EXCELing with Mathematical Modeling

Prof. Sandip Banerjee
Department of Mathematics
Indian Institute of Technology Roorkee (IITR)

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Lecture – 36 (Discrete Models: Difference equations I)

Hello welcome to the course EXCELing with mathematical modelling.

So far we have been discussing the models which are continuous.

So, today we switch to discrete models. Now, what are discrete models?

So, in continuous type modeling which we often implement, we use differential equations.

Sometimes it is not the best approach.

You will face many situations where you are describing the state of a process for all real time values and you will find that the process is too long if you use the continuous model or the process is too much calculative if you use the continuous model.

So, in such cases we avoid the continuous one and switch to discrete models.

So, sometimes even due to the complexity of the model, we avoid the continuous one obviously depends from model to model and then we use this discrete time mathematical model.

Let us start with an example which will clear it.

Say suppose you consider bouncing of a ball.

So, if I take the situation that it starts from some initial height, the first bounce the second bounds and so on.

So, this is the height and this is the time. So, this is t_1, this is t_2, this is t_3, and so on.

We know that the differential equation which will satisfy this motion is given by

$$\frac{d^2s}{dt^2} = -g$$

and if we integrate it, we will get

$$s(t) = -\frac{1}{2}gt^2 + v_0t + s_0\ , \quad t_0 \leq t \leq t_1,$$

means in this particular interval and your height are these.

Now given this v_0 and s_0 that is the initial velocity and the initial height, we will be able to calculate the time t_1.

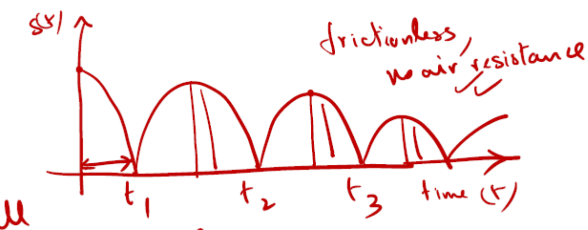
Let us consider the second bounds. You have the same equation

$$s(t) = -\frac{1}{2}gt^2 + v_1 t + s_1 , \qquad t_1 \le t \le t_2$$

but now you have a different constant or different initial values.

Now it is v_1 and s_1 which depends on v_0 and s_0 and also we have assumed that this is frictionless or no air resistance but we assume one thing that the ball loses one-fourth of its energy after the first bounce.

So, this means if in the second bounce it will loses a one-fourth of the energy that is left and the process continues.



Discrete Process
Sequence of heights.
The ball must retain $\frac{3}{4}$ of its energy from one max-height to the next max-height.
It must rise back to the max. height

The ball loses $\frac{1}{4}$ of its energy after the first bounce.

$$\frac{d^2s}{dt^2} = -g$$

$$s(t) = -\frac{1}{2}gt^2 + v_0 t + s_0 \quad t_0 \le t \le t_1$$

v_0 $\quad s_0$ $\quad t_1$

$$s(t) = -\frac{1}{2}gt^2 + v_1 t + s_1 , \quad t_1 \le t \le t_2$$

$$s(t) = -\frac{1}{2}gt^2 + v_n t + s_n , \quad t_n \le t \le t_{n+1}$$

nth time the ball hits the ground.

$$S_1 = \frac{3}{4}\boxed{S_0}$$

$$S_2 = \frac{3}{4} S_1$$

$$\boxed{S_{n+1} = \frac{3}{4} S_n} \quad n = 0, 1, 2, \ldots$$

So, in each of the step you have to calculate this v_1 and s_1 which depends on v_0 and s_0 and also on this particular property.

So, in general if we write this equation will be treated like this

$$s(t) = -\frac{1}{2}gt^2 + v_n t + s_n , \qquad t_n \le t \le t_{n+1}$$

So at t_n is the nth time the ball hits the ground.

So if you see that you have a substantial calculations of v_i's and s_i's to get the height that the ball reaches after each bounce.

So this is a continuous time model.

Now let us see what happens if we switch to our discrete model.

The same situation only this case we consider the discrete process.

So we concentrate on the sequence of heights that is this height, this height and this height and so on.

So, the ball loses one-fourth of its energy, there is no air resistance.

So, the ball must retain three-fourth of its energy from one maximum height to the next maximum height.

And from physics this will mean that with this energy it must rise back to the maximum height.

And what is going to be that height after the first bounce?

It is $S_1 = \frac{3}{4} S_0$.

This is after the first bounce.

After the second bounce, it is $S_2 = \frac{3}{4} S_1$.and so on.

So, $S_{n+1} = \frac{3}{4} S_n$.where n starts from 0, 1, 2 and so on and S_0 is your initial height.

So you can see that if we process the same thing in a discrete way then your calculation is much much simpler and you can get the answer in the very simple way and very easy to understand whereas in this continuous process it becomes a bit cubism and it is a long calculation.

So, that is why sometimes it is preferable that you use the discrete way of modeling or discrete process than the continuous one.

Now, what are the tools you will be using while modeling this discrete process?

So, in the continuous one you know we use differential equation, in discrete case we will be using difference equation.

Now you may not be familiar with difference equation. So, let us go through it. So, this difference equation it is sometimes referred to as the recurrence relation.

So, it is an equation which involves the difference and one can define a difference equation as a sequence of numbers that are generated recursively using a rule to the previous number in the sequence.

Now, let us see what it means. We take an example.

For example, you considered this sequence of numbers, say, 0, 1, 3, 6, 10, 15, 21.

These are known as sequence of triangular numbers.

and they are obtained from the difference equation,

$$T_{n+1} - T_n + (n+1) \quad \text{with} \quad T_0 = 0.$$

So, this is a recurrence relation or a difference equation which generates a sequence of numbers known as the triangular numbers.

So if I substitute n = 0,

$$T_1 = T_0 + 1 = 0 + 1$$

which matches with this.

You put n = 1,

$$T_2 = T_1 + (1+1) = 1 + 2 = 3$$

matches with this and so on.

Similarly, you may have heard the famous Fibonacci sequence.

So, here the recurrence relation is

$$F_n = F_{n-1} + F_{n-2}\,, F_0 = 0, F_1 = 1,$$

the sequence is 0, 1,1,2, 3, 5, 8, 13, 21 and so on.

So, as usual if you want to generate the Fibonacci sequence you have to use this recurrence relation or this difference equation and here two initial values.

So, I will start with n = 2 and if you put the value here

$$F_2 = F_1 + F_0 = 0 + 1 = 1.$$

I put n = 3, I get

$$F_3 = F_2 + F_1 = 1 + 1 = 2,$$

which matches with this and you can generate the rest of the sequence.

So, let us now take a few difference equation and see how we can find their solution.

But before that, how a difference equation is formulated?

So, suppose you have a relation like this where your C is an arbitrary constant.

It is exactly like differential equation suppose you have $y = cx.$ and if I ask you, you find the differential equation.

So what you will do?

You will eliminate this arbitrary constant c and find a relation between x, y and $\frac{dy}{dx}$.

It is exactly the same way we do, but here we will not differentiate. What we will do is we take one increment.

$$u_n = cn + 5, \quad \text{where c is arbitrary constant} \qquad \dots (1)$$

$$u_{n+1} = c(n+1) + 5 \qquad \dots (2)$$

So now you eliminate c for equation 1 and equation 2. So you calculate c and substitute it here.

So that will give you

$$\Rightarrow u_{n+1} = \frac{u_n - 5}{n}(n+1) + 5$$

$$\Rightarrow \quad u_{n+1} = \frac{nu_n + u_n - 5n - 5 + 5n}{n}$$

$$\Rightarrow \quad nu_{n+1} = (n+1)u_n - 5$$

We now take another example. You have

$$u_n = A2^n + B(-3)^n \qquad \dots (1)$$

where A, B are arbitrary constants. We have to obtain the difference equation by eliminating the arbitrary constants A and B.

So, what you do is you first write what is

$$u_{n+1} = A2^{n+1} + B(-3)^{n+1} \qquad \dots . (2)$$

And since we have two arbitrary constants you have to again write

$$u_{n+2} = A2^{n+2} + B(-3)^{n+2} \qquad \dots . (3)$$

So, I will eliminate and find out the values of A and B from (2) and (3) and substitute it in (1).

So, (2) and (3) can be rewritten as

$$u_{n+1} = 2A2^n - 3B(-3)^n \qquad \ldots (4)$$

$$u_{n+2} = 4A2^n + 9B(-3)^n \qquad \ldots . (5)$$

Now, we are to solve these two equations for A and B. So, you multiply equation (4) by 2 and equation (5) by 1, to obtain,

$$2u_{n+1} = 4A2^n - 6B(-3)^n$$

$$u_{n+2} = 4A2^n + 9B(-3)^n$$

Subtracting we get,

$$B = \frac{u_{n+2} - 2u_{n+1}}{15(-3)^n}$$

Now you can substitute the value of B here and calculate the value of A and if you do that you will get

$$u_{n+1} = 2A2^n - 3B(-3)^n = 2A2^n - \frac{3(u_{n+2} - 2u_{n+1})}{15}$$

$$\Rightarrow \; 2A2^n \; = \; u_{n+1} + \frac{u_{n+2} - 2u_{n+!}}{5}$$
$$= \; \frac{5u_{n+1} + u_{n+2} - 2u_{n+1}}{5}$$

$$\Rightarrow A = \frac{3u_{n+1} + u_{n+2}}{10(2)^n}$$

So, now I have to substitute this value of A and this value of B in equation (1) to get the required difference equation and if I do that, I will get

$$u_n = \frac{3u_{n+1} + u_{n+2}}{10} + \frac{(u_{n+2} - 2u_{n+1})}{15}.$$

So, if I simplify them, I will get this as

$$u_{n+2} + u_{n+1} - 6u_n = 0.$$

This is the required difference equation after eliminating the arbitrary constants A and B. Now, this difference equation can be linear or it can be nonlinear.

So, this is a linear equation, this is a nonlinear equation and the nonlinearity is here.

Difference Equation

Linear: $x_{n+1} = 2x_n - 7$

Non-linear: $x_{n+1} = 2x_n^2 - 7$

Consider the equation

$$x_{n+1} = \alpha_n x_n + \beta_n$$

If α_n is equal to α and β_n is equal to β, where both of them are constants, then we say that this difference equation is autonomous.

However, if they are dependent on n, then it is called non-autonomous.

So, an autonomous equation is $x_{n+!} = 2x_n + 7$ and if I want the non-autonomous one, I will make this as a $x_{n+1} = \frac{2x_n}{n} + \frac{7}{n+1}$.

So, this is a non-autonomous one where the constants are dependent on n. Next, we solve two special difference equation which we will be requiring in our modelling, modelling of discrete cases.

Consider,

$$c_0 u_n + c_1 u_{n-1} + c_2 u_{n-2} = f(n).$$

So, this is called a linear difference equation with constant coefficients.

If $f(n) = 0$, we say that the equation is homogeneous otherwise non-homogeneous, and the order of this difference equation is the largest argument minus the smallest.

Now what is the largest argument?

That is this n. What is the smallest argument? That is n – 2.

So, order $= n - (n - 2) = 2$. So order of this difference equation is 2.

Now let us find the solution of homogeneous equation.

So the equation is of the form $u_n = ku_{n-1} \implies u_n - ku_{n-1} = 0$.

So as you can see there is no function of n and hence it is a homogeneous equation.

You can put this in this form also.

So there is no f(n) here and that value of f(n) is 0.

So this is a homogeneous equation.

Solution of Homogeneous Equation

$$\boxed{u_n = k\,u_{n-1}} \qquad u_n - k\,u_{n-1} = \boxed{0}$$

$n=1$, $\quad u_1 = k\,u_0$

$n=2$, $\quad u_2 = k\,u_1 = k\,k\,u_0 = \boxed{k^2 u_0}$

$n=3$, $\quad u_3 = k\,u_2 = k \cdot k^2 u_0 = k^3 u_0$

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$$\boxed{u_n = k\,u_{n-1} = k^n u_0}$$

The general solution of the difference equation $u_n = k\,u_{n-1}$ is $\boxed{u_n = C k^n}$, where C is the arbitrary constant.

So, basically what you need to remember is that if your difference equation is of the $u_n = ku_{n-1}$, then the solution is $u_n = Ck^n$, where C is an arbitrary constant.

So, with this we stop today for the first part of this difference equation.

In the second part, we will be taking more examples and more different kind of difference equation.

Till then, bye-bye.