EXCELing with Mathematical Modeling

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Week – 08
Lecture – 39 (Stability Analysis I)

Hello, welcome to the course EXCELing with Mathematical Modelling.

Today we will be talking about the stability analysis of difference equations.

To start with if you consider the continuous case, say,

$$\frac{dx}{dt} = f(x)$$

and if I want to find the equilibrium solution of this particular system, I say that there is no change of x with respect to time and if there is no change with respect to x, no change of x with respect to time, I put

$$\frac{dx}{dt} = 0$$

and hence your $f(x) = 0$, and you solve this particular equation and you get say $x = x^*$.

Now, let us see what happens in case of difference equation.

So, if you consider the difference equation, say,

$$u_n = au_{n-1} + b$$

So, from $n - 1 \rightarrow n,$ so this is

$$\text{generation}(n-1) \rightarrow \text{generation}(n).$$

If there is no change from generation n-1 to generation n, then we say that you have an equilibrium solution.

So, what do you do to find the equilibrium solution of this particular equation?

Since there is no change, your $u_n = u^*$ and at the same time your $u^* = u_{n-1}$ because there is no change from n-1 to n. So, you replace both your

$$u_n = u^* = u_{n-1}$$

And if you substitute it there, you will get

$$u^* = au^* + b \Longrightarrow (1-a)u^* = b \Longrightarrow u^* = \frac{b}{1-a}, \quad (a \neq 1)$$

So while writing this equation you have to mention the condition that $a \neq 1$.So this is the equilibrium solution of this difference equation.

So once again that if you want to find the difference equation of this particular difference equation you have to consider that there is no change from this generation to this generation.

For example, I can rewrite this, I can change this equation as

$$u_{n+1} = au_n + b$$

So, in this particular case from $\text{generation}(n) \rightarrow \text{generation}(n+1)$ and if there is no change in the solution if u^* be that solution we say that u^* is the equilibrium point or fixed point or steady state point, and then you replace this and this with the u^* and you solve for the u^* like you have done it here and you get the solution to be

$$u^* = \frac{b}{1-a}, \quad (a \neq 1)$$

Now let us move to the stability of this particular difference equation

$$u_n = au_{n-1} + b, (a \neq 1)$$

We have already shown that $u^* = \frac{b}{1-a}$ $(a \neq 1)$ is the equilibrium solution or steady state solution or fixed point.

The equilibrium point u^* will be stable if number one it is stable, if $|a| < 1$, it is unstable, if $|a| > 1$ and if it is $|a| = 1$, no conclusions, further investigation is necessary.

So, without the proof you just remember this theorem or statement that if you have a difference equation of the form

$$u_n = au_{n-1} + b, (a \neq 1)$$

Then the system will be stable about the equilibrium point u^* if $|a| < 1$ $\quad(-1 < a < 1)$ and unstable if $|a| > 1$, $\quad (a < -1, a > 1)$, and at the point $a = -1$ and $a = 1$, you have no conclusions.

Now with this there is also one more point you have to remember that if

$$a > 0 \quad \text{with} \quad |a| < 1$$

$\Longrightarrow$the solution converges monotonically to u^*.

And if $a < 0$ with $|a| < 1$, the solution oscillates or solution converges to u^* with oscillation.

So, the first point is that if $|a| < 1$ then clearly it is stable but at the same time if a > 0, it will converge monotonically to u^* that will it will be something like this, and if it is a < 0, then it will converge but with some oscillations like this.

Now let us take an example. Let

$$x_{n+1} = a(x_n - 1)$$

So let us specify that $a = \frac{4}{5}$

So, to first find the equilibrium solution you put it $x_n = x_{n+1} = x^*$ and substitute, you get

$$x^* = a(x^* - 1) \Rightarrow x^* = \frac{4}{5}(x^* - 1)$$
$$\Rightarrow 5x^* = 4x^* - 4 \Rightarrow x^* = -4$$

The value of

$$a = \frac{4}{5} \Rightarrow |a| < 1$$

and at the same time a>0, so, this will imply the system is stable about $x^* = -4$, this is the equilibrium point. Your value of $a > 0$, so it will converges monotonically to -4.

So, by monotonically means either it is increasing or it is decreasing, it depends on the initial value that you have chosen.

The second case, the same equation only let me change the value of $a = -\frac{4}{5}$. So now it is

$$x_{n+1} = -\frac{4}{5}(x_n - 1)$$

So if you want to solve this again for the equilibrium solution, there is no change in the generations. So x_n and x_{n+1}, they remain constant. Let it be $x^* = x_{n+1} = x_n$.

You substitute again and get

$$x^* = -\frac{4}{5}(x^* - 1) \Rightarrow 5x^* = -4x^* + 4 \Rightarrow 9x^* = 4$$
$$\Rightarrow x^* = \frac{4}{9} \approx 0.444.$$

The value of $a = -\frac{4}{5}, |a| = \left|-\frac{4}{5}\right| < 1$. and a<0.

So, in this particular point the equilibrium $x^* = 0.444$ is stable and since you're a<0, so it will move towards the equilibrium point with oscillations.

Now, let us see the solution of these two difference equation when we solve them numerically with the help of Microsoft Excel.

So, I already have the equation

$$x_{n+1} = ax_n + b$$

So, the value of

$$a = \frac{4}{5} = 0.8$$

and the value of

$$b = -\frac{4}{5} = -0.8$$

So, let us see I put it here n=0 this will be equal to 0+1 and let us draw drag it to some values say up to 30.

Now this value, so if I put n=0 here, I get

$$x_1 = ax_0 + b$$

and so I need an initial value

$$x(0) = 6$$

and then I calculate this next value.

So, I put n = 0 here to calculate

$$x_1 = ax_0 + b = 0.8 \times 6 - 0.8$$

This a is constant and this b is a constant.

So I put a $ sign. So this is the value and then I drag it, to say 30 values.

So, I choose this. So, if you recall the equilibrium points which you get here is -4 and here I have calculated till -3.99 few more and you will get minus

So now I go to this insert, go to this chart and plot this.

So I get the chart, I can change the chart name, I can write it some like x(n+1)

And if you want, because I don't like this grid lines, I will remove the grid lines.

I can get the legends if you want and I can get axis title.

So here it is n and here it is x(n+1), and you get a nice curve like this the solution of this difference equation.

If I want to move to the next part where you have solution as $-\frac{5}{4}$ so in that case it will be the value of $a = -4/5$ and the value of $b = 4/5$.

So if you want to change that so this becomes a=-0.8 and this becomes b=0.8.

So, now I want to solve this again with these two values. So, I just highlight them.

Same this value is 0. This value is 0+1 and I drag it to the next 30 steps.

Let me increase the font size. So let's make it 20.

Now this value again I take the initial value x(0)=6 that is equal to so now the value of a=-0.8 and the value of b=0.8. So this is

$$x_{n+1} = ax_n + b$$
$$x_1 = ax_0 + b$$
$$x_1 = -0.8 \times 6 + 0.8$$

Your b is a constant and your c is also a constant. So, if I drag up to the next 30 values and I plot them.

So, if I plot them, I go to insert, click this and this chart and I get this value, that here the equilibrium point is 0.444 and the system is stable.

So, the meaning is no matter from the initial condition which is close to 0.444 because this is a local stability.

So, if you start from that position it is going to reach 0.444.

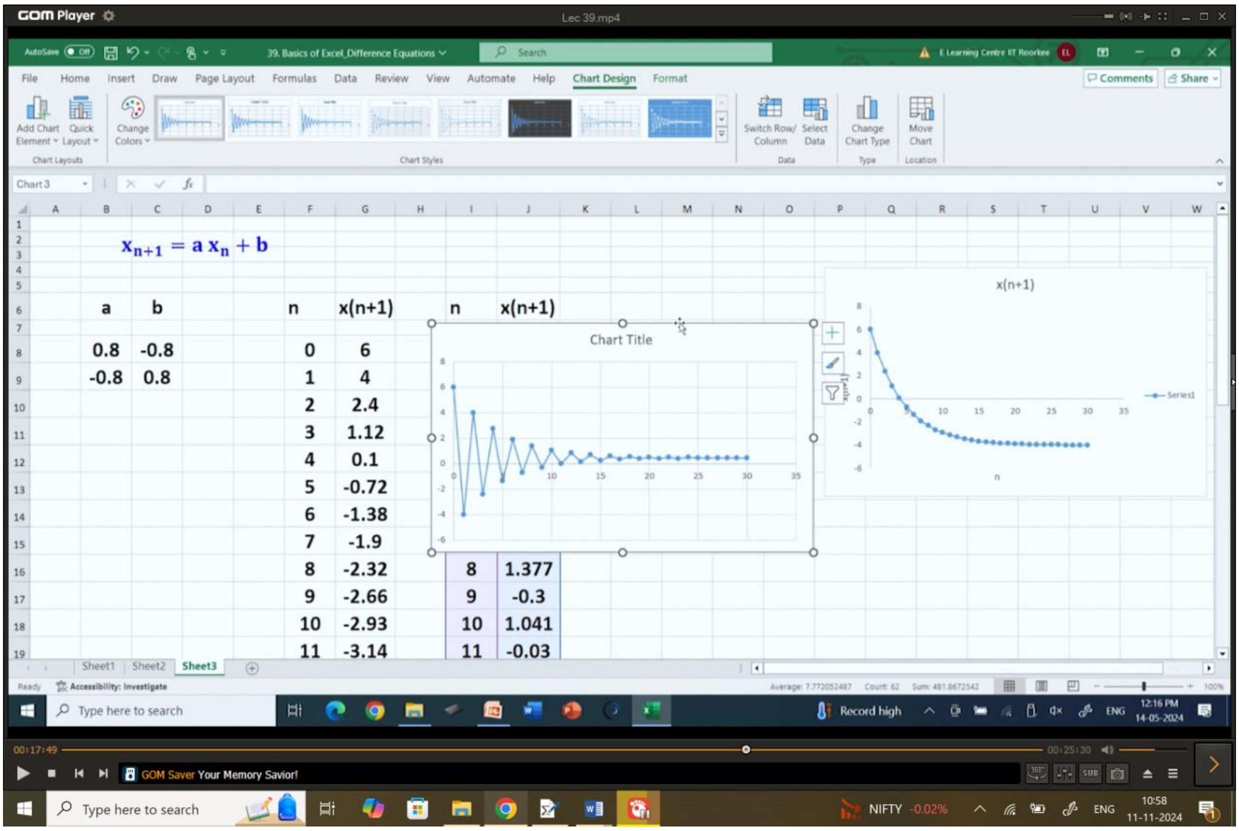
So, if I see the values, I can see that here it is 0.451, a few more it will reach 0.444, but you get the idea that this slowly this curve is reaching the value 0.444, it is a very small deviation here.

So in this particular case when your

$$|a| < 1 \text{ and } a < 0$$

so as the theory suggests that there will be a damping oscillation and the value will slowly decrease and reach the equilibrium point.

So that is exactly what this theory says and we have got this numerical we have got the solution of this particular difference equation in the forms of stability and which is supported by our numerical results.



So, in the similar manner if we take another equation say which is of the form

$$x_{n+1} = \frac{5}{4}(x_n - 1).$$

So if I solve this particular equation, I will get

$$x_{n+1} = x_n = x^*$$

$$\Rightarrow x^* = \frac{5}{4}(x^* - 1)4x^* \Rightarrow 4x^* = 5x^* - 5 \Rightarrow x^* = 5$$

And if I now look at the value of $a = \frac{5}{4},\ |a| = \left|\frac{5}{4}\right| > 1,$ and $a > 0$

So, if this happens then your solution is unstable and it will diverge monotonically from the equilibrium point.

However, if you take your $a = -\frac{5}{4}$ in this particular case again $|a| > 1$. So, the system is unstable, but a < 0, then it will move away from the equilibrium solution, but with oscillations. So, it will be an increasing oscillation.

Let us quickly check this numerically with the help of Microsoft Excel.

Let me rewrite a=1.25 and the value of b=-1.25. So, if you want to calculate this is the n, this is the x_{n+1}, we start with 0 here and this value $x(0) = 6$, this is equal to 0+1 and I drag it to the next 30 values.

So, this will be equal to your

$$x_1 = ax_0 + b$$
$$= 1.25 \times 6 - 1.25$$

Your value of a is a constant and value of b is also a constant.

And you drag till. We just increase the font size.

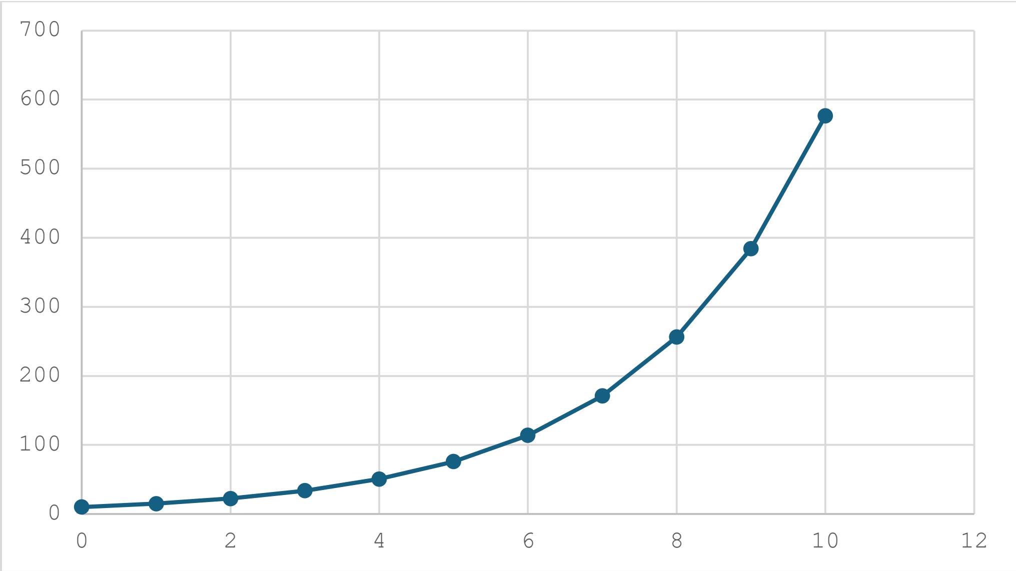
Let us make it 20 bold middle and if you want to draw the graph insert go to the graph and this particular graph you get. So as you can see as the series suggested that this will go unbounded away from the equilibrium value.

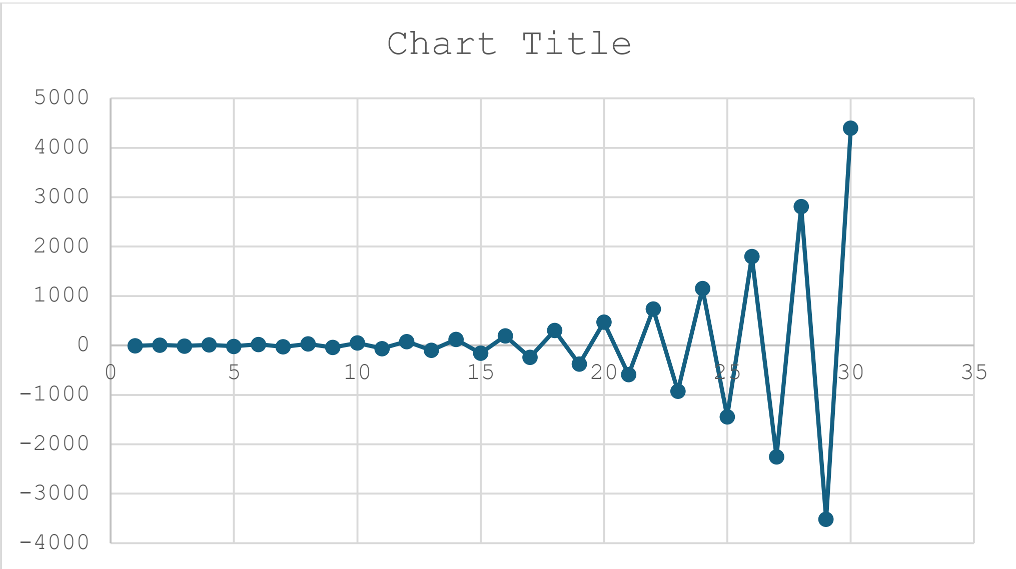
In the similar manner if we quickly change this value to minus. and this particular value to plus.

So if we change this particular value to a=-1.25 and this particular value b=1.25.

So as the theory suggested there will be an oscillation and it will move away from the equilibrium solution with increasing magnitude of the oscillations.

So with this we come to an end of this lecture where we have seen the equilibrium solution, equilibrium point and stability of a difference equation of the form $x_{n+1} = ax_n - b\ (or + b)$, where for various values of a and b's you have seen the dynamics how the dynamics of the solution will change.





In my next lecture we will again do this stability analysis but for two difference equation and non-linear difference equations.

Till then bye-bye.