EXCELing with Mathematical Modeling

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Lecture – 40 (Stability Analysis II)

Hello, welcome to the course EXCELing with Mathematical Modeling.

We will be continuing with our discussion of the stability analysis of difference equations.

So, today we start with system of linear difference equations.

So we consider a homogeneous system of the form

$$u_{n+1} = \alpha u_n + \beta v_n,$$
$$v_{n+1} = \gamma u_n + \delta v_n,$$

homogeneous linear difference equation. In matrix form

I can put it in the matrix form which can be written as

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix},$$

which can be written as

$$w_{n+1} = Aw_n$$

where

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \text{ and } w_n = \begin{pmatrix} u_n \\ v_n \end{pmatrix}.$$

Now as you can see that (0,0) is the only solution of this homogeneous system, provided

$$|A| \neq 0.$$

So now for the stability, I state the theorem, which says that if λ_1 *and* λ_2, two real and distinct eigenvalues of the coefficient matrix A, then your equilibrium point (0,0) will be stable if $|\lambda_1| < 1$ and $|\lambda_2| < 1$. It will be unstable if $|\lambda_1| > 1$ and $|\lambda_2| > 1$, and no definite conclusion if both $|\lambda_1| = 1$ *and* $|\lambda_2| = 1$.

However, if one of them is positive and one of them is negative, we get, what is called a saddle.

So, I repeat the theorem that if you have λ_1 and λ_2 are two real and distinct eigenvalues. By distinct, I mean that $\lambda_1 \neq \lambda_2$. Then (0,0) the only equilibrium point of this homogeneous system will be stable if $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$.

It is unstable if $|\lambda_1| > 1$ and $|\lambda_2| > 1$, and it is saddle if one of them $|\lambda_1| > 1$(or $|\lambda_1| < 1$) and $|\lambda_2| < 1$(or $|\lambda_2| > 1$). We consider this as case 1.

In case 2, we take $\lambda_1 = \lambda_2 = \lambda^*$. So, they are real and equal. So, the eigenvalues of the coefficient matrix are real and equal.

So, (0,0) the equilibrium point is stable if $|\lambda^*| < 1$ and unstable if $|\lambda^*| > 1$.

And finally, the case 3.

If they are complex conjugate. So, $\lambda_1 = a + ib$ and $\lambda_2 = a - ib$.

So, the roots are complex and since there will be two roots in this case that is two eigenvalues they must appear in complex conjugates.

So, (0,0) is the equilibrium point it will be stable focus or spiral

if $|a \pm ib| < 1$, it is an unstable focus or spiral if $|a \pm ib| > 1$.

Now in all these cases if the modulus of the eigenvalues becomes equal to 1, we cannot get any conclusion further investigations are necessary.

So, this is the case for stability for a homogeneous linear system.

Now, let us see what happens when it becomes non-homogeneous, that is, you have

$$w_{n+1} = Aw_n + b$$

Here it is in the form of matrix, so you can assume that

$$w_{n+1} = \begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix}, A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \text{ and } b = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

.So, if I assume that my w^* is the equilibrium point, then there is no change in the generation.

So, $w^* = w_{n+1} = w_n$

So I will replace this by w^* and you get

$$w^* = Aw^* + b$$

And I can write this as

$$Aw^* - w^* + b = 0 \text{ (null matrix)}$$

Suppose, I introduce a new variable we call this change of origin say z_n about the equilibrium point w^*, that is, $w_n = z_n + w^*$. So, I will substitute this value and get

$$z_{n+1} + w^* = A(z_n + w^*) + b$$

$$\Rightarrow z_{n+1} = Az_n + Aw^* - w^* + b$$

and we have already seen that $Aw^* - w^* + b = 0$, So,

$$z_{n+1} = Az_n$$

So, the conclusion is that if you have a system of non-homogeneous linear difference equation, it can be converted to a homogeneous linear difference equation.

And the theorem which holds theorem for stability which holds for homogeneous linear difference equation is also true for non-homogeneous linear difference equation.

So, where the problem is given in terms of homogeneous equation or non-homogeneous equation provided they are linear the same theorem holds good.

Let me prove this with the help of an example.

I have a system of linear homogeneous difference equation in the form

$$x_{n+1} = -x_n - 4y_n$$
$$y_{n+1} = x_n - y_n$$

So, if you have (x^*, y^*) as your equilibrium solution. So by equilibrium solution, I mean that there would not be change from n to n+1 and hence I can replace this $x_{n+1} = x_n = x^*$ and similarly with the variable y that is $y_{n+1} = y_n = y^*$ So I substitute it here and you get

$$x^* = -x^* - 4y^* \text{ and } y^* = x^* - y^*$$

$$\Rightarrow 2x^* + 4y^* = 0 \text{ and } x^* - 2y^* = 0$$

So, (0,0) is the only solution provided determinant of this coefficient matrix which is

$$\begin{vmatrix} 2 & 4 \\ 1 & -2 \end{vmatrix} = -4 - 4 = -8 \neq 0$$

So, (0,0) is the only solution or the unique solution of this system and (0,0) is the equilibrium point.

Now, you check for stability and for that I need the coefficient matrix

$$A = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$$

So I need the eigenvalue of this that can be written as

$$\begin{vmatrix} -1-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix} = 0$$

This implies

$$(\lambda + 1)^2 + 4 = 0 \Rightarrow (\lambda + 1)^2 = -4$$
$$\Rightarrow (\lambda + 1)^2 = 4i^2 \Rightarrow \lambda + 1 = \pm 2i$$
$$\Rightarrow \lambda = -1 \pm 2i$$

So, the roots are complex, you take the $|\lambda|$ and you get $|-1 \pm 2i| = \sqrt{1+5} = \sqrt{5} > 1$.

So, this implies the given system is unstable.

So what you have done here is that you have been given two difference equations. You find the equilibrium solution which means that there is no change from n to n+1 and hence you substitute $x_{n+1} = x_n = x^*, y_{n+1} = y_n = y^*$ and you get this particular set.

You simplify this and you find out this set of equation whose only solution is (0,0) because this determinant is not equal to zero.

So, (0,0) is the unique equilibrium solution of this system of difference equation, and then you find the matrix A and find the eigenvalues and you see that the eigenvalues are both complex $-1 \pm 2i$, you find the $|-1 \pm 2i| = \sqrt{5} > 1$, which implies that the given system is unstable.

Now, let us take the second example, say non-homogeneous equation

$$x_{n+1} = 0.75x_n - y_n + 1000$$

$$y_{n+1} = -0.5x_n + 0.25y_n + 1500$$

So, you do not have to convert this into homogeneous linear difference equation.

We have already proved that whatever is true for linear homogeneous system is also true for linear non-homogeneous system.

So, we just state that (x^*, y^*) be the equilibrium solution and since there is no change from the definition of equilibrium solution from n to n+1, we replace this $x_{n+1} = x_n = x^*, y_{n+1} = y_n = y^*$. So, this will be

$$x^* = 0.75x^* - y^* + 1000,$$

$$y^* = -0.5x^* + 0.25y^* + 1500$$

Now you solve these two equations which are quite easy to solve and you will get your $(x^*, y^*) = (2400,400)$.

This is again a unique system; all you have to do is simplify this. If I do that, I will get

$$0.25x^* + y^* = 1000,$$

$$0.5x^* + 0.75y^* = 1500.$$

and you can easily check that this determinant

$$\begin{vmatrix} 0.25 & 1 \\ 0.5 & 0.75 \end{vmatrix} \neq 0$$

and hence your $(2400,400)$ will be the equilibrium solution and it is unique.

And then you find the coefficient matrix which is

$$\begin{pmatrix} 0.75 & -1 \\ -0.5 & 0.25 \end{pmatrix}.$$

The eigenvalues are given by

$$\begin{vmatrix} 0.75-\lambda & -1 \\ -0.5 & 0.25-\lambda \end{vmatrix} \neq 0 \Rightarrow \lambda_1 = 1.25 \text{ and } \lambda_2 = -0.25.$$

Now, you see the roots are real but opposite signs.

So, what you do is according to the previous theorem you calculate $|\lambda_1| = |1.25| > 1$ and $|\lambda_2| = |-0.25| = 0.25 < 1$.

So, one of the modulus of one of the eigenvalue is greater than 1 and another one is less than 1 and the conclusion is the equilibrium point $(2400,400)$ is a saddle.

So, far we have been dealing with linear difference equations, now let us move on to the non-linear difference equation.

So, a non-linear difference equation will be of the form

$$x_{n+1} = f(x_n)$$

but $f(x_n)$ will contain the non-linear terms. And, as usual to find the equilibrium point you have to consider that there is no change in the generation from n to n+1 and you replace this by some x^* for equilibria, you will have $x^* = f(x^*)$.

Now to see that what will happen or what kind of condition we will get for stability analysis. And for stability analysis we give a small perturbation about the equilibrium point, which mean that you put

$$x_n = \epsilon_n + x^*.$$

So, you substitute it here and you get

$$\epsilon_{n+1} + x^* = f(\epsilon_n + x^*).$$

and by Taylor series expansion we will get

$$\epsilon_{n+1} + x^* = f(\epsilon_n + x^*) = f(x^*) + \epsilon_n f'(x^*) + \text{higher order terms.}$$

We keep up to the first order as ϵ_n is small,

$$\epsilon_{n+1} + x^* = f(x^*) + \epsilon_n f'(x^*) + \text{higher order terms}$$

$$\epsilon_{n+1} + x^* \approx f(x^*) + \epsilon_n f'(x^*)$$

Now, at the equilibrium point $x^* = f(x^*)$, so this two values are equal and hence they cancels and you are left with

$$\epsilon_{n+1} \approx \epsilon_n f'(x^*) \qquad (1)$$

Now this solution it will decrease, so I name it as (1), so (1) will decrease

if $|f'(x^*)| < 1$

and will increase f if $|f'(x^*)| > 1$

and no definite conclusion if we get $|f'(x^*)| = 1$.

So, the theorem is that the equilibrium point x^* of the non-linear equation

$$x_{n+1} = f(x_n)$$

is stable if $|f'(x^*)| < 1$ and unstable if $|f'(x^*)| > 1$ and no definite conclusion if $|f'(x^*)| = 1$.

If we generalize this theorem for two equations say of the form

$$u_{n+1} = f(u_n, v_n),$$
$$v_{n+1} = g(u_n, v_n),$$

Suppose you have two non-linear difference equation to get the equilibrium point you have to solve

$$u^* = f(u^*, v^*)$$

and

$$v^* = g(u^*, v^*)$$

So, just like the previous one you replace

$$u_{n+1} = u^* = u_n = x^*, and \;\; v_{n+1} = v^* = v_n$$

because there is no change in the generation from n to n+1.

And then you calculate the Jacobian matrix which is

$$A = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix}$$

at the equilibrium point (u^*, v^*) and once you do that you calculate the eigenvalues and your (u^*, v^*) is stable if both the eigen values $|\lambda_1| < 1$ and $|\lambda_2| < 1$ and unstable if both the eigenvalues of A, $|\lambda_1| > 1$ and $|\lambda_2| > 1$.

So, it is similar kind just like the linear one. So, you have to just remember these as a formula or as a theorem so that you can apply them to the problems.

Now quickly let us take an example, say I have a set of non-linear difference equation

$$x_{n+1} = x_n + 2.5y_n - 0.1{x_n}^2 - 1,$$
$$y_{n+1} = y_n + \frac{5}{x_n} - 1$$

Now, to get the equilibrium solution as we have told that you have to replace

$$x_{n+1} = x_n = x^* \quad \text{and} \qquad y_{n+1} = y_n = y^*$$

You substitute here and you get

$$x^* \quad = x^* \quad + 2.5y^* - 0.1x^{*2} - 1,$$
$$y^* = y^* + \frac{5}{x^*} - 1$$

So from this equation

$$y^* = y^* + \frac{5}{x^*} - 1$$

you can see that

$$\frac{5}{x^*} = 1 \Longrightarrow \; x^* = 5$$

you pluck this value here and you get your $y^* = 1.4$.

So, you have (5,1.4) as your equilibrium point and you can easily check that this equilibrium point is unique and once you do that then you calculate the Jacobian matrix.

So, what you do is replace say state this as f and this as your g. So, your matrix

$$A = \begin{pmatrix} 1 - 0.2x^* & 2.5 \\ -\frac{5}{x^{*2}} & 1 \end{pmatrix}$$

And when you substitute the value of x^* and y^* here in this particular case it is only x^*, you will get this matrix to be

$$A = \begin{pmatrix} 0 & 2.5 \\ -0.2 & 1 \end{pmatrix}$$

So, now you have to just find the eigenvalue of this matrix by solving

$$\begin{vmatrix} -\lambda & 2.5 \\ -0.2 & 1 - \lambda \end{vmatrix} = 0$$

$$\Longrightarrow \lambda = \frac{1}{2} \pm \frac{i}{2}$$

If you take the modulus, then it is

$$|\lambda| = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{1}{2}} < 1$$

So, this implies that (5,1.4) the equilibrium point is stable, and to be more precise it is a stable spiral or stable focus.

So, with that we come to the end of this particular lecture where you have seen the stability analysis of a system of difference equation whether it is a linear or whether it is a non-linear.

In my next lecture we will be taking up some interesting discrete models and their corresponding analysis.

Till then bye-bye.