EXCELing with Mathematical Modeling

Prof. Sandip Banerjee

Department of Mathematics

Indian Institute of Technology Roorkee (IITR)

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Lecture – 47 (Forensic Model)

Hello welcome to the course EXCELing with mathematical modeling.

Today we will be talking about forensic model. For that, we need to know this law, Newton's law of cooling. So the whole theory is based on this law.

So what does that law tells?

It says the rate of change of the temperature of an object is proportional to the difference between its own temperature and the ambient temperature.

So, by ambient temperature we mean temperature of its surroundings.

So, if you want to mathematically write this.

So the rate of change of temperature is proportional to the difference.

So the rate of change in discrete case is given by this

$$T_{n+1} - T_n$$

It is the temperature after nth generation or after nth time and the difference between the (n+1)th time.

So this is the rate of change and it is proportional to the difference between its own temperature, that is, your T_n and the ambient temperature that is its surroundings which we can take as some S. So it is some $S - T_n$ and this is proportional to

$$T_{n+1} - T_n \propto S - T_n$$

So you can write

$$T_{n+1} - T_n = k(S - T_n)$$

Now one can ask that, okay, if I write

$$T_n - S$$

what will happen?

Well, nothing will happen.

It is proportional to the difference. So whether you write this one first or this one first really does not matter because it will be taken care by this constant of proportionality.

If for one case it comes positive, for the other case it will come negative.

So, it will be adjusted. So, you do not have to worry about this.

You can take any one of them as the first, the second one as the second.

So, here your T_n is the temperature of the object after some time say some n minutes can be an hours also, S is the surrounding temperature, and k is the constant of proportionality.

Now, suppose you have a cup of coffee in the room, say the temperature is 190°F and say after 1 minute it becomes 180°F after 1 minute.

So, your equation was

$$T_{n+1} - T_n = k(S - T_n).$$

So, in this particular case, S is the surrounding temperature or the room temperature which we take as S=70°F.

Now I put say n=0.

So you get

$$T_1 - T_0 = k(S - T_0)$$

So your $T_1 = 180$, your $T_0 = 190$, k, you do not know, this is your S=70 and your $T_0 = 190$.

So you get this particular equation,

$$180 - 190 = k(70 - 190)$$

$$\Rightarrow -10 = k(-120) \Rightarrow k = \frac{10}{120} \Rightarrow k = \frac{1}{12}$$

So, you can substitute it here

$$T_{n+1} - T_n = \frac{1}{12}(S - T_n) \Rightarrow T_{n+1} - T_n = \frac{1}{12}(70 - T_n)$$

$$\Rightarrow T_{n+1} = T_n + \frac{70}{12} - \frac{T_n}{12}$$

$$\Rightarrow T_{n+1} = \frac{11}{12}T_n + \frac{70}{12}$$

This equation, it is of the form

$$u_{n+1} = au_n + b.$$

You can either recall the solution or but it is quite easy to derive.

So,

$$u_{n+1} = au_n + b$$
$$u_n = au_{n-1} + b$$
$$u_{n-1} = au_{n-2} + b$$
$$\dots\dots\dots\dots$$

So we use that, you substitute this u_n here, so

$$u_{n+1} = au_n + b = u_{n+1}$$
$$= a(u_{n-1} + b) + b$$
$$= a^2u_{n-1} + ab + b = a^2u_{n-1} + b(a+1)$$
$$= a^2(au_{n-2} + b) + b(a+1)$$
$$= a^3u_{n-2} + a^2b + b(a+1) = a^3u_{n-2} + b(a^2 + a + 1)$$
$$\dots\dots\dots\dots\dots\dots$$
$$u_{n+1} = a^{n+1}u_0 + b(a^n + a^{n-1} + a^{n-2} + \cdots + a + 1)$$

So if I now compare it with this particular equation, I will get

$$T_{n+1} = \left(\frac{11}{12}\right)^{n+1} T_0 + \frac{70}{12}\left(\frac{1-a^{n+1}}{1-a}\right), \quad a = \frac{11}{12} < 1$$

$$\Rightarrow T_{n+1} = \left(\frac{11}{12}\right)^{n+1} 190 + \frac{70}{12}\left(\frac{1-\left(\frac{11}{12}\right)^{n+1}}{1-\left(\frac{11}{12}\right)}\right)$$

$$\Rightarrow T_{n+1} = \left(\frac{11}{12}\right)^{n+1} 190 + 70 - 70\left(\frac{11}{12}\right)^{n+1}$$

$$\Rightarrow T_{n+1} = 70 + 120\left(\frac{11}{12}\right)^{n+1}$$

So this is the equation which you get for a cup of coffee which was kept at the surrounding temperature which is the room temperature at 70°F

So what happens in the long run? So as your $n \longrightarrow \infty$. you see that your $T_{n+1} \longrightarrow 70$, it approaches the room temperature because

$$\frac{11}{12} < 1, \text{hence } \left(\frac{11}{12}\right)^{n+1} \longrightarrow 0 \text{ and } T_{n+1} \longrightarrow 70.$$

So, we now look into the numerical solution. The equation which we will be solving is

$$T_{n+1} = \frac{11}{12}T_n + \frac{70}{12}$$

If I want to change the font size, make it 20. So, I just and I make it bold.

So, this is the equation which shows a cup of coffee is kept in a room and we want to see in the long run at what temperature the coffee becomes.

If you want to type this in equation, so you have to go to this insert and you have to click this equation and you will see such a thing will appear.

So, if I want Tn plus 1, I will go here and I will click this one for the subscript.

So, this is your t and this will be your n+1 and that is equal to T_n.

If you need a fraction, you have to go here fraction and click this, this will give

$$T_{n+1} = \frac{11}{12}T_n + \frac{70}{12}$$

So basically this equation written in mathematical form, if you want to increase the font size, I can make it 20 and bold and I can put it here.

So the same equation in the mathematical form by going to insert and this clicking this equation.

Let's now solve this difference equation.

So we put n here, T_n here at time 0.

This value is 190°F. I just increase the font to 20 and middle the values.

So, this is equal to this plus 1, say let us calculate this one, this is equal to $\frac{11}{12}$. So, $\frac{11}{12}$, put it in a bracket, and this is

$$T_1 = \frac{11}{12}T_0 + \frac{70}{12}$$

I also put this one in a bracket and end. So, let me now drag this, little more.

So, as you can see say up to 100 values.

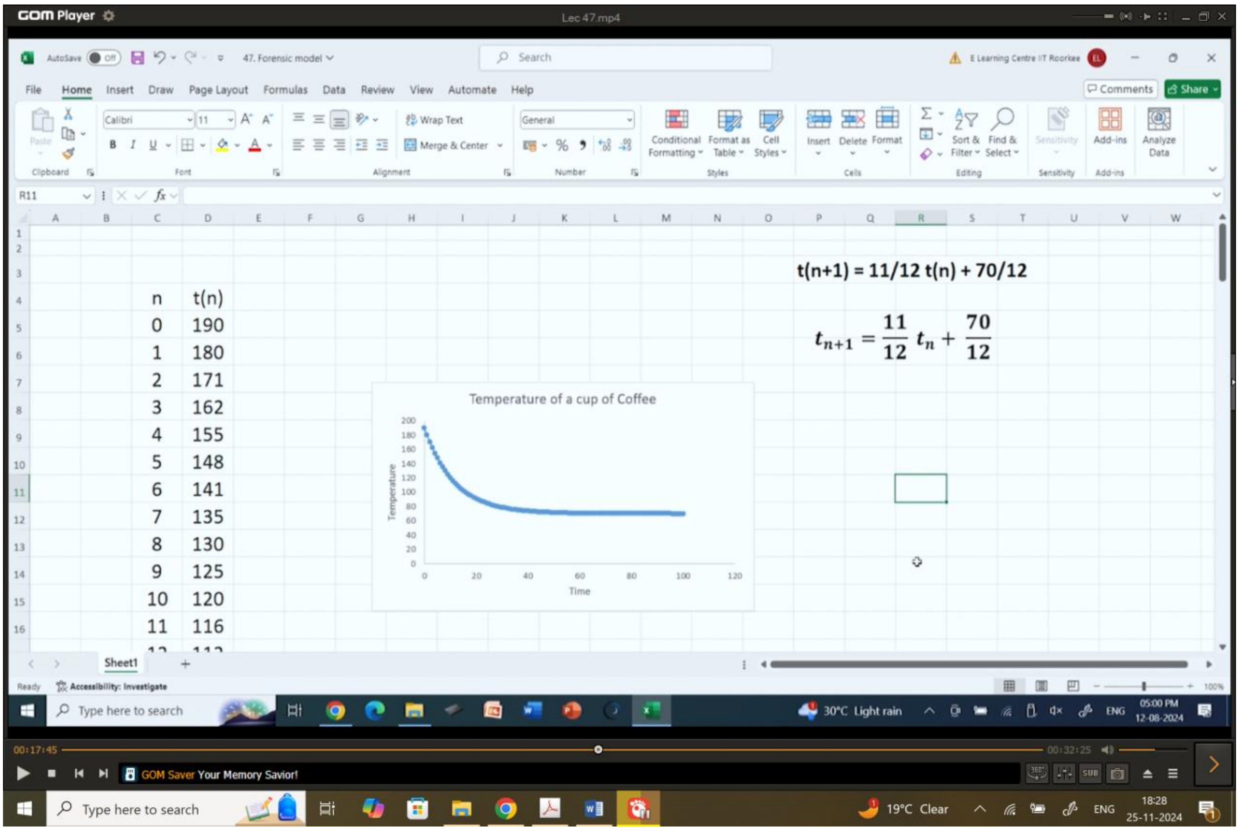
So, as your time becomes large the cup of coffee reaches the value Fahrenheit which was the room temperature.

So, we now plot this graph and we will get a clear picture here.

So, I highlight them, I go to insert, click the chart and thus.

So, I remove the grid lines, I put access title. x-axis is the time and y-axis is the temperature and temperature of a cup of coffee.

So you can see that with time this approaches the steady state 70°F which matches with our analytical result. So this is what the application of Newton's law of cooling.



Now let us see how this law can be applied to a forensic model.

So, what does this forensics do generally when there is murder or any sort of crime this team goes there and try to collect evidences.

So, this particular law will be used to calculate the approximate time of murder and how it is done.

So, let us formulate the problem.

A group of investigators they got a call or the police get a call that there has been a murder in the neighbourhood and say at 5 am the team of investigators arrive.

So, they took the temperature of the body, and they find that it is 87.5°F.

So, then the body is removed and after 1 hour when the body is to be shifted to somewhere.

So, after 1 hour they again took the temperature and it is 80.4°F.

So, the normal body temperature of a alive medicine can be approximately taken as 98.6°F. So, now the question is find the time when the victim was murdered.

We will be using this Newton's law of cooling to find an approximate time when the victim was murdered and this is exactly what is done in the real life scenario also.

So, once again the gist of the problem is that there is been a murder in the neighbourhood, the police got a call and the first team of investigator arrives at 5 am, they took the temperature of the body which was 87.5°F. The body is removed after 1 hour when they again took the temperature which is 80.4°F. The normal temperature of a person when he is alive or she is alive is 98.6°F.

So, you have to find an approximate time when the victim was murdered.

So, let us start with the Newton's law of cooling which says

$$T_{n+1} - T_n = k(S - T_n)$$

So, if I put n=0, I get

$$T_1 - T_0 = k(S - T_0)$$

$$\Longrightarrow T_1 = T_0 + Sk - kT_0 = (1-k)T_0 + kS$$

So, if I want n=1, so

$$\begin{aligned} T_2 &= T_1 + k(S - T_1) = (1-k)T_1 + kS \\ &= (1-k)[(1-k)T_0 + kS] + kS \\ &= (1-k)^2 T_0 + (1-k)kS + kS \\ &= (1-k)^2 T_0 + (1+1-k)kS \end{aligned}$$

$$\Longrightarrow T_2 = (1-k)^2 T_0 + (1+1-k)kS$$

So, in the similar manner,

$$T_3 = (1-k)^3 T_0 + [1 + (1-k) + +(1-k)^2]kS$$

$$\dots \dots \dots \dots \dots \dots.$$

$$T_n = (1-k)^n T_0 + [1 + (1-k) + +(1-k)^2 + \cdots + (1-k)^{n-1}]kS$$

So, this is a simple GP series and we get

$$\begin{aligned} T_n &= (1-k)^n T_0 + \frac{1-(1-k)^n}{1-(1+k)} kS \\ &= (1-k)^n T_0 + \frac{1-(1-k)^n}{k} kS \end{aligned}$$

$$T_n = (T_0 - S)(1-k)^n + S$$

So, some sort of formula.

Now, let us calculate the time for model.

So, I rewrite

$$T_n = (T_0 - S)(1-k)^n + S$$

So, after the murder the forensic team arrives after say P minutes.

So, the first equation

The initial temperature of the body is T_0 =98.6 and the surrounding temperature let us take as S=72.

So, if I substitute this in this equation, so just at the point of murder the temperature of the body is 98.6 minus the surrounding temperature.

$$87.5 = T_P = (98.6 - 72)(1-k)^P + 72$$

So, after P minutes this temperature was found to be 87.5 because this information was already given here.

The first one was 87.5 and the second one was 80.4 and after 1 hour.

if I write it in minutes, this is P+60.

The temperature of this T_0 remains the same, the surrounding remains the same,

$$80.4 = T_{P+60} = (98.6 - 72)(1-k)^{P+60} + 72$$

So now you have to solve this.

So from the first equation, what we will get is

$$26.6(1-k)^P = 87.5 - 72 = 15.5$$

and from this equation this is from 1 and from 2 you will get

$$26.6(1-k)^{P+60} = 80.4 - 72 = 8.4$$

Okay so now what you do is you just divide.

So I will divide and get

$$\frac{26.6(1-k)^{P+60}}{26.6(1-k)^P} = \frac{8.4}{15.5} = 0.542$$

And this cancels with this part. So, you are left with

$$(1-k)^{60} = 0.542$$

So you take log both sides and simplify and you will get

$$(1-k) = 0.9898 \Rightarrow k = 0.0101$$

So you got the value of k and you substitute it here to solve for P.

So, k=0.0101 and

$$(1-0.0101)^P = \frac{87.5-72}{26.6} = 0.5827$$

$$\Rightarrow (0.9898)^P = 0.5827$$

And you take log on both sides, so you will get

$$P = \frac{\log(0.5827)}{\log(0.9898)}$$

and this value will approximately

$$P = \frac{\log(0.5827)}{\log(0.9898)} = 52 \text{ minutes.}$$

So, what is this 52 minutes?

So, if you recall after the murder the forensic team arrive after P minutes.

So, after 52 minutes of the murder, they arrived and the time was that time was 5 AM.

So, if you subtract 52 minutes from their time of arrival, what you get is an approximate time of murder.

So, they arrived at 5 AM, you subtract 52 minutes from there and you get 4:08 AM.

So this gives you the approximate time of murder.

So this is how you use this Newton's law of cooling to solve the approximate time of murder in case of this forensic team uses this particular law which they do actually.

So summing up, in this lecture, we see the application of Newton's law of cooling with two examples.

The first one is about the temperature of a hot coffee which reaches the room temperature over time and supported by numerical results.

The second one is much more interesting where we use the law to find the approximate time of murder.

In my next lecture, we will be talking about some discrete drug delivery model.

Till then, bye-bye.