling.

Today, we will be talking about an interesting social science problem, mathematical model of love affair. So, we call it a Laila Majnu model and this **n** is silent.

So, this was first introduced this kind of model by Strogers in his famous book non-linear dynamics and chaos and later, such kind of modified model was given by Rinaldi in this particular paper.

So, this model can be generalized for any famous couple like Romeo and Juliet, Cleopatra and Mark Anthony. We call it Majnu Laila model or Laila Majnu model which is taken from a romantic poem by Nizami of Ghazi and inspired by an Arab legend.

So, let us look into the model. So, it is a mathematical model of love affair.

So, how do you define love or how you explain love? So, that can be referred to a variety of different feelings, states and attitudes ranging from generic pleasure to intense interpersonal attraction.

So, we consider a mathematical model between two individuals in a romantic relationship we call it Majnu's, this **n** is silent and Laila, So, we consider they are modern Laila's and Majnu's.

So, this is the rate of love for Majnu and this is the rate of love for Laila. So, on what these love depends? So, the first is on his own feelings. So, suppose a boy sees a girl and he started liking her. So, he is encouraged by his own feelings. So, that is taken care by this first term.

The rate is a . Similarly, for the Laila's also. So, she is first encouraged by its own feeling which is at the rate c . And then, how much he or she is encouraged from the feeling of the other opposite sex.

So, in this case what kind of signals this the girl sent suppose you look at her whether she is looking back you asked her for coffee whether she is accepting or whether she is declining that sort of encouragement is given by this particular term which is happening at a rate b . The same goes for the Laila's also that this is the part where she is encouraged by her own feeling and this is the part where she is encouraged by Majnu's response to her feeling.

Now this is the interesting part which we name them as say some function f and g . So this is some sort of appeal.

Layla-Majnun model

The word love can refer to a variety of different feelings, states, and attitudes ranging from generic pleasure to intense interpersonal attraction.

We consider a mathematical model of love between two individuals in a romantic relationship.

Majnu

$$\frac{d}{dt} M = a M + b L + f$$

Layla

$$\frac{d}{dt} L = c L + d M + g$$

appeal

g

So, when in the very first you a person say the Majnu's look into the Laila's, so he must be looking how she looks or how she dresses, how she is carrying over any background from the family. So, those kinds are taken care by this appeal f and in the similar manner for the Laila's other than the looks, they look for some social status, whether he owns a property or a car, how much money he can spend on a date. So, those kind of things are taken care by this function g . So, if you now want to look at a mathematical model, this will look something like this:

$$\frac{d}{dt} M(t) = a M(t) + b L(t) + f, \quad \frac{d}{dt} L(t) = c L(t) + d M(t) + g$$

Here, $M(t)$ represents Majnu's love for Laila and $L(t)$ represents Laila's love for Majnu,

$a \rightarrow$ Majnu is encouraged by his own feeling

$b \rightarrow$ Majnu is encouraged by Laila's feeling

$c \rightarrow$ Laila is encouraged by her own feeling

$d \rightarrow$ Laila is encouraged by Majnu's feeling.

Please note that I have not put any sign on a , b , c , d , f and g . So, the negative sign on a will mean that Majnu is discouraged by his own feeling. So, the positive and the negative sign here represents the love and the hate relationship.

Now, let us look into the analysis. So, as explained before this a, b, they are the encouragement of particular Majnu for his own feelings or encouraged by the Laila's feelings.

And similarly, c and d is the rate at which Laila is encouraged by her own feeling and encouraged by Majnu's feelings. So as explained before this f, it is the reaction of Majnu towards Laila's appeal. For example, how she looks, how she dresses, the way she walks and talks and show also her social background.

Similarly, g is the reaction of Laila towards Majnu's appeal. For example, other than the looks, whether he owns a bike or a car, the amount of money he can spend on a date, whether he owns a property or not, also the social background and etc. This can be also be positive and can be negative.

Now, let us look into the analysis.

$$\frac{d}{dt}M(t) = a M(t) + b L(t) + f = F_1$$

$$\frac{d}{dt}L(t) = c L(t) + d M(t) + g = F_2$$

If I want the equilibrium solution,

$$a M(t) + b L(t) + f = 0, \quad c L(t) + d M(t) + g = 0$$

You can solve this by many methods. I use method of cross multiplication.

$$\frac{M}{(cf - bg)} = \frac{L}{df - ag} = \frac{1}{ac - bd}$$

$$\Rightarrow M^* = \frac{cf - bg}{ac - bd} > 0 \text{ and } L^* = \frac{df - ag}{ac - bd} > 0,$$

provided, $cf - bg > 0, df - ag > 0$ and $ac - bd > 0$

So, for stability analysis, you have to find the Jacobian matrix, that is,

$$A = \begin{pmatrix} \frac{\partial F_1}{\partial M} & \frac{\partial F_1}{\partial L} \\ \frac{\partial F_2}{\partial M} & \frac{\partial F_2}{\partial L} \end{pmatrix} = \begin{pmatrix} a & b \\ d & c \end{pmatrix}$$

The eigen-values are given by

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} a - \lambda & b \\ d & c - \lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - (a + c)\lambda + ac - bd = 0.$$

Now if I want this to be stable, both my eigenvalues should be negative. So this is the sum of the roots, this is the product of the roots. If both the eigenvalues are negative, the sum of the root has to be less than zero and the product of the root has to be greater than zero.

So this is the condition that this particular system representing the Laila Majnu model will be stable if $a + c < 0$, $ac - bd > 0$

Now let us look into various cases.

Case I: ($a = 0, b = 3, f = 0, c = 0, d = 2, g = 0$)

With these values of the parameters, we get the simplified equations as

$$\frac{d}{dt}M(t) = 3L(t), \quad \frac{d}{dt}L(t) = 2M(t)$$

Now what does this mean?

So, this means that the rate of change of Majnu's love that is this one depends on the encouragement of Laila and the rate of change of Laila's love will depend on the encouragement of Majnu.

If I want to see the dynamics, first I have to find the eigenvalues

$$|A - \lambda I| = \begin{vmatrix} 0 - \lambda & 3 \\ 2 & 0 - \lambda \end{vmatrix} = 0$$

and hence, the characteristic equation is

$$\lambda^2 - 6 = 0 \Rightarrow \lambda = \pm\sqrt{6}$$

So the solution will be

$$M(t) = A_1 e^{\sqrt{6}t} + A_2 e^{-\sqrt{6}t}$$

$$L(t) = \frac{d}{3dt}M(t) = \frac{\sqrt{6}(A_1 e^{\sqrt{6}t} - A_2 e^{-\sqrt{6}t})}{3}$$

As your t becomes large from here you see that your

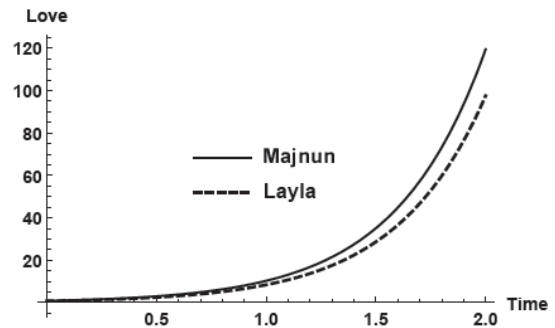
$$M(t) = A_1 e^{\sqrt{6}t}$$

and similarly your

$$L(t) = \frac{\sqrt{6}}{3} A_1 e^{\sqrt{6}t}$$

So if we plot this with some initial conditions so you will be getting something like this that is some sort of exponential solutions, which is exponentially increase.

So, this is what we called exponential love. So, this kind of love generally it is seen at the school level or it is very early teen level where they say I love you exponentially, not a very realistic situation but it happens and hence this kind of dynamics you get from this love affair model when you put this particular set.



So, what you are trying to do is you are choosing this the value of a, b, c, d's in such a manner that you can relate these cases to some real life scenario.

Case II: ($a = -2, b = 1, f = 2, c = -1, d = 1, g = 1$)

$$\frac{d}{dt}M(t) = -2M(t) + L(t) + 2, \quad \frac{d}{dt}L(t) = -L(t) + M(t) + 1$$

Now, what does this mean?

So, you can see that the rate of change of love for Majnu, it is a negative and then a positive. Similarly, for Laila also, it is first negative and then positive. So, some sort of forgetfulness is appearing here and this due to which the partner lose interest due to his or her own feeling.

However, when there is a reciprocate from the other partner then that love rekindles. So, this is the case in both this equation and both of them have the positive appeal which also attracts towards each other.

So, let us see what kind of dynamics this gives when you find the eigenvalues. So, your coefficient matrix

$$A = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}.$$

The eigen-values are given by

$$|A - \lambda I| = 0 \Rightarrow (\lambda + 2)(\lambda + 1) - 1 = 0$$

$$\Rightarrow \lambda^2 + 3\lambda + 1 = 0$$

$$\Rightarrow \lambda = \frac{-3 \pm 2.24}{2}$$

Since, $\lambda_1, \lambda_2 < 0$, the system is stable.

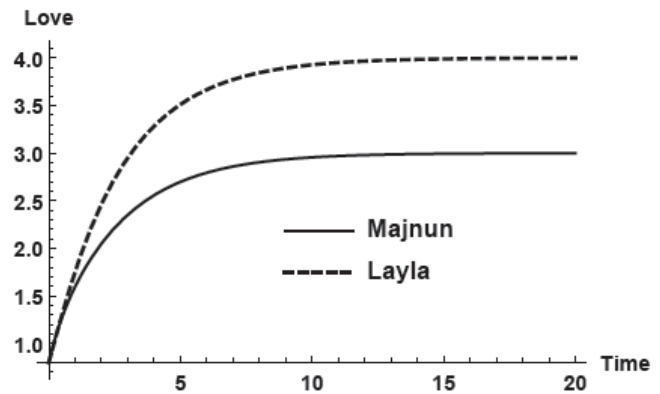
Now, let us look into the interpretation of this model,

$$\frac{d}{dt}M(t) = -2M(t) + L(t) + 2, \quad \frac{d}{dt}L(t) = -L(t) + M(t) + 1.$$

So, as I told you that $-2M(t)$ this will mean that some forgetful or forgetting process which give rise to loss of interest in the partners. And this kind of behavior generally is seen in a long distance relationship where some sort of out of sight, out of mind kind of thing happens and the partners lose interest from their own feeling. If there is a response from the other partners, then the love blooms again and your relationship continues and this instinct is sensitive to partner's appeal, as both of them are positive.

So, in this if you plot the graph, you will get this kind of dynamics which says that you start with some sort of relationship and slowly their love grows and ultimately it becomes a plateau.

So, the take home conclusion is that this kind of model it describes a romantic relationship for cautious lovers with a positive appeal which are capable of establishing a steady love relationship as you can see from the graph.



Case III: ($a = 0, b = 3, f = 0, c = 0, d = -2, g = 0$)

So, if you pluck all these values here, you get your differential equation like

$$\frac{d}{dt}M(t) = 3L(t), \quad \frac{d}{dt}L(t) = -2M(t)$$

Now, what does that mean? So, here it means that the love of Majnu is encouraged by the feeling of Laila because it is positive. We use the word Laila is a fickle lover. So, you can see that the more Majnu loves Laila but since it is a negative sign she runs away and hide.

So, the appeal from the person Majnu does not affect Laila. So, basically some sort of heading kind of relationship and we call it is a Fickle number.

So, if I want to see the dynamics of this kind of model, so first I have to find the eigenvalues and your coefficient matrix or Jacobian matrix will be this

$$A = \begin{pmatrix} 0 & 3 \\ -2 & 0 \end{pmatrix}.$$

The eigen-values are given by

$$\begin{vmatrix} 0 - \lambda & 3 \\ -2 & 0 - \lambda \end{vmatrix} = 0 \Rightarrow \lambda = \pm i\sqrt{6}$$

The general solution is

$$M(t) = C_1 \cos(\sqrt{6}t) + C_2 \sin(\sqrt{6}t)$$

$$L(t) = \frac{1}{3} \frac{dM(t)}{dt} = \frac{\sqrt{6}(C_2 \cos(\sqrt{6}t) - C_1 \sin(\sqrt{6}t))}{3}$$

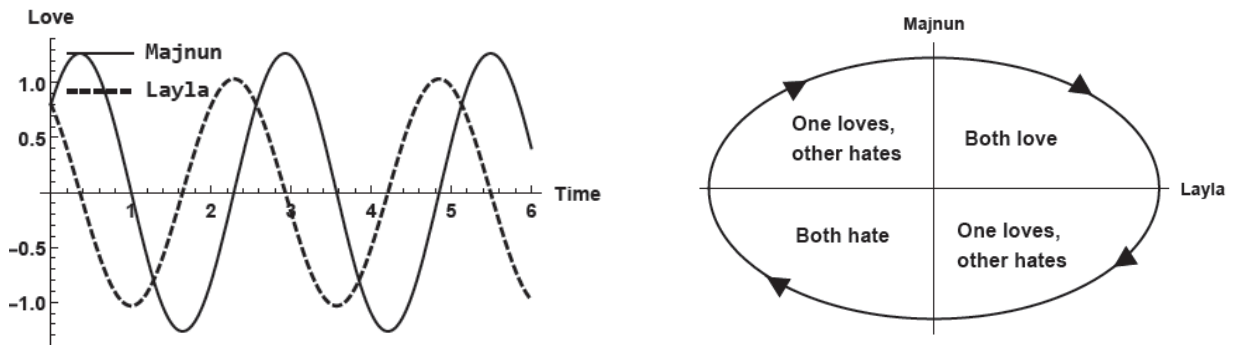
where C_1 and C_2 are arbitrary constants. Since, it contains the sine and cosine obviously there will be a periodic solution.

Suppose you want to put an initial condition, say, $M(0) = 1, L(0) = 0$, then

$$M(t) = \cos(\sqrt{6}t), L(t) = -\sqrt{6}/3 \sin(\sqrt{6}t)$$

So if you plot them, so once more time what you see here is that the love of Majlou is encouraged by the feeling of Laila. So, if she loves him back, his love warms up and if she doesn't, he grows cold.

On the other hand, your Laila is a fickle lover. So, the more Majnu loves her, she runs away and hides. Lila finds it strangely attractive when Majnu gets discouraged and backs off and that explains this negative sign. So, if you plot this you will get a curve like this.



So, you can see that there is a love-hate kind of relationship that goes on between Laila and Majnu. Now, if you want to eliminate t from here, you will get something like

$$M(t)^2 + \frac{9}{6}L(t) = 1$$

So some sort of ellipse which you get. So if you plot that and then try to explain, you can see that your Majnu and Laila, their love will be positive in only this one quadrant. So one quarter of the time is where both of them love each other. However, in the all the three quarters there is a love-hate relationship, mainly, this is love and hate, this is both hate, this is again love and hate and this is due to the fickleness of Laila.

So, sometimes this kind of relationships are also seen and this captures that kind of dynamics.

Case IV: $(a = 3, b = 2, f = 0, c = -3, d = -2, g = 0)$

$$\frac{d}{dt}M(t) = 3M(t) + 2L(t), \quad \frac{d}{dt}L(t) = -3L(t) - 2M(t)$$

In this particular case you can see that this is totally positive, that is, Majnu's love depends on his own feeling and also from the feeling from Laila. So, in both these cases it is positive. However, in case of Laila neither she is encouraged by her own feeling neither she is encouraged by the feeling from Laila.

So, lots of cases happen like this that when one partner or one boy or girl started liking the other one is not necessarily that they get reciprocate back.

So, if you want in this particular case I have taken the appeals to be zero.

And if you want to calculate the eigenvalues, your coefficient matrix is

$$A = \begin{pmatrix} 3 & 2 \\ -2 & -3 \end{pmatrix}$$

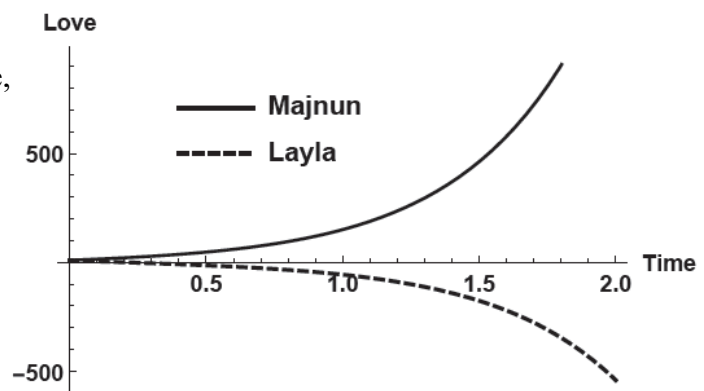
The eigen-values are given by

$$\begin{aligned} |A - \lambda I| = 0 &\Rightarrow \begin{vmatrix} 3 - \lambda & 2 \\ -2 & -3 - \lambda \end{vmatrix} = 0 \\ \Rightarrow (\lambda - 3)(\lambda + 3) + 4 = 0 &\Rightarrow \lambda^2 - 9 + 4 = 0 \\ \Rightarrow \lambda = \pm\sqrt{5} \end{aligned}$$

So one eigenvalue is positive and the other is negative. What you will be getting is a saddle point.

And if you plot them, you will get something like this. So, which means that when two peoples they are exactly opposite, they end up in an odd kind of relationship where one loves and the other hates.

So, with this we come to an end for this lecture on social science where we have taken a model for love affairs and you see that in these four various cases, we are able to capture many of the dynamics that is seen in the real life scenario of love affair.



In my next lecture, we will be shifting to discrete models with the help of difference equation and till then bye-bye.