EXCELing with Mathematical Modeling

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Lecture – 37 (Discrete Models: Difference equations II)

Hello welcome to the course EXCELing with mathematical modelling.

In my previous lecture, we were talking about the discrete models and the tools that we will be using to solve those discrete models, mainly, the difference equations.

In this lecture, we continue with this.

So, the next difference equation that we will be solving is

$$u_n = au_{n-1} + b.$$

If you recall, we have already solved the difference equation of the form

$$u_n = au_{n-1}$$

and we have shown that the general solution is

$$u_n = C\,k^n,$$

where C is an arbitrary constant.

Now we have the same form but only now we have a constant b. Let us see what is the solution of this particular difference equation. So, if you write this

$$u_n = au_{n-1} + b$$

So, what I am going to do is I will be replacing

$$u_{n-1} = au_{n-2} + b\ ,$$

$$u_{n-2} = au_{n-3} + b,$$

and so on.

Therefore,

$$u_n = \ au_{n-1} + b\ = a(au_{n-2} + b) + b$$

$$= a^2u_{n-2} + ab + b\ =\ u_{n-2} + b(a + 1)$$

Now, in the next step I will replace this u_{n-2} and you will get

$$u_n = a^2 u_{n-2} + b(a+1)$$

$$= a^2(a u_{n-3} + b) + b(a+1)$$

$$= a^3 u_{n-3} + a^2 b + b(a+1)$$

$$= a^3 u_{n-3} + b(a^2 + a + 1)$$

Now, if we continue this, we get

$$u_n = a^n u_0 + b(a^{n-1} + a^{n-2} + \cdots + a^2 + a + 1) \quad ---- \ (A)$$

Now this is a known GP series and if I want to find the sum of this, it is

$$u_n = a^n u_0 + b\left(\frac{a^n - 1}{a - 1}\right) \ , a > 1$$

$$u_n = a^n u_0 + b\left(\frac{1 - a^n}{1 - a}\right) \ , a < 1$$

And if a = 1, then you just directly substitute in (A) and get

$$u_n = u_0 + nb, \qquad a = 1.$$

So, again you have to remember these two form of difference equation. One is $u_n = a u_{n-1}$ and another is $u_n = a u_{n-1} + b$.

Most of the discrete models that we will be doing, will have a formation of this kind of difference equation and if you remember the solution you can quickly write them and go for further analysis otherwise you have to solve them in this methods that has been shown here.

The next difference equation that we will be solving is **Second order homogeneous linear difference equations,** which is of the form

$$a_0 u_n + a_1 u_{n-1} + \cdots + a_n u_{n-2} = 0. \quad \ldots (1)$$

Let $u_n = c\,k^n$, $(c \neq 0)$ be a solution of equation (1). So you substitute it here and you will get

$$a_0 c k^n + \ a_1 c k^{n-1} + a_2 c k^{n-2} = 0$$

$$\Rightarrow c k^{n-2}(a_0 k^2 + a_1 k + a_2) = 0 \ \ (c k^{n-2} \neq 0)$$

$$\Rightarrow a_0 k^2 + a_1 k + a_2 = 0$$

This is called the auxiliary equation.

And you have to solve this auxiliary equation to get the roots of this equation. It is just a quadratic equation you can use any formula to get

$$k = \frac{-a_1 \pm \sqrt{{a_1}^2 - 4a_0a_1}}{2a_0}.$$

So, let the roots be m_1 and m_2.

Case 1: m_1 and m_2 are real and distinct.

Then, the solution will be the general solution will be of the form

$$c_1 m_1^n + c_2 m_2^n,$$

where your c_1 and c_2 are arbitrary constants.

Case 2: m_1 and m_2 are real and equal; $m_1 = m_2 = m$.

The general solution is given by

$$(c_1 + c_2 n) m^n,$$

where your c_1 and c_2 are arbitrary constants.

And finally, the

Case 3: Roots are imaginary.

Since it is a quadratic equation, it will have two roots and if the roots are imaginary, they will appear in conjugate pairs. Let $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$. So, the solution will be of the form

$$c_1(\alpha + i\beta\ )^n + c_2(\alpha - i\beta)^n \ \ldots (B)$$

But, we put it in a more simplified form, we put $\alpha = r\cos\theta$ and $\beta = r\sin\theta$.

If you substitute it in (B), you will get

$$c_1 r^n(\cos\theta + i\ \sin\theta)^n + c_2 r^n(\cos\theta - i\ \sin\theta)^n$$

Using De Moivre's theorem, we get

$$c_1 r^n(\cos n\theta + i\ \sin n\theta) + c_2 r^n(\cos n\theta - i\ \sin n\theta)$$

$$= r^n[(c_1 + c_2)\cos n\theta + (ic_1 - ic_2)\sin n\theta]$$

So if I take

$$= r^n[(c_1 + c_2)cosn\theta + (ic_1 - ic_2)sin\theta]$$

I replace this whole thing by another constant, then

$$= r^n(A_1cos\theta + A_2sin\theta)$$

where this r is given by

$$r^2 = \alpha^2 + \beta^2 \Longrightarrow r = \sqrt{\alpha^2 + \beta^2}$$

$$\text{and} \;\; \tan\theta = \frac{\beta}{\alpha} \Longrightarrow \theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$$

We now take an example where we solve the equation

$$u_{n+2} - u_{n+1} - 6u_n = 0 \quad --- (1).$$

Initial conditions: $u_0 = 3, u_1 = 4$

Let. $u_n = ck^n$ $(c \neq 0)$ be a solution of (1). So, you substitute it and get

$$ck^{n+2} - ck^{n+1} - 6ck^n = 0 \Longrightarrow ck^n(k^2 - k - 6) = 0$$

$$\Longrightarrow k^2 - k - 6 = 0 \Longrightarrow k = 3, -2.$$

So, the roots are real and distinct and this implies general solution.

$$u_n = A(-2)^n + B(3)^n,$$

where A and B are arbitrary constants.

Now, these two conditions are given which we have to use to find the values of A and B. So,

$$u_n = A(-2)^n + B(3)^n$$

$$u_0 = 3, u_1 = 4$$

which implies

$$A(-2)^0 + B(3)^0 = 3 \Longrightarrow A + B = 3 \;\; \dots (2)$$

and

$$A(-2) + B(3) = 4 \;\; \Longrightarrow \;\; -2A + 3B = 4 \;\; \dots (3)$$

We now solve (2) and (3). So we multiply equation(2) by 2 and (3) equation by 1 and add them. So we get

$$2A + 2B = 6$$

$$-2A + 3B = 4$$

Adding we get

$$5B = 10 \implies B = 2.$$

Therefore,

$$A = \frac{3B - 4}{2} = 1,$$

and you get your required solution as

$$u_n = (-2)^n + 2(3^n)$$

Let us move on to another example,

$$u_{n+2} + 4u_{n+1} + 4u_n = 0 \quad \dots (3)$$

Initial condition: $u_0 = 2, u_1 = -6$

Let $u_n = ck^n$ $(c \neq 0)$ be a solution of (3). Then

$$ck^n(k^2 + 4k + 4) = 0$$

So, clearly $c \neq 0, k^n \neq 0$

$$\implies (k + 2)^2 = 0 \implies k = -2, -2.$$

We have a repeated root. So, if you have a repeated root then the general solution is given by

$$u_n = (A + Bn)(-2)^n$$

So, you are given $u_0 = 2$, this will imply

$$(A + B \times 0)(-2)^0 = 2 \implies A = 2$$

$$u_1 = -6 \implies (A + B \times 1)(-2)^1 = -6 \implies A + B = 3$$

$$\implies B = 3 - A = 3 - 2 = 1 \implies B = 1$$

The required solution is

$$u_n = (2 + n)(-2)^n.$$

Suppose you are asked to find u_n , if $u_0 = 0, u_1 = 1$, and

$$u_{n+2} + 16u_n = 0 \quad ...(1)$$

Let $u_n = ck^n \ (c \neq 0)$ be a solution of (1). So, you will get

$$ck^{n+2} + 16ck^n = 0 \Longrightarrow k^2 + 16 - 0 \Longrightarrow k = \pm 4i$$

The roots are purely imaginary, their real part is zero but if I write that formula you will get it in the form $r^n(c_1 \cos n\theta + c_2 \ \sin n\theta)$. So if I substitute, we get

$$\alpha + i\beta = 4i$$

$$\alpha - i\beta = -4i$$

$$\Longrightarrow r = \sqrt{\alpha^2 + \beta^2} = \beta = 4 \text{ and } \tan\theta = \frac{\beta}{\alpha} = \infty \Longrightarrow \theta = \frac{\pi}{2}$$

So, if I now substitute it here I will get

$$u_n = 4^n[c_1 \cos\left(n\frac{\pi}{2}\right) + c_2 sin\left(n\frac{\pi}{2}\right)]$$

So this is the general solution.

But we have been given two conditions that is $u_0 = 0, u_1 = 1$,.

So if we use this,

$$u_0 = c_1 = 0 \Longrightarrow c_1 = 0$$

and your

$$u_1 = 1 \Longrightarrow 4c_2 \sin\left(\frac{\pi}{2}\right) = 1$$

.

$$\Longrightarrow 4c_2 \sin\left(\frac{\pi}{2}\right) = 1 \Longrightarrow c_2 = \frac{1}{4}$$

The solution of this particular difference equation is

$$u_n = 4^n\left(\frac{1}{4}\right)\sin\left(\frac{n\pi}{2}\right)$$

$$\Longrightarrow u_n = 4^{n-1}\sin\left(\frac{n\pi}{2}\right).$$

We next look into some graphs some solutions the visualization of solutions of some of a particular form of difference equation.

So, you will see that solutions of homogeneous linear difference equation with constant coefficients. So when we were discussing this we have assumed that the constants which is $(a_0,\ a_1,\ a_2)$ they are all constant coefficients.

So they are a combination of the basic expression $u_n = ck^n$

So mostly your general solution will be of this form for the models that we will be doing in this course.

Now, what will happen to the solution of this difference equation depending on this value of k. So, suppose $k > 1$.

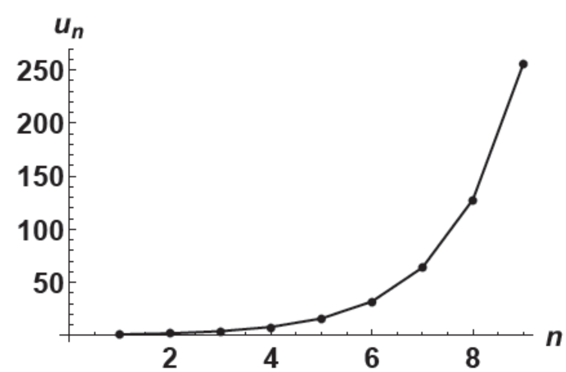
So, if your $k > 1$ you have this $u_n = ck^n$, you will get a solution like this.

Now, what is the conclusion from this solution?

Obviously you put some arbitrary value of c or some conditions will be given from where you will get this value of c and then you will plot this graph and when you plot this graph the dynamics will depend on the value of k. So

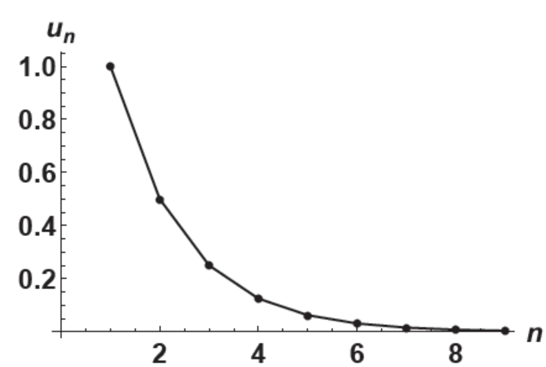
Case 1: $k > 1$

So it is clear that the solution becomes unbounded as n increases. Since this is a solution of a discrete equation, you can see that it is given 2, 4, 6, 8. So, it starts with 0, 1, 2, 3, 4, 5, 6 like that. So if $k > 1$ our conclusion is the solution becomes unbounded as your n increases.

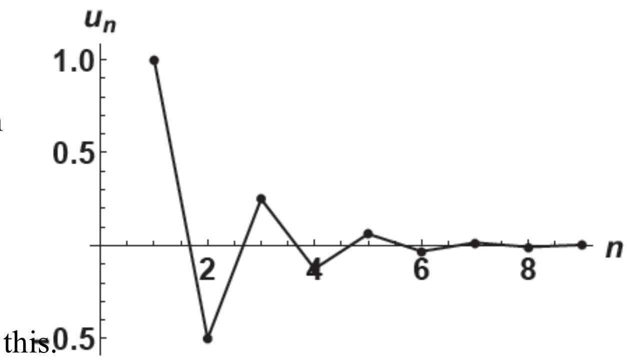


Case 2: $0 < k < 1$

So you can see it starts from some initial value and slowly decreases to zero. So your k^n goes to zero as n increases because you have the solution $u_n = ck^n$, it goes to 0 as n increases and as such your u_n will also decrease and goes to zero. So, if your value of $0 < k < 1$ you will get a curve like this and the conclusion from the curve is as your n increases your u_n decreases.

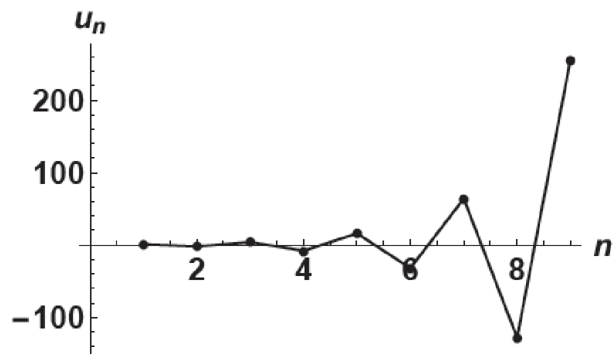


Case 3: $-1 < k < 0$.



So as you can see this is some sort of oscillation and a damping one. So it starts with some large value and slowly it damps and goes to zero. So your conclusion is k^n oscillates between positive and negative values with diminishing magnitude. So, as you can see that it starts from some higher value and slowly it goes down like this. So, if it is a discrete one I have to join like this and it goes some sort of damping oscillation.

Case 4: $k < -1$.



In this case, you can see again it oscillates between the negative and positive values but with increasing n, the magnitude increases. So, in this case k^n oscillates between positive and negative values with increasing magnitude.

So, the reason we are doing this 4 cases is that that when you get the value of k lying between these ranges you have to remember the curve. So, that you can easily draw the curve and you can conclude from that curve with respect to the model.

Otherwise you have to plot them using Microsoft Excel and get the same kind of curve.

Now some marginal points, so what happens if your $k = 1$, you will get because your solution was un equal to c times k to the power n, so you will get un equal to some constant.

If your $k = 0$, you will get un equal to 0 and if your $k = -1$, then you will get it is an oscillatory solution it will oscillates between $-c \ and \ c$.

So, summing up in the last two lectures, we have learned the basics of the difference equation, how to solve the difference equation and its graphical visualization.

In the next lecture, we will be learning how to solve this difference equations with the help of Microsoft Excel.

Till then bye bye.