EXCELing with Mathematical Modeling
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Lecture – 44 (Lake Pollutant Model)

Hello, welcome to the course EXCELing with Mathematical Modelling.

Today we will discuss about some lake pollutant model.

Now you must have noticed in which area you staying if there is a lake, how dirty it gets. People throw things there, plastic bottles, plastic sheets and other dirt and it gets polluted.

So we take one such example where there are two neighbouring lakes and how the pollution there is happening and what can we do about its control?

So, let us take say two lakes A and B. So, the problem is defined like this that you have U_n and U_n.

They are the total amount of pollutants in lake A and in lake B respectively after some year n. Now, every year 38% of the pollutant from lake A and 13% of the pollutant from lake B are removed.

However, the pollutant that is removed from lake A is added to lake B due to the flow of water from lake A to lake B. And also it is assumed that 3 tons of pollutants are directly added to lake A and 9 tons of the pollutants are directly added to Lake B.

So, now you have to formulate a model on the base of this particular scenario.

So, I have two lakes, Lake A and Lake B and it says that 38 percent of the pollutant taken from Lake A, 13 percent from Lake B. However, this 38 percent it goes to Lake B due to some flow of water.

So from Lake B if I take out 13% which is 0.13 and from Lake A I am taking out 38% 0.38 and it goes directly to Lake B. So that is the first part which has been told.

Now it is also assumed that 3 tons of pollutants is directly added to Lake A. So from here it is 3 tons and 9 tons to Lake B. So, from here 9 tons.

So, this is you can say some sort of schematic diagram which happens between Lake A and Lake B. So, you have $U_n$which is the total amount of pollutant in Lake A and V_n, which is the total amount of pollutant in Lake B after year n. So, if I want to write the equation for $U_n$then I have to take into account the previous year's pollutant.

Then 0.38 of the pollutant goes out, but 3 tonnes are being added.

So,

$$U_n = U_{n-1} - 0.38U_{n-1} + 3 \quad \text{—— (1)}$$

that is our equation 1.

When it comes to lake B, it is the pollutant from the previous year, then this 38% removed from Lake A will go to Lake B, so plus 0.38 UN minus 1.

Then 13% is removed from Lake B, it does not come to Lake A, so it is minus 0.13 of VN minus 1 and 9 tons are being added every year. Hence,

$$V_n = V_{n-1} + 0.38U_{n-1} - 0.13V_{n-1} + 9 \quad \text{—— (2)}$$

So, this 2 now gives the mathematical model of this pollutant in lake A and lake B and let us look into their solutions. So, let me rewrite this

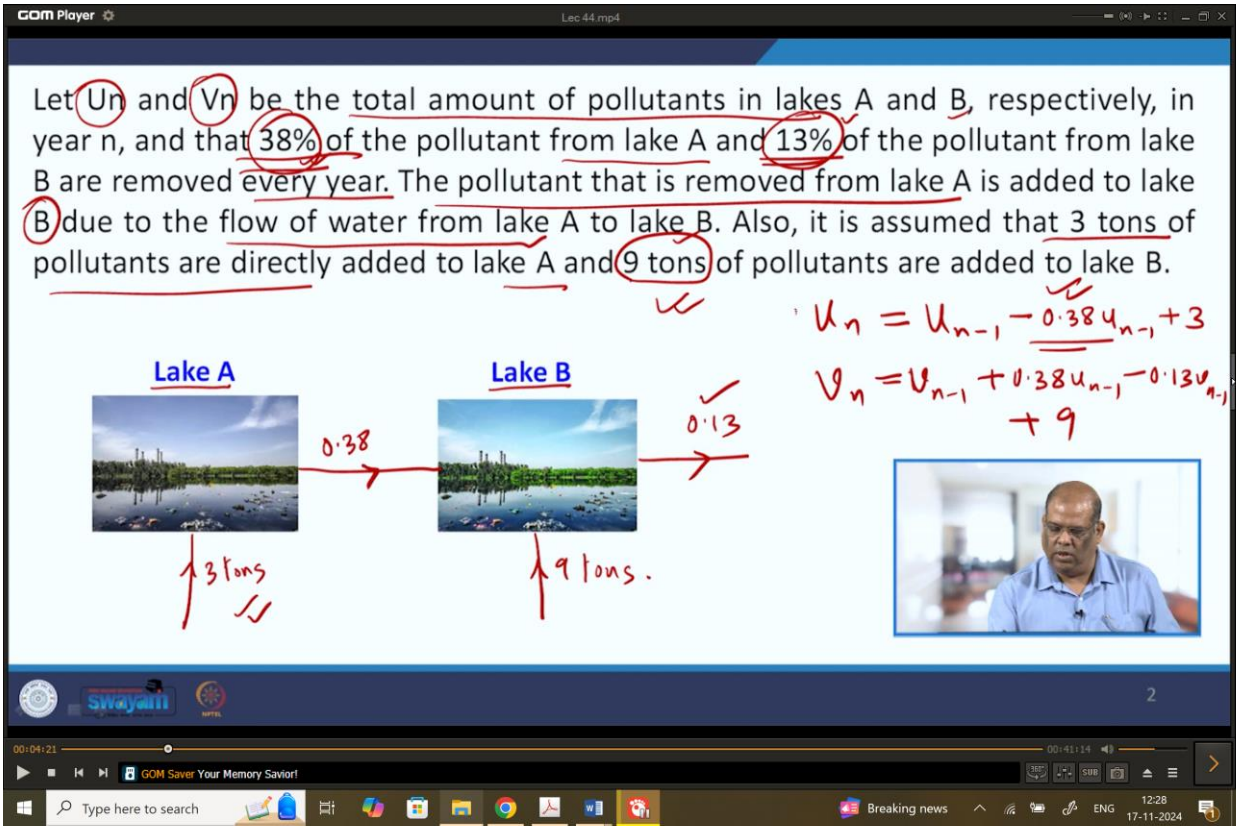
$$U_n = U_{n-1} - 0.38U_{n-1} + 3$$

$$V_n = V_{n-1} + 0.38U_{n-1} - 0.13V_{n-1} + 9$$

So, once again this is the equation for lake A pollutants and this is the equation for lake B pollutants

So, in the lake A 38% is removed 3 tons have been added and in lake B this 38% comes to lake B. So, this has been added 13% is removed.

So, minus 0.13 and 9 tons has been added and these are the pollutants from the previous year.



Now, let us look into the equilibrium solution.
So, by the definition of equilibrium solution there will be no change from n-1 to n. So no change from n-1 generation to n generation and in such case you will substitute

$$U_n = U_{n-1} = U^*$$

and

$$V_n = V_{n-1} = V^*$$

So, you substitute this in the original model and you will get

$$U^* = U^* - 0.38U^* + 3$$

and

$$V^* = V^* + 0.38U^* - 0.13V^* + 9$$

So, from this equation this and this cancels and you get

$$-0.38U^* + 3 = 0$$

which gives

$$U^* = \frac{3}{0.38} = 7.9$$

Now from here V star cancels and you will get

$$0.13V^* = 0.38U^* + 9$$

So, your

$$V^* = \frac{0.38U^* + 9}{0.13}$$

from here U star is equal to 7.9.

So, you substitute this value here and you get

$$V^* = \frac{0.38 \times 7.9 + 9}{0.13} = 92.3$$

So, I have the equilibrium solution as $(U^*, V^*) = (7.9, 92.3)$

Let us now look into the stability of this model.
So, to find the stability, let me again rewrite the equation, it is

$$U_n = U_{n-1} - 0.38U_{n-1} + 3$$

$$V_n = V_{n-1} + 0.38U_{n-1} - 0.13V_{n-1} + 9$$

So, if I simplify them, I will get

$$U_n = 0.62U_{n-1} + 3$$

$$V_n = 0.38U_{n-1} + 0.87V_{n-1} + 9$$

So, the coefficient matrix

$$A = \begin{pmatrix} 0.62 & 0 \\ 0.38 & 0.87 \end{pmatrix}$$

Now, I have to find the eigenvalues and for that it is

$$|A - \lambda I| = 0$$

which will give me

$$\begin{vmatrix} 0.62 - \lambda & 0 \\ 0.38 & 0.87 - \lambda \end{vmatrix} = 0$$

And this will give me

$$(\lambda - 0.62)(\lambda - 0.87) - 0 = 0$$

So, from here it can be easily seen that $\lambda_1 = 0.62$ and $\lambda_2 = 0.87$ the two values of lambda which we name as λ_1 and λ_2.

If you now take the modulus which is less than 1 take the modulus is less than 1 and you can say the system is stable about the equilibrium point $(U^*, V^*) = (7.9\,, 92.3)$

So this model which we have created according to the given problem is stable.

We now check this numerically with the help of Microsoft Excel.

So we have this lake pollutant model and so I put the value of N value of U_n and value of V_n. Initially let us take the initial value is of the pollutant is 25.

So, let me change the font size to 20, put some borders and make it in the middle, okay.

So, after simplification the equation is

$$U_n = 0.62U_{n-1} + 3$$

and

$$V_n = 0.38U_{n-1} + 0.87V_{n-1} + 9$$

This we have already seen while simplifying and checking for the stability.

So, the first one is increase them by 1 which is equal to this cell plus 1 and then I drag it to the next 35 values.

So, this number of values in your hand you can choose them accordingly and this value is equal to….

So, I am calculating first this particular equation.

So, if I put in equal to 1 here, this is 0.62 u 0 plus 3. So, exactly that is what I am going to do. This is equal to 0.62 multiplied by u0 which I have taken to be 25 plus 3 and enter.

This value is equal to 0.38 multiplied by un which is un minus 1 rather u0 which is this plus 0.87 multiplied by V0 which is this plus 9.

So, now I drag these two values to this much then I highlight them pressing the shift and the arrow keys.

So I choose all these 34 values, 35 values, go to insert, go to scatter and since it is a discrete case, I will choose this one and I get a figure like this.

So as you can see that this is approaching the value 92.3 here and this is approaching the value 7.9.

Since the system is stable, so from whatever initial condition you start, they will reach the equilibrium point and it's a local stability.

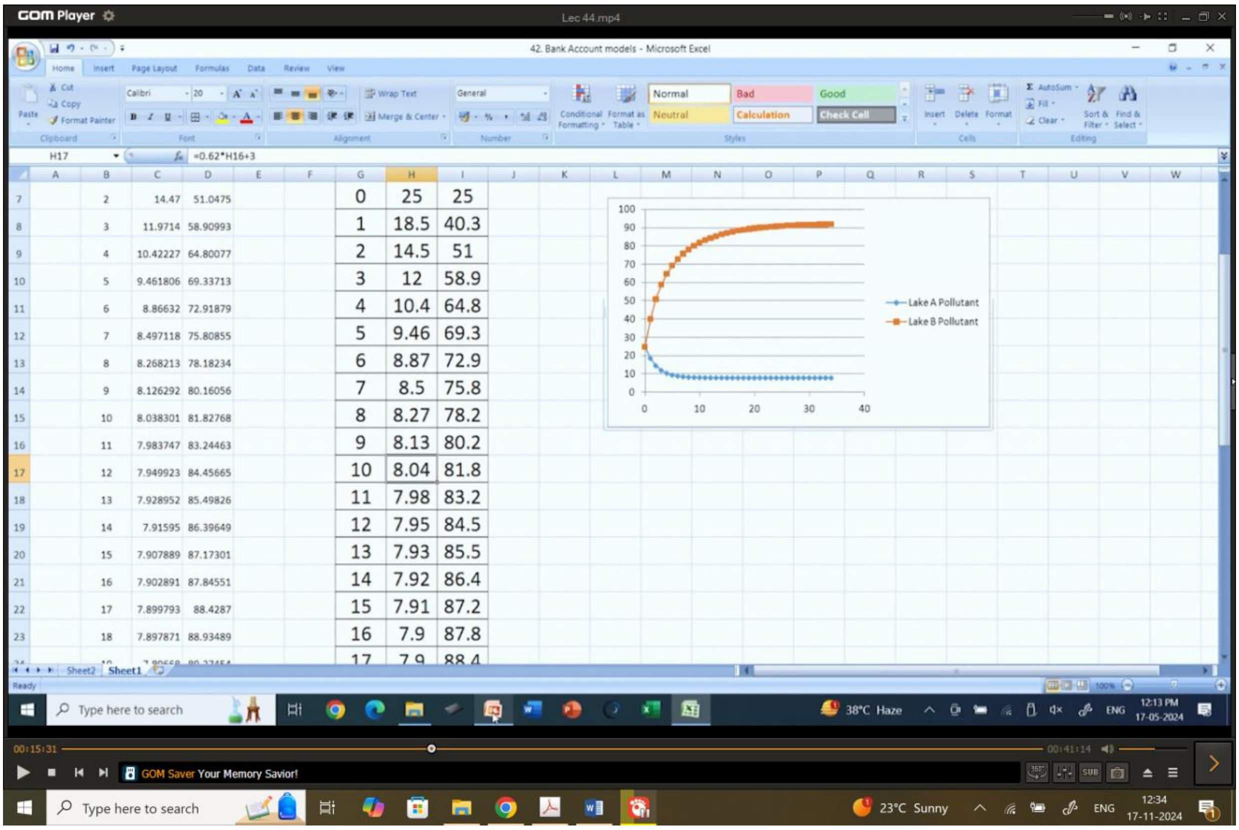
Obviously, you can just change the series from here.

You go to design, you go to select data and here the series one, edit, you write lake
A pollutant okay go here edit like B and you get

So let us move back to the discussion.

So as we have proved analytically that your lake A pollutant will have the equilibrium value as 7.9 and the lake B will have the equilibrium value 92.3.

So as you can see since the system is stable they start from 25 which is the initial condition and both of them are reaching their respective equilibrium values which is 7.5 and 92.3 respectively.



Let us now put some more clause in this particular model. So as you know that the aim is that how to keep the lakes less polluted.

So a team decided okay now our equilibrium solution should be 10 tons for lake A and 30 tons for lake B. So, how much of the pollutant now you can throw or you can restrict that amount of pollutant for lake A and lake B to be thrown.

So which was previously the 3 tons and the 9 tons.

So now let us take them as x tons and y tons.

So you have the previous equation as

$$U_n = 0.62U_{n-1} + 3$$

tons and

$$V_n = 0.38U_{n-1} + 0.87V_{n-1} + 9$$

So, this was the previous model.

Now, a committee has formed and they observed and let us say the bring down the equilibrium solution to be 10 and 30 and for that you cannot throw 3 and 9 turns in the lake bring me the new values of x and y. So now your equation will be

$$U_n = 0.62U_{n-1} + x$$

and

$$V_n = 0.38U_{n-1} + 0.87V_{n-1} + y$$

So what is known to you? You have known the equilibrium solution. So you put

$$U_n = U_{n-1} = U^* = 10$$

and

$$V_n = V_{n-1} = V^* = 30$$

So, we will be plucking this value here and solve for x and y. So, you will get

$$U^* = 0.62U^* + x$$

this will give

$$x = 0.38U^* = 0.38 \times 10 = 3.8$$

tons and

$$V^* = 0.38U^* + 0.87V^* + y$$

so your

$$y = 0.13V^* - 0.38U^* = 0.13 \times 30 - 0.38 \times 10 = 3.90 - 3.80 = 0.1$$

So, y equal to 0.1 tons.

So, basically now you are supposed to put 3.8 tons to lake A and 0.1 to lake B so that your pollutants are less and they maintain the equilibrium solution of 10 and 30 tons respectively.

We now look into the chemical pollution in a lake model.

So, consider a lake say area 2 kilo meters square and average depth say 10 meter it has a river flowing through it.

So, if you consider this to be the lake and this is the river which is flowing through it and say at a rate of 10000 meter cube per day.

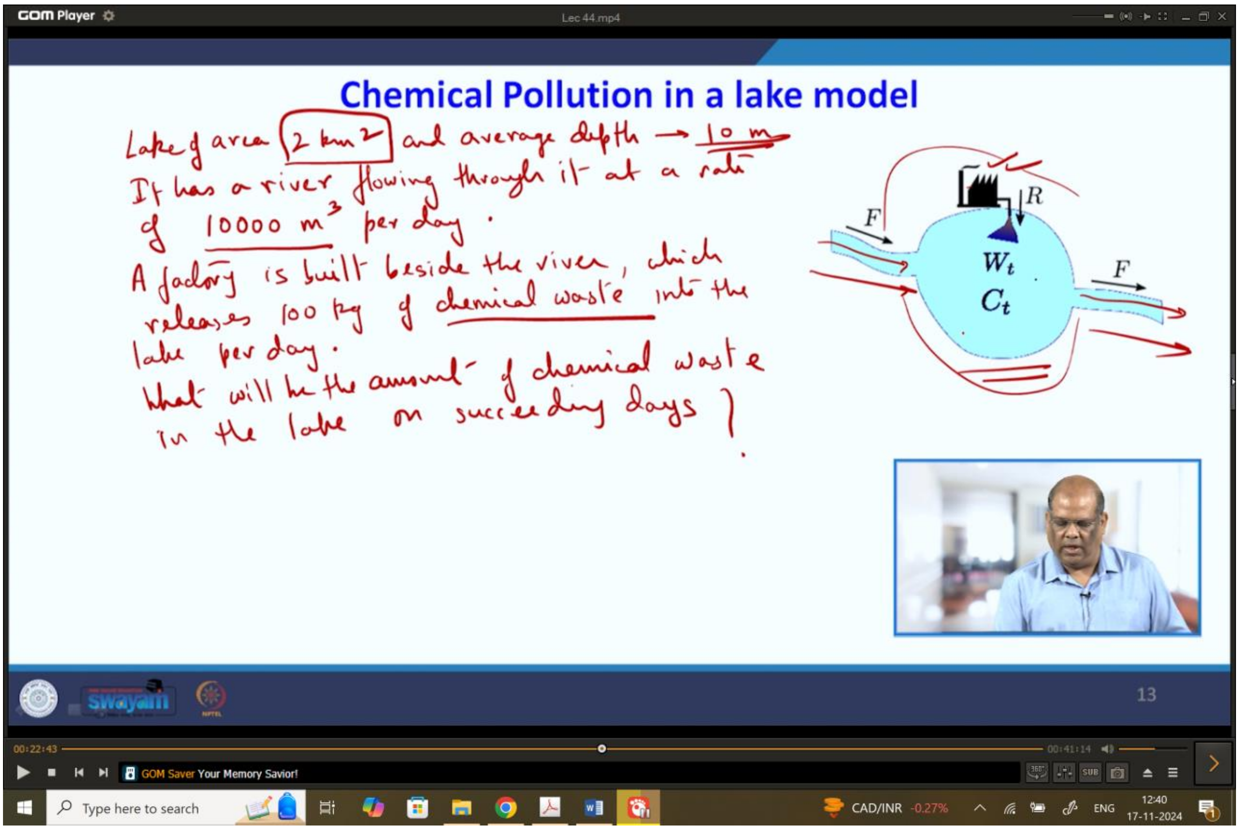
Then there is a factory is built beside the river is the factory which releases 100 kg of chemical waste into the lake per day.

So, what we want to find out is what will be the amount of chemical waste in the lake on succeeding days.

So, you want to find out what will be the chemical waste in the lake in 30 days, 40 days, 1 month and so on.

So, this is the lake and which has an area of 2 kilo meters square. It has a depth of 10 meter.

A river flows through it flowing in and flowing out at a rate of 10,000 meter cube per day. There is a factory which deposits this 100 kg of chemical waste into this lake. We want to find out through a model that what will be the amount of this chemical waste in successive days.



So, let us define the variables say W_t that is the chemical waste in the lake on day t and t is the time measured in days, C_t is the concentration of chemical waste.

V is the volume of the lake, F is the daily flow of water through the lake and R is the daily release of chemicals.

So, with these variables, let us now formulate a model. So, the first thing we want is what is the

$$\text{Lake Volume} = \text{Area} \times \text{depth}$$

and the area is 2 kilo meters square multiplied by 10 meter, we have to convert it to meter.

So, this is going to give $2 \times 10^6 \times 10$ meter cube.

So, 2×10^7 meter cube. So, that is the volume of the lake.

Next is the flow through the lake or out of the lake that is the water is flowing in and going out what is that flow denoted by F say this is denoted by V. So, that F is given to be 10000 meter cube which is 10 to the power 4 meter cube and amount of chemical released in a day, we denote it by R and it is 100 kg.

Now, since the volume of the lake is fixed, the chemical concentration which is denoted by C_t is a known multiple of the total chemical waste which is denoted by W_t in the lake.

So, in kg per meter cube the concentration is

$$C_t = \frac{W_t}{V} = \frac{W_t}{2 \times 10^7}$$

Now for chemical waste we look into the change in the amount of chemicals which is given by

$$W_{t+1} - W_t = R$$

if there is no river flow through it.

But however, in this case there is a river flow and hence some chemicals will be removed and flow downstream.

So, the amount of chemical waste that leaves the lake each day is equal to the amount of water that leaves the lake that day multiplied by the concentration of chemical waste in the water.

Now this problem has a twist, it says that there is a river flowing through it.

And because of that this waste it will not remain constant.

So, some chemical waste will leave the lake each day because of the flow.

And hence the amount of chemical waste that leaves the lake each day is equal to the amount of water that leaves that lake multiplied by the concentration of the chemical waste in that water which we have to calculate now.

So flow of chemicals out of the lake that = concentration C_t of chemicals in kg per meter cube × daily flow of water through lake in meter cube.

So, this will be C_tF kg. But you have shown that $W_t = VC_t$

So, basically this $C_t = \frac{W_t}{V} = \frac{W_t}{2\times10^7}$.

So, this is equal to W_t by V into F. So, F by V into W_t which is the flow of chemicals out of the lake.

So, therefore, change in chemicals per day that is equal to amount added per day multiplied by amount removed. So, change is

$$W_{t+1} - W_t = R - \frac{F}{V}W_t$$

So, this is gives you the model of chemical pollutant in a lake where there is a flow of river.

Now, to get the equilibrium solution, we assume there is no change in T and T plus 1 and we substitute that to be some W^*. So,

$$W^* - W^* = R - \frac{F}{V}W^*.$$

This cancels and this gives $W^* = \frac{RV}{F}$

And if you want to calculate numerically, you know $W^* = \frac{100\times2\times10^7}{10^4} = 200000$ kgs.

So this is the equilibrium solution of the chemical waste.

Now to find the stability, so let us write this function as, so you can write this as

$$W_{t+1} = W_t + R - \frac{F}{V}W_t$$

So, let us put

$$f(W) = W_t + R - \frac{F}{V}W_t$$

So,

$$f'(W) = 1 - \frac{F}{V}$$

So, the system will be stable if $\left|1 - \frac{F}{V}\right| < 1$

So, you have everything fixed value $\left|1 - \frac{10^4}{2\times10^7}\right| = 0.9995$ which is less than 1 and hence the system is stable.

And if you now want to solve this numerically, so I will substitute the values of R, F and V and you will get

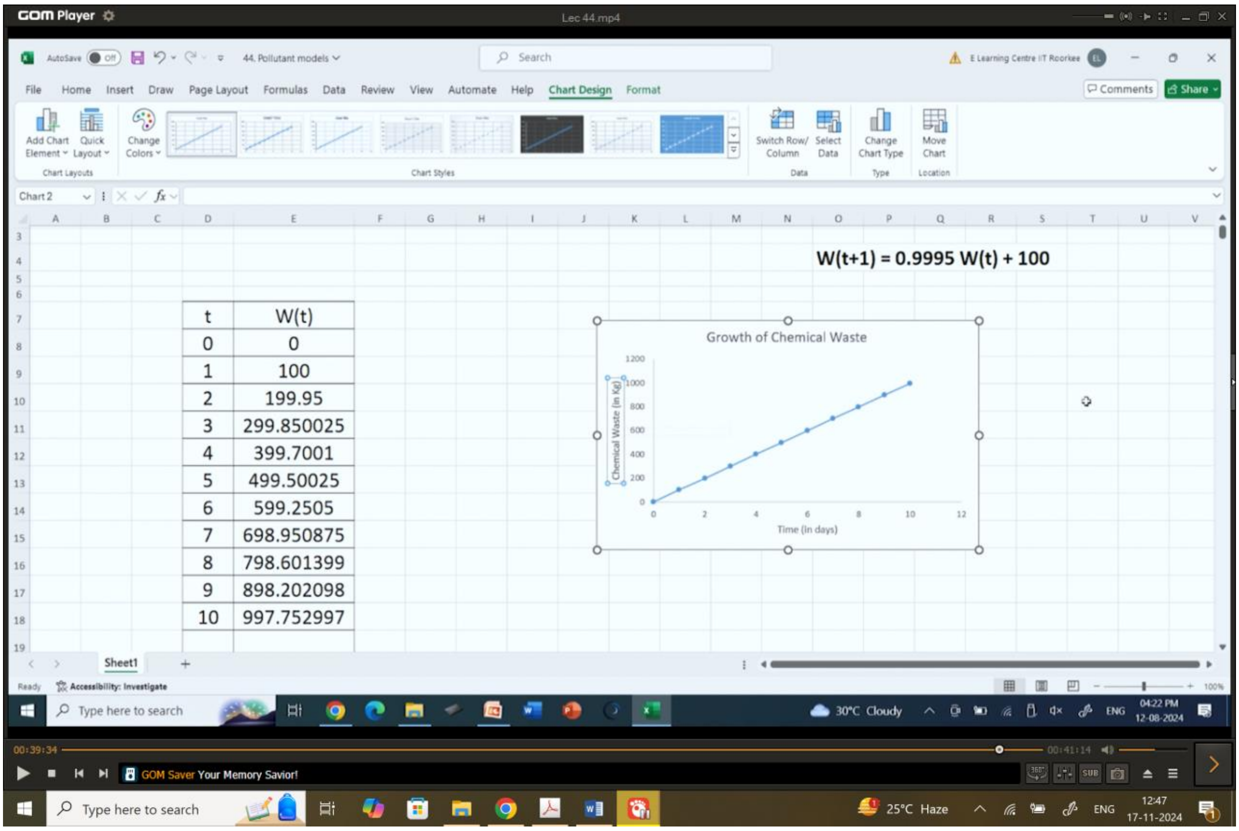
$$W_{t+1} = \left(1 - \frac{F}{V}\right)W_t + R$$

So, already you have calculated that your F is 10 to the power 4, your V is 2 into 10 to the power 7 and 1 minus F by V is gives 0.9995 W_t and this is value is 100.

$$W_{t+1} = 0.9995W_t + 100$$

So, let us see the solution of this model numerically say we want to find what will be the chemical waste after 10 days.

Chemical waste after 10 days taking the initial value is 0. So, at time t equal to 0 there is no waste.



Let's now solve this equation using Microsoft Excel.

So, I already have an Excel sheet opened here.

The equation which I need is

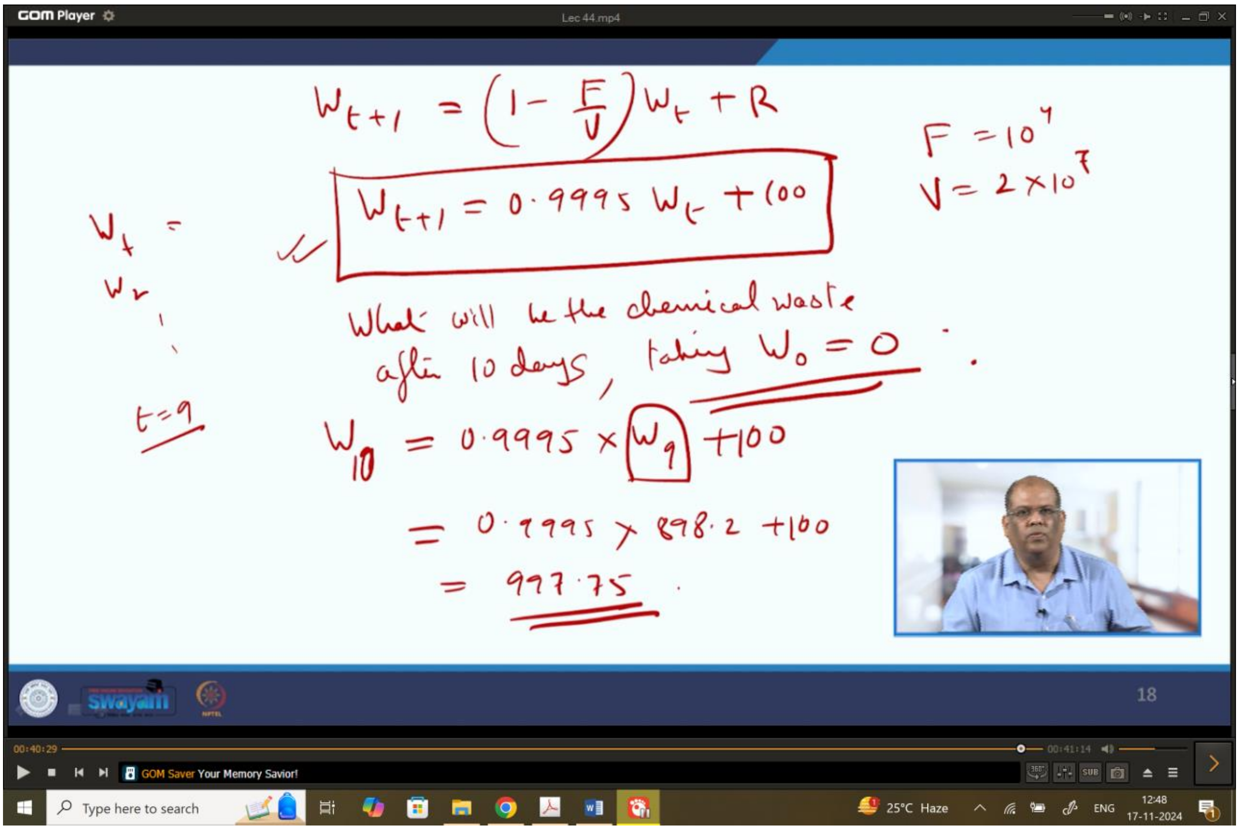
$$W_{t+1} = 0.9995W_t + 100$$

So, I just increase the font size.

So, I need to solve this one with the initial condition that at time t equal to 0 the value is 0.

So, let us first put this is equal to 0 plus and let us drag it to say 10 days and this is going to be this is equal to 0.9995 multiplied by W0 plus 100.

So I just drag it to 10 days.



So you can see the waste after 10 days is 997.75.

If you want to plot, highlight this to go to insert, go to chart and click this. So the chart title is growth of chemical waste. If you want to remove the grid lines, you click this. If you want access title, you just click here.

This is your time and this is your chemical waste in kg.

So, we now look into the analytical equation and if you want to find the amount of chemical waste after 10 days from this particular equation.

So, all you have to put is T equal to 9.

So, you can calculate what is W10 is equal to 0.9995 multiplied by W9 plus 100.

So, basically you have to start with W1, W2 and go on calculating and you will see that this W9 is coming to be 898.2 plus 100 and this is equal to 997.75 and this matches exactly with the numerical solution.

So, with this we come to the end of this lecture where we have discussed the lake pollutant models.

In my next lecture, we will be talking about the dynamics of alcohol model.

So, this is one of the most interesting model.

Till then, bye-bye.