EXCELing with Mathematical Modeling

Prof. Sandip Banerjee

Department of Mathematics

Indian Institute of Technology Roorkee (IITR)

Week – 10

Lecture – 46 (Discrete Predator- Prey Model)

Hello, welcome to the course EXCELing with Mathematical Modeling.

Today, we will be discussing about linear predator prey model.

So, we consider the predator, which is the tiger and the prey which is the deer.

So, we assume that in a forest, you have tigers and you have deer, the tiger kills their prey the deer for food and your T_n and D_n be the respective population of tigers and deer at the end of year n. Now, while constructing the model, we make some assumptions and the assumptions are as follows.

So, deer is the only source of food for the tigers.

Okay, there is nothing else, only the deer and tiger is the only predator for deer.

So, it is very, very well defined that the tigers will eat the deer and deer is the only source of food for the tiger.

Now, the deer population will grow exponentially if there is no tiger and without the deer population, the tiger will die due to starvation.

So, that is the second assumption and the rate at which the tiger population grows that will increase with the presence of deer population and the rate at which the deer population decreases with the presence of tiger population.

This is some sort of obvious because when the tiger will find a deer obviously it will try to kill it for its food and its population will increase and at the same time when the deer dies obviously the population of the deer decreases.

Now let us formulate the model so you have T_{n+1}, the tiger population and D_{n+1}, the deer population.

This T_n is the tiger at year n, $D_n$the deer at year n.

Now, without the deer the tiger will die due to starvation say at a rate α and with the presence of deer the tiger will grow, say, at the rate of β.

Similarly, in case of deer if there is no tiger the deer will go exponentially say at a rate γ and when there will be a presence of tiger obviously it will eat the deer and hence the deer population will decrease at a rate δ.

$$T_{n+1} = T_n - \alpha T_n + \beta D_n$$

$$D_{n+1} = D_n + \gamma D_n - \delta T_n$$

So here α is the rate at which the tiger population dies if there is no food in this case deer and β is the rate at which the tiger population grows, when it comes in contact or when it kills and eat the deer.

Similarly, γ is the rate at which deer population grows if there is no tiger and δ the rate at which the deer population decreases in presence of tigers.

So, these are the four parameters α, β, γ and $\delta > 0$, and $0 < \alpha, \delta < 1$.

So this is the restriction on the parameters.

So once this model is defined, let us now look into its equilibrium solution. So for equilibria, there should not be any change from n generation to n+1 generation, hence we take both the values

$$T_{n+1} = T_n = T^*$$

$$D_{n+1} = D_n = D^*$$

and you substitute them, to get

$$T^* = T^* - \alpha T^* + \beta D^*$$

$$D^* = D^* + \gamma D^* - \delta T^*$$

This cancels and you are left with

$$-\alpha T^* + \beta D^* = 0$$

$$\gamma D^* - \delta T^* = 0$$

If you write it in a matrix form, this will be

$$\begin{pmatrix} -\alpha & \beta \\ -\delta & \gamma \end{pmatrix} \begin{pmatrix} T^* \\ D^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This is a homogeneous equation because the right hand side is 0 and it will have a unique solution $T^* = 0, D^* = 0$, provided

$$\text{Det} \begin{pmatrix} -\alpha & \beta \\ -\delta & \gamma \end{pmatrix} \neq 0.$$

$$\implies -\alpha\gamma + \beta\delta \neq 0$$

and then we have a unique solution (0,0).

So, (0,0) is your equilibrium point.

So, once you get the equilibrium solution, you now check for its stability. So, you have the equation

$$T_{n+1} = T_n - \alpha T_n + \beta D_n$$

$$D_{n+1} = D_n + \gamma D_n - \delta T_n$$

So, the coefficient matrix or the Jacobian matrix in this case will be

$$A = \begin{pmatrix} 1-\alpha & \beta \\ -\delta & 1+\gamma \end{pmatrix}$$

So I can take this is as

$$T_{n+1} = T_n - \alpha T_n + \beta D_n = f_1$$

$$D_{n+1} = D_n + \gamma D_n - \delta T_n = f_2$$

and your matrix

$$A = \begin{pmatrix} \frac{\partial f_1}{dT} & \frac{\partial f_1}{dD} \\ \frac{\partial f_2}{dT} & \frac{\partial f_1}{dD} \end{pmatrix} \text{ at } (T^*, D^*)$$

$$\Rightarrow A = \begin{pmatrix} 1-\alpha & \beta \\ -\delta & 1+\gamma \end{pmatrix} \text{ at } (0,0)$$

So, now you have to find the eigenvalue of this matrix and this will give you

$$det\begin{pmatrix} 1-\alpha-\lambda & \beta \\ -\delta & 1+\gamma-\lambda \end{pmatrix} = 0,$$

where λ is your eigenvalue. This implies

$$\begin{vmatrix} \lambda-(1-\alpha) & \beta \\ -\delta & \lambda-(1+\gamma) \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - (2-\alpha+\gamma)\lambda + (1-\alpha)(1+\gamma) + \beta\delta = 0$$

$$\Rightarrow \lambda = \frac{(2-\alpha+\gamma) \pm \sqrt{(2-\alpha+\gamma)^2 - 4((1-\alpha)(1+\gamma)+\beta\delta)}}{2}$$

$$\Rightarrow \lambda = \frac{(2-\alpha+\gamma) \pm \sqrt{(1-\alpha+1+\gamma)^2 - 4(1-\alpha)(1+\gamma) - 4\beta\delta)}}{2}$$

$$\Rightarrow \lambda = \frac{(2-\alpha+\gamma) \pm \sqrt{(\alpha+\gamma)^2 - 4\beta\delta)}}{2}$$

So, this is the generalized version of the eigenvalues in terms of the parameters α, β, γ and δ. Let us now see how this will behave if you take the numerical values of the parameters.

So, if you put $\alpha = 0.5$, $\beta = 0.4$, $\gamma = 0.1$ and $\delta = 0.17$.

So, your coefficient matrix or Jacobian matrix is

$$A = \begin{pmatrix} 1-\alpha & \beta \\ -\delta & 1+\gamma \end{pmatrix}$$

If you substitute these values and you get

$$A = \begin{pmatrix} 0.5 & 0.4 \\ -0.17 & 1.1 \end{pmatrix}$$

and

$$|A - \lambda I| = 0 \Longrightarrow \begin{vmatrix} 0.5-\lambda & 0.4 \\ -0.17 & 1.1-\lambda \end{vmatrix} = 0$$

$$\Longrightarrow (0.5-\lambda)(1.1-\lambda) + 0.068 = 0$$

If you simplify this, you will get

$$\lambda^2 - 1.6\lambda + 0.618 = 0$$

and you can solve $\lambda_1 = 0.948$ and $\lambda_2 = 0.652$.

So as you can see, one of the root is λ_1, another root is λ_1, modulus of them

$$|\lambda_1| < 1, |\lambda_2| < 1.$$

So, this implies (0,0) is stable equilibrium point.

However, if you change the values, say, $\alpha = 0.5$, $\beta = 0.4$, $\gamma = 0.1$ and $\delta = 0.05$.

In this particular case, you will see your coefficient matrix is

$$A = \begin{pmatrix} 0.4 & 0.5 \\ -0.05 & 1.1 \end{pmatrix}$$

and you can easily calculate the eigenvalue to be $\lambda_1 = 1.06$ and $\lambda_2 = 0.535$

So, if you notice now your be $|\lambda_1| = |1.06| > 1$ and be $|\lambda_2| = |0.535| < 2$, which implies the system is unstable at (0,0).

Now let us look into the numerical solution using Microsoft Excel.

We just see that whether our analytical result it matches with the numerical one.

So I open the excel sheet I will quickly copy this one. So you have the n, you have the tiger population and the deer population.

So I start with 0 and the second one will be 0+1 and say I calculate the next 30 values.

So let us take initial population, say, T(0)=500 and this is D(0)=200.

So this is equal to, so in this particular case, let us start with the first one your $\alpha = 0.5$, $\beta = 0.4$, $\gamma = 0.1$ and $\delta = 0.17$. This is equal to say which is $T_0 + h$.

So if you look into this equation it is

$$T_{n+1}\ if\ n = 0,$$

this is,

$$T_1 = T_0 - \alpha T_0 + \beta D_0 = T_{n+1} = (1 - \alpha)T_0 + \beta D_0$$

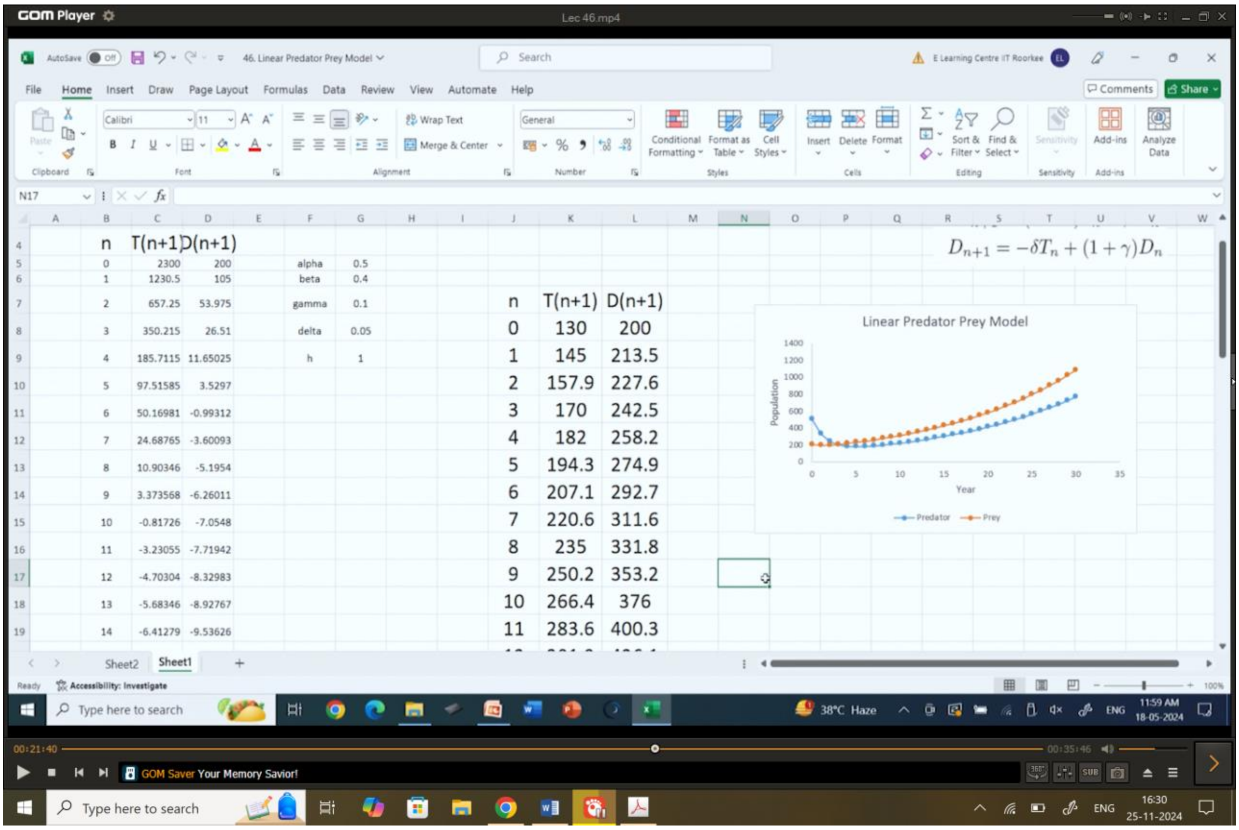
So, α is constant and your β is also a constant.

Similarly, if you calculate the deer population, it is D_0

$$D_1 = D_0 + \gamma\, D_0 - \delta T_0$$

G0 L8 which is constant that is your γ and G8 γ is G7 sorry so this will be constant and this is your δ which is again constant.

So, now I drag these values up to 30 of them and let us now plot them just before that I make the font a bit bigger and in the middle, go to insert, go to this chart and take this chart.



So, I will get it this one change in the title linear predator prey model and highlight this series go to select data then this series edit the first one is the predator and the second one is the prey okay.

So if I want to remove the grid lines, if I want an axis title, I will put it here.

So this is your time or you can put year and this is your population and you get this.

So, you can see that with this parameter values we have already proved that the equilibrium (0,0) is stable and hence if you have started from (500,200) it approaches to (0,0).

Now, let us see what happens if I change this value to δ =0.05.

And immediately you see that the system becomes unstable and it increases unboundedly from starting from T(0)=500 and D(0)=200.

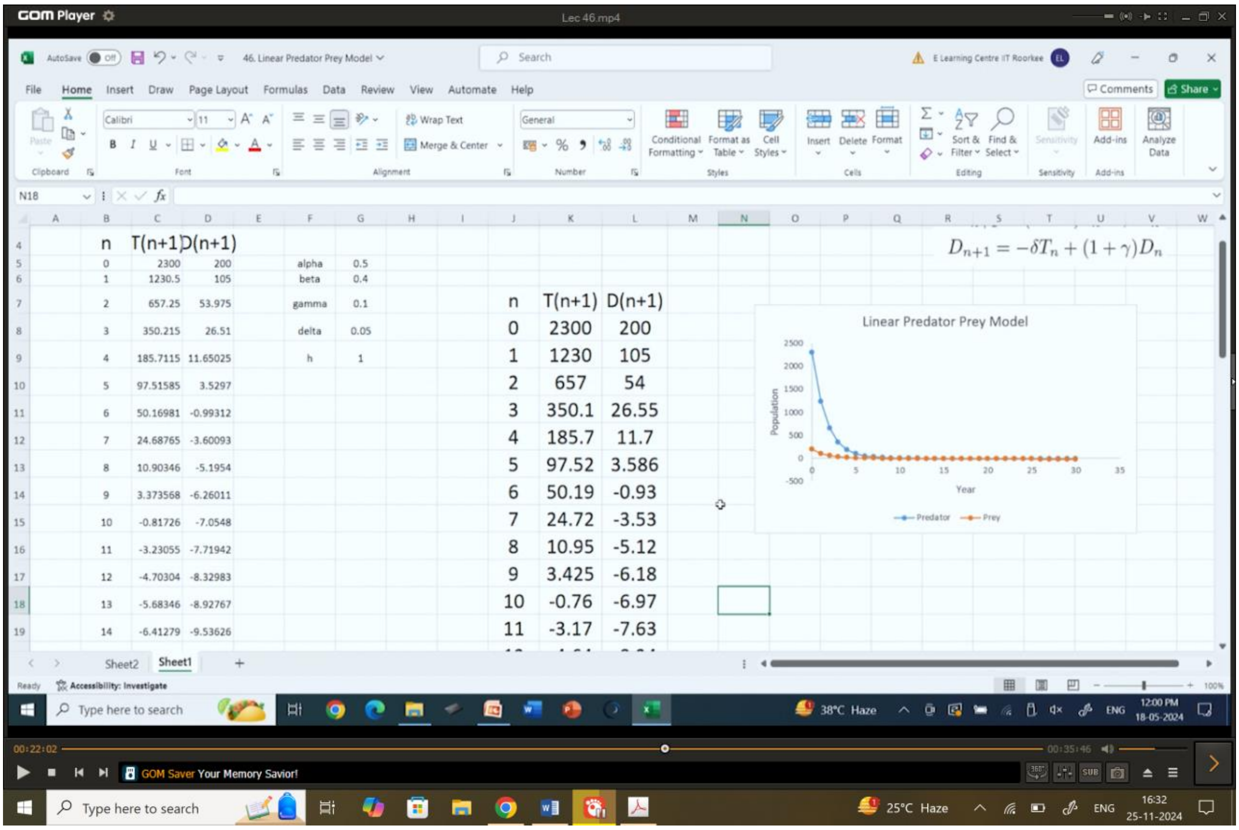
So, this matches which are analytical results.

Now, quickly what will happen if I change the initial condition. So, this system is now unstable and I am playing with the initial condition.

I make this T(0)=130, let us see okay still it is unstable and it is increasing unboundedly.

How about if I increase them to a very large number of tigers, T(0)=2300.

So, you see that again they both goes to (0,0).



What I am trying to say is in an unstable condition if you change the initial conditions the dynamics changes because since the system is not stable it will not reach the equilibrium point always from wherever you start.

Let us now go back to the slides quickly recapitulate the numerical results.

Case 1: Your $\alpha = 0.5, \beta = 0.4, \gamma = 0.1$ and $\delta = 0.17$.

This is the case where (0,0) is stable and you see that you have started from 500 and 200 and you reach the (0,0), your γ value changes from 0.17 to 0.05 and analytically we have seen that the system is not stable rather unstable.

So, again you have started from (500,200) and you see that your graph increases unbounded.

In the third case, I take T(0)=150, your D remains the same D(0)=200. These values remains the same which means your system is unstable and in this particular graph you also see the same thing you started from 150 and 200 and the increases unboundedly.

And finally I make the initial population of the tigers to be T(0)=2300 and you see that both the deer population and the tiger population goes to (0,0).

So it is obvious because with so many tigers they will eat the deer and when there is no deer to eat obviously the tiger population will also die due to starvation.

We now modify this linear model in a non-linear model.

So, our model is

$$T_{n+1} = T_n - \alpha T_n + \beta D_n$$

$$D_{n+1} = D_n + \gamma D_n - \delta T_n$$

So to make it non-linear, we introduce the interaction between the tiger and the deer.

So when there is an interaction of the tiger and the deer, obviously the tiger eats the deer at a rate

$$T_{n+1} = T_n - \alpha T_n + \beta D_n T_n$$

$$D_{n+1} = D_n + \gamma D_n - \delta T_n D_n$$

So, there was a negative sign and which is then added up to the population of tiger due to reproduction.

So, if we now look into this model, our first thing is the equilibrium solution. That means there is no change in the n and n+1 and we substitute that to be

$$T_{n+1} = T_n = T^*$$

$$D_{n+1} = D_n = D^*$$

So, we put the value here and you get

$$T^* = T^* - \alpha T^* + \beta D^* T^* \Longrightarrow T^*(-\alpha + \beta D^*) = 0$$

$$\Longrightarrow T^* = 0, D^* = \frac{\alpha}{\beta}$$

Now, from this equation you will get

$$D^* = D^* + \gamma D^* - \delta T^* D^* \Longrightarrow D^*(\gamma - \delta T^*) = 0$$

Now, when $T^* = 0$, we are using this. This will imply so this word vanishes $\gamma D^* = 0$,

$$\Longrightarrow D^* = 0, \text{ as } \gamma \neq 0.$$

So, (0,0) is the one of the equilibrium solution and when we put

$$D^* = \frac{\alpha}{\beta}$$

we get from here

$$\frac{\alpha}{\beta}(\gamma - \delta T^*) = 0 \Longrightarrow T^* = \frac{\gamma}{\delta}$$

So, $\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$ is another equilibrium solution. Let us now look into the stability analysis.

So, I have

$$T_{n+1} = T_n - \alpha T_n + \beta T_n D_n = f_1(T, D)$$

$$D_{n+1} = D_n + \gamma D_n - \delta T_n D_n = f_2(T, D)$$

where

$$f_1(T, D) = T - \alpha T + \beta T D$$

$$f_2(T, D) = D + \gamma D - \delta T D$$

So, this will give

$$\frac{\partial f_1}{\partial T} = 1 - \alpha + \beta D, \text{ and } \frac{\partial f_1}{\partial D} = \beta T,$$

$$\frac{\partial f_2}{\partial T} = -\delta T, \text{ and } \frac{\partial f_2}{\partial D} = 1 + \gamma - \delta T$$

So the matrix

$$A = \begin{pmatrix} \frac{\partial f_1}{dT} & \frac{\partial f_1}{dD} \\ \frac{\partial f_2}{dT} & \frac{\partial f_1}{dD} \end{pmatrix} = \begin{pmatrix} 1 - \alpha + \beta D & \beta T \\ -\delta T & 1 + \gamma - \delta T \end{pmatrix} \text{ at } (T^*, D^*)$$

So, let us now check the values at the equilibrium points.

The first $(T^*, D^*) = (0,0)$. So, if we substitute it here, we will get

$$A = \begin{pmatrix} 1-\alpha & 0 \\ 0 & 1+\gamma \end{pmatrix}$$

So, clearly eigen values are $1-\alpha$ and $1+\gamma$. Now, the parameter values α, β, γ and $\delta > 0$ all positive and this value $1+\gamma > 1$

So, if I take $|1-\alpha| < 1$ and $|1+\gamma| > 1$. So, at the equilibrium point, this particular model is unstable.

Let us check the non-zero equilibrium point.

So, you have

$$A = \begin{pmatrix} 1-\alpha+\beta D^* & \beta D^* \\ -\delta T^* & 1+\gamma-\delta T^* \end{pmatrix}$$

where your

$$T^* = \frac{\gamma}{\delta}, \qquad D^* = \frac{\alpha}{\beta}$$

If I substitute

$$A = \begin{pmatrix} 1-\alpha+\beta\dfrac{\alpha}{\beta} & \beta\dfrac{\alpha}{\beta} \\ -\delta\dfrac{\gamma}{\delta} & 1+\gamma-\delta\dfrac{\gamma}{\delta} \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ -\gamma & 1 \end{pmatrix}$$

So, if you want the eigenvalues is

$$|A-\lambda I| = 0 \Longrightarrow \begin{vmatrix} 1-\lambda & \alpha \\ -\gamma & 1-\lambda \end{vmatrix} = 0$$

This will be

$$(1-\lambda)^2 + \alpha\,\gamma = 0 \Longrightarrow (\lambda-1)^2 = -\,\alpha\,\gamma$$

$$\Longrightarrow (\lambda-1)^2 = \alpha\,\gamma i^2 \Longrightarrow \lambda - 1 = \pm i\sqrt{\alpha\,\gamma}$$

$$\Longrightarrow \lambda = 1 \pm i\sqrt{\alpha\,\gamma}$$

So, we have a complex eigenvalue with positive real part I have to take the modulus of this

Again you can see that this will be

$$|\lambda| = \left|1 \pm i\sqrt{\alpha\,\gamma}\right| = \sqrt{1+\alpha\,\gamma} > 1$$

and this α, β, γ and $\delta > 0$ and $0 < \alpha, \gamma < 1$ and hence the

$$|\lambda| = \left|1 \pm i\sqrt{\alpha\,\gamma}\right| = \sqrt{1+\alpha\,\gamma} > 1$$

and hence this system is also unstable at the non-zero equilibrium point.

So, this model the non-linear one is not a very good model as you can see that whatever equilibrium points we have, the whole model is unstable no matter whatever the parameter values are.

So, summing up, in this lecture we learned about the linear preparator model where (0,0) is the only equilibrium point.

We perform linear stability analysis about this (0,0) equilibrium point.

Our numerical results matches with the analytical results.

We then modify this model to a non-linear one.

Our analysis shows that the model is always unstable, no matter what the parameter values are.

In my next lecture, we will be talking about forensic model, which is an application of Newton's law of cooling.

Till then, bye-bye.