EXCELing with Mathematical Modeling
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Lecture – 41 (Population Models)

Hello, welcome to the course EXCELing with Mathematical Modelling.

Today we will discuss about discrete population models.

Let us start with an example.

You consider a population of Royal Bengal tigers in Sundarban which is located in West Bengal and survey shows that there is a decrease of 3% in the population of these tigers per year.

Now if you want to model this using difference equation, this is how it will go.

that your population in the next year will depend in the population

$$p_{n+1} = p_n - \frac{3}{100} p_n.$$

So, very simple model where this gives you the number of tigers after n years.

This is the number of tigers in the n years.

And if you just solve them,

$$p_{n+1} = p_n - 0.03\, p_n = 0.97 p_n$$

So, if somebody asks that what will be the number of tigers after 4 years.

For that you need an initial count of the tiger.

So, let us say $p_0 = 150$

$$p_1 = 0.97 p_0 = 0.97 \times 150 = 146.$$

$$p_2 = 0.97 p_1 = 0.97 \times 146 \approx 141.$$

Similarly, you can calculate what is your p_3 and your p_4.

$$p_3 = 137$$

$$p_4 = 133$$

Now, if somebody wants they can calculate in this manner also that

$$p_4 = 0.97 p_3 = 0.97\ 0.97 p_2 = (0.97)^2\ 0.97 p_1 = (0.97)^3 0.97 p_0 = (0.97)^4 p_0$$

So, this gives you that if you have no information about p_1, p_2, p_3 , p_4 , still you can calculate what is the number of tigers after 4 years just by knowing the initial condition p_0.

And if you substitute here this 150,

$$p_4 = 0.88529 \times 150 \approx 133$$

So, a very simple discrete case model using a difference equation where you can calculate the population. In this case, it is the Royal Bengal tiger in Sundarban in certain years.

Let us take another example.

Suppose, in any population of persons, it grows each year by 2%, but then there is an immigration. This occurs at a constant rate of 3000 per year.

So, by immigration we mean that you move out of the one place to another place.

So, from that particular place at a constant rate 3000 moves out to some other city, but the population grows at a constant rate of 2 percent.

Now, if you want to model this using difference equation, again p_{n+1}, this population plus there is an increase of 2%, but with a constant immigration of 3000 per year.

$$p_{n+1} = p_n + \frac{2}{100} p_n - 3000$$

$$p_{n+1} = 1.02\, p_n - 3000$$

Now, say, population after 10 years we need to know.

So, we have already done equation of the form

$$x_{n+1} = a\, x_n + b$$

So, this falls exactly in that category and let us quickly recall this.

$$x_{n+1} = a\, x_n + b = a\,(a\, x_{n-1} + b) + b$$
$$= a^2 x_{n-1} + b(a+1)$$

$$= a^2(a x_{n-2} + b) + b(a+1)$$

$$= a^3 x_{n-2} + b(a^2 + a + 1)$$

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$$x_{n+1} = a^{n+1} x_0 + b(a^n + a^{n-1} + - - - - +1)$$

$$= a^{n+1} x_0 + b\left(\frac{a^{n+1} - 1}{a - 1}\right), \qquad a > 1$$

Here,

$$a = 1.02, \qquad b = -3000, \qquad n = 9$$

because I need the value p_{10}, that is, the population after 10 years. So, if I substitute this here, I get

$$p_{10} = (1.02)^{10} \times 150000 + (-3000)\left(\frac{(1.02)^{10} - 1}{1.02 - 1}\right)$$

$$= 182849 - 32849 = 150000$$

So, the conclusion is that, okay, you have a growth of 2% in the population with a constant movement of 3000 people moving out of the city, even after 10 years the population remains constant to 1.5 lakh.

Well, the problem is chosen like that such that you get this answer 1.5 lakhs, but otherwise there will be an increase and the decrease in the population if I change this number 3000.

So, let us move to some non-linear models now, non-linear population models.

We call them density dependent population growth. As the name suggested they are density dependent and the very first in the line is discrete logistic growth.

We have done this logistic growth many times. So, I will just discuss the model. The equation is also looking familiar.

$$x_{n+1} = r\, x_n \left(1 - \frac{x_n}{k}\right)$$

So, r is your intrinsic growth rate, k is your carrying capacity and if you want to find the equilibrium solution you have to replace

$$x_{n+1} = x_n = x^*$$

Substitute the values here and find the equilibrium points as

$$x^* = 0, \qquad r\left(\frac{r-1}{k}\right), \qquad r > 1$$

and then you can check for stability at these equilibrium points.

Since we have done this logistic growth so many times, I am not going in details of this analysis, I leave that to you.

Let us move on to another kind of density dependent population growth and the name is Richer's model, namely,

$$x_{n+1} = \alpha\, x_n\, e^{-\beta x_n}, \qquad \alpha, \beta > 0$$

So, here α is your maximum growth rate and β is the inhibition growth which means that this is the factor or the parameter which boosts the growth which increases the growth and this is the factor with the negative sign which actually keeps the growth down. So your α and β are positive parameters.

Now if you want to see what are their equilibrium points. So as usual you have to submit substitute

$$x_{n+1} = x_n = x^*$$

because there is no change from n to n+1, that is the definition of this equilibrium point and you have

$$x^* = \alpha x^* e^{-\beta x^*} \Rightarrow x^*\left(\alpha e^{-\beta x^*} - 1\right) = 0$$

$$\Rightarrow x^* = 0, \qquad \text{and} \quad \alpha e^{-\beta x^*} = 1$$

$$\Rightarrow \alpha = e^{\beta x^*} \Rightarrow \ln\alpha = \beta x^* \Rightarrow x^* = \frac{\ln\alpha}{\beta}.$$

Let us now look into the stability analysis. So our model was in the form

$$x_{n+1} = \alpha\, x_n\, e^{-\beta x_n}$$

$$f(x) = \alpha\, x_n\, e^{-\beta x}$$

$$f'(x) = \alpha\, e^{-\beta x} - \alpha\beta\, e^{-\beta x}$$

Now, you have the two equilibrium points

$$x^* = 0 \text{ and } x^* = \frac{\ln\alpha}{\beta}$$

Now,

$$|f'(x^* = 0)| = |\alpha| < 1 \Rightarrow -1 < \alpha < 1$$

$$\text{But, } \alpha > 0, \Rightarrow 0 < \alpha < 1$$

So, this is the condition for stability for the equilibrium point $x^* = 0$

Similarly, if I take

$$\left|f'\left(x^* = \frac{\ln\alpha}{\beta}\right)\right| = \left|\alpha.\frac{1}{\alpha} - \alpha\beta\frac{\ln\alpha}{\beta}.\frac{1}{\alpha}\right| = |1 - \ln\alpha|$$

For stability,

$$|1 - \ln\alpha| < 1 \Rightarrow -1 < 1 - \ln\alpha < 1$$

Now,

$$-1 < 1 - \ln\alpha \Rightarrow \ln\alpha < 2 \Rightarrow \alpha < e^2$$

and

$$1 - \ln\alpha < 1 \Rightarrow \ln\alpha > 0 \Rightarrow \alpha > e^0 = 1$$

$$\Rightarrow 1 < \alpha < e^2$$

So, this is the condition for stability for the equilibrium point $x^* = \frac{\ln\alpha}{\beta}$.

Now, you can easily check that for $1 < \alpha < e$ you will see that $f'\left(x^* = \frac{\ln \alpha}{\beta}\right) > 0$ and for $e < \alpha < e^2$, we get $f'\left(x^* = \frac{\ln \alpha}{\beta}\right) < 0$.

So this means that the model is stable in both the cases but in first case it will monotonically approach the stable point and in second case it is going to be an oscillatory behaviour but approach the equilibrium point.

Now let us look into the numerical solution of this Richer's model.We use the same Microsoft Excel

So I have taken $\alpha = 1.535$ and $\beta = 0.000783$ and your initial value is 50.

So my equation is

$$x_{n+1} = \alpha\, x_n\, e^{-\beta x_n}$$

So if I want it here, this will be equal to α which is this multiplied by x_n which is this initial value multiplied by exponential of -β times x_n.

Now we will just put dollars in place of the constant, your α is a constant.

So I put dollar notation, then your G4 is x_n exponential, again your β is a constant.

So I put a dollar sign and end.

So I get some values, drag till 50 of them or say 40 of them, just increase the font size.

So, let us plot this.

If I want to a discrete look, go there, go to this and we choose this scatter and go here, yes.

So, this is what we get for this Richer's model.

If I change the value of this α because α lies between 1 and e^2 if you want a different dynamics, that also can be done.

We now look into the solution of this Richards model by changing the parameter values namely the value of α.

So, I have already the equation here and let me now increase this by 1 and drag it to the next 30 values, 3 more, next we calculate the initial condition here is 50.

So, this is equal to α which is a constant, so I put a dollar here, multiplied by the x_n, which is this value, multiplied -β multiplied by x_n.

Since β is a constant, I put dollar here, and you get the value.

So I drag it till 30 of them and next we plot it.

So, I highlight these values, go to insert, go to this charts and choose this.

If you want to name the title, so this Richards model, I don't like the grid lines, so I will remove this if you want access title, so this is your n and this is your x_n.

And you get the figure for the parameter values α equal to 6.535 and value of β remains the same which is 0.000783 and you see the behavior of the curve is oscillatory and reaching an equilibrium value.

The next is we again change this parameter value of α. we now make it as 15.535 and see what's the dynamics is. So, again I increase this value by 1.

I take it to the next 30 values and then calculate this x_n in a similar manner.

So, this is equal to α which is a constant multiplied by x_n multiplied by exponential minus β which is again a constant multiplied by x_n and we get the value.

So, if I plot this, I again highlight these values, I go to insert, charts and I get the plot.

So, in the similar manner Richards model, I remove the grid lines, I want to put the axis title, so I put it n and x_n.

So this behavior shows that it is some sort of unstable situation which we get by replacing the value of α 15.535.

Now there is a shortcut also.

So suppose in this previous figure, I want to change the value of α directly here.

So, all I have to do is I have to just replace this value by 15.535.

So, if I replace this value by 15.535 and just click it outside, you see that all these values are automatically changed and you get the value which is exactly the graph which is exactly same like here.

So, that is another way of solving with the help of Microsoft Excel.

So, if we see the result, this is the numerical figures which you will get.

So, the very first one is for α equal to 1.535 and your β is equal to 0.000783

So, initial value it started from say 50, your x0 is 50 and you reach a steady equilibrium point.

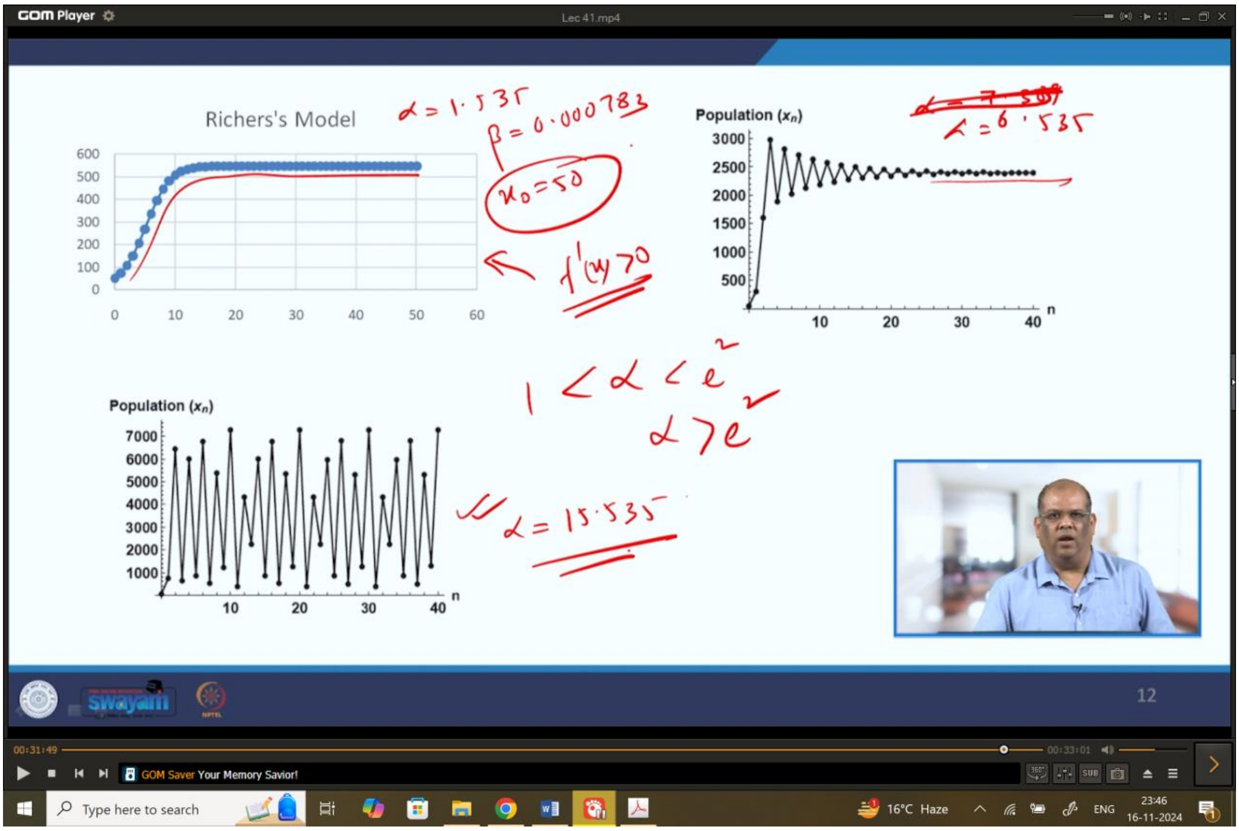
Whereas when your $f'(x) > 0$ then you get this.

However, if it is less than 0, then we get an oscillatory behaviour.

So, it starts with 50, it jumps up to some level and then there is an oscillation, a damped one and slowly reaching the equilibrium point.

However, if your α does not lie between 1 and e^2, if your α is greater than e^2, you get an unstable equilibrium and in this case your figure will be like this.

So, in this case your α is 7.389 sorry this is 6.535 this is your α and in this case your α is equal to 15.535.



So, this gives a density dependent population model namely Richards model where you have so many interesting dynamics.

So, summing up today we have learned about this population growth models both linear and non-linear and density dependent models.

We have learned about the logistic growth about the Richards growth model.

We analyse the model both analytically as well as numerically we have solved with the help of Microsoft Excel.

So, in my next lecture we will be talking about some banking models which will involve various savings schemes, loan problem, retirement saving problem.

Till then bye-bye.