EXCELing with Mathematical Modeling

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Week – 10

Lecture – 49 (Lanchester’s Combat Model)

Hello welcome to the course EXCELing with Mathematical Modeling.

Today, we will be talking about Lanchester's combat model.

In the continuous case, we have done this, but let us see how this model behaves in the discrete case.

F. W. Lanchester is a British Engineer who formulated a series of mathematical problems that used to predict the outcome and the number of soldiers surviving in a given battle during First World War.

This model it helped in better planning, prediction of the battles and their possible outcomes and was a huge success in doing the First World War. So, he was the one to develop one of the first mathematical model for analyzing combats whose greatest strength lies in its simplicity.

So, let us look into the model.

So in the discrete case, he has taken a team A and a team B. So this is a team A and this is a team B. They are having combat among each other.

So let A_n be the number of soldiers of team A and B_n be the number of soldiers of team B. So, this is the remaining number of soldiers after time n.

There is an assumption here as every model needs an assumption. So, the assumption is the combat loss rate of both the teams is proportional to the size of their respective enemies.

Let us see what does this mean?

It means that the combat loss rate of team A is proportional to the size of team B. Now if you formulate the model using this assumption, so

$$A_{n+1} = A_n - \beta B_n + r_1$$

So, what is this beta? This is the fighting effectiveness of team B.

Similarly,

$$B_{n+1} = B_n - \alpha A_n + r_2$$

So, α is the fighting effectiveness of team A.

So, when two teams or two group of enemies are fighting among each other in combat, so what happens is the loss of one team depends on how efficient the other soldiers of the other team and that is why it depends on the loss of soldiers of team A will depend on the effectiveness of the soldier of team B and similarly the loss of team B will depend on the effectiveness of the soldiers of team A.

Now what is this r_1?

r_1 is the reinforcement for team A and r_2 is the reinforcement for team B.

So, by reinforcement it means either people or supplies or both.

So, supplies of food, supplies of ammunition and obviously supply of extra soldiers.

So, once you get the model you look for the equilibrium solution. We replace

$$A_{n+1} = A_n = A^*$$
$$B_{n+1} = B_n = B^*$$

you put the values here.

So you get

$$A^* = A^* - \beta A^* + r_1$$
$$B^* = B^* - \alpha A^* + r_2$$

This gives

$$B^* = \frac{r_1}{\beta} \text{ and } A^* = \frac{r_2}{\alpha}$$

$\left(\frac{r_2}{\alpha}, \frac{r_1}{\beta}\right)$is the equilibrium solution and the α, $\beta, r_1, and\ r_2$,they are all positive constants.

So, once you have the equilibrium solution let us look into the stability about this equilibrium point. So, you have the model

$$A_{n+1} = A_n - \beta B_n + r_1$$

$$B_{n+1} = B_n - \alpha A_n + r_2$$

The coefficient matrix say $A = \begin{pmatrix} 1 & -\beta \\ -\alpha & 1 \end{pmatrix}$

So this is the coefficient matrix the eigenvalues will be given by

$$|A - \lambda I| = 0$$

which means

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -\beta \\ -\alpha & 1-\lambda \end{vmatrix} = 0 \Longrightarrow (1-\lambda)^2 - \alpha\beta = 0 \Longrightarrow (\lambda - 1)^2 = \alpha\beta$$

You could have written $(\lambda - 1)^2$ also both gives the same result.

This is equal to

$$\lambda - 1 = \pm\, \alpha\beta \Longrightarrow\ \lambda = 1 \pm\ \alpha\beta$$

Now one thing is, we already mentioned that $r_1 > 0, r_2 > 0, \alpha > 0,\ \beta > 0$, but at the same time α, β are the fractions, so it will lie between 0 and 1, that is,

$$0 < \alpha, \beta < 1.$$

So, if I take

$$|\lambda_1| = |1 +\ \alpha\beta|$$

clearly

$$|\lambda_1| = |1 +\ \alpha\beta| > 1$$

and

$$|\lambda_2| = |1 -\ \alpha\beta| < 1$$

So, one of the eigenvalue

$$|\lambda_1| = |1 +\ \alpha\beta| > 1$$

and

$$|\lambda_2| = |1 -\ \alpha\beta| < 1$$

so your equilibrium point gives a saddle.

Now let us look into the numerical solution of this particular model for which we use the Microsoft Excel.

Okay, so I already have the equations here, the value of $\alpha = 0.2$, the value of $\beta = 0.3$, this I make it $\alpha = 0.3$, this I make it $\beta = 0.2$, $r_1 = 3$ and $r_2 = 2$.

So, what does that mean that the effectiveness of soldier of team A is more than the effectiveness of the soldier of team B and the reinforcement of A also is more than of team B. So, let us now calculate these values say the initial value of A(0)= 100 and this is say B(0)=75.

Okay, to solve it numerically I already have the equations here of the model and the initial value of A(0)=75 and that of B(0)=100. The value of α =0.3, value of β=0.2, $r_1 = 3$ and $r_1 = 2$.

Now what does this mean?

You can see that value of α is greater than the value of β which means that fighting effectiveness of the soldiers of team A is greater than the fighting effectiveness of the soldiers of team B. And the reinforcement for team A is also greater than the reinforcement of team B.

So, initially this team A has 75 soldiers and team B has 100 soldiers.

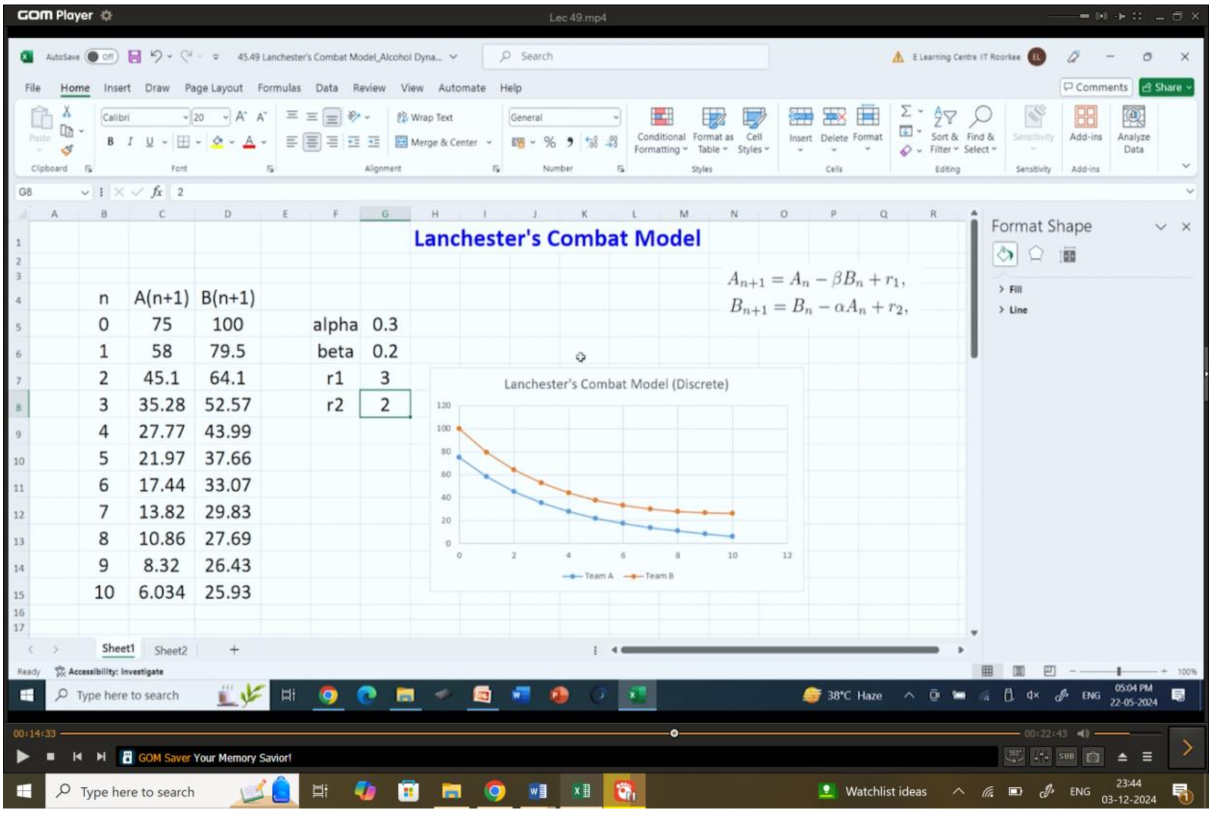
Let us now calculate the next value.

This is equal to An which is this minus beta multiplied by Bn which is 100 and beta is a constant plus r1 again a constant and this value is equal to Bn which is 100 minus alpha which multiplied by An which is this alpha is a constant and plus r2 again a constant.

So, now you highlight these two and drag up to say 10 values. So, if I want the graph of this, I will highlight them, go to insert, go to the chart and get the value.

So, you change the title Lanchester's combat model (discrete) and the series you go to select data series edit you just write Team A, Team B.

So, in this model what you see here is the team B has started from 100, team A has started from 75 though the fighting effectiveness of team A is more than the team B, also the reinforcement of team A is more than team B but still they are winning because their numbers are high. So, the rate at which the team B killing team A is much faster even though their actual fighting effectiveness is low.



Now you can change the values and you can see what happens for other cases also.

For example, if I make this B(0)=75, that is, both teams start with the same number of soldiers where I have kept all these things same, you see a different scenario.

You see now the team A is winning and team B goes to zero. Basically, here you have to stop.

These are the negative values, will not be counted. So, somewhere here the team B lost the battle. If you again want to change the value of alpha and beta suppose I make this as α=0.2 with this A(0)=75 and this as B(0)=100, then you see now the team B they are winning and team A now almost at around 6 time units they goes to zero.

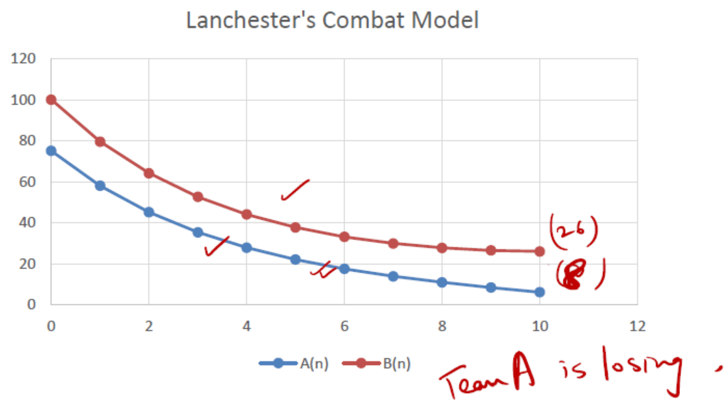
So, there are the various scenarios that can be drawn from this particular model and the strategies of the war they just take out various cases so that they have a winning strategy.

Let us go back to the slides. So, let me one more time explain the solutions and the graphical representation.

Case 1: $\boldsymbol{\alpha = 0.3, \beta = 0.2, r_1 = 3, r_2 = 2, A(0) = 75, B(0) = 100.}$

So, fighting effectiveness for A is greater than the fighting effectiveness of B. Reinforcement for A is greater than the reinforcement of B. So, in that particular case you get these two curves and as you can see this is for team A.

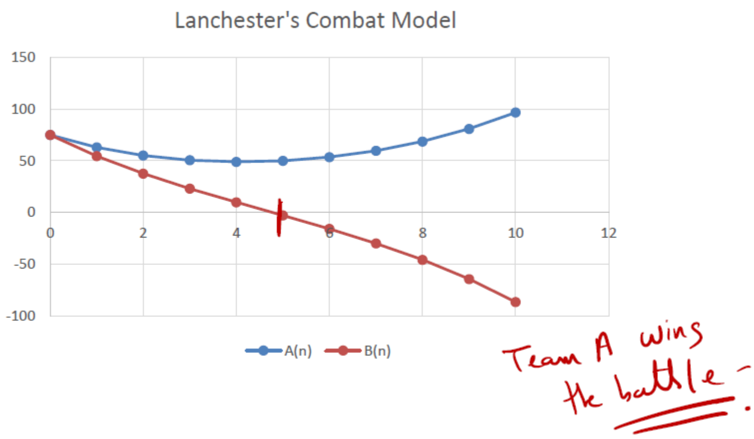
The number of soldiers, this is for team B, the number of soldiers and team A is losing. Here they are approximately 26 number of soldiers and here it is approximately 10, looks like 8.



So, in case 1 when the even the effectiveness of team A is greater than team B when the reinforcement is greater than the team B but due to the number the initial values your team B is winning.

Case 2: $\boldsymbol{\alpha = 0.3, \beta = 0.2, r_1 = 3, r_2 = 2, A(0) = 75, B(0) = 75.}$

In case 2 both the teams have the same initial number. Here also, team A effectiveness of their fighting is more than team B the reinforcement is more than team B and obviously here your team A wins because now the number of both team A(0)=B(0)=75.



So somewhere here, your team B, all of them dies and team A wins the battle. So, as told before, it is because of the number team B won the battle in the first case and here when the numbers are same, team A wins the battle.

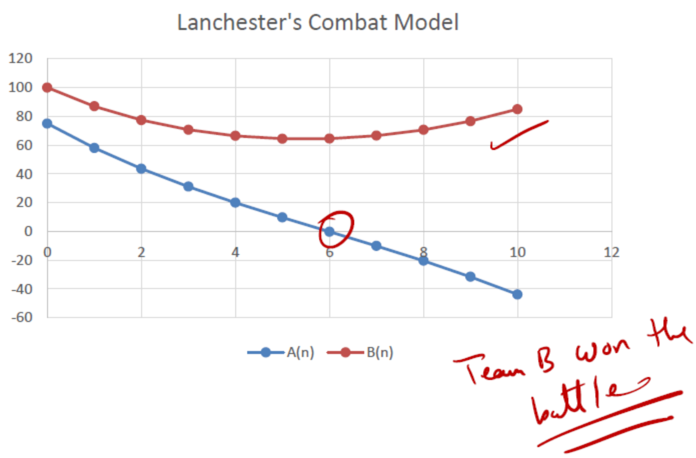
So, in both the cases the teams will try that the team B must have more numbers than the team A and team A will try that say that their number is same as team B such that they win the battle.

Case 3: $\boldsymbol{\alpha = 0.2, \beta = 0.2, r_1 = 3, r_2 = 2, A(0) = 75, B(0) = 100.}$

In case 3, we have kept the same initial condition that is the team A has A(0)=75, team B is B(0)=100 but their fighting effectiveness is same $\alpha = \beta = 0.2$.

However, team A has more reinforcement than team B. So, what happens in this case?

You see because again of the number, though they have the same fighting effectiveness, team B won the battle and team A lose at some time point.

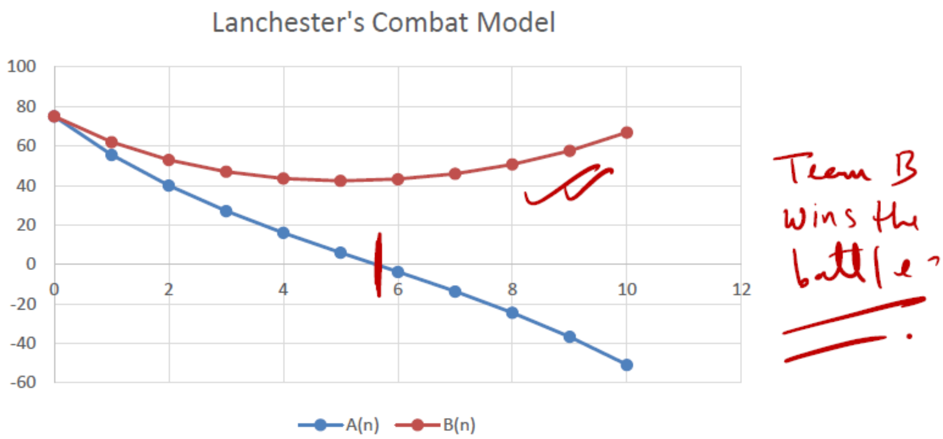


So, here even though they have the advantage of more reinforcement but their number is less due to which team B won the battle.

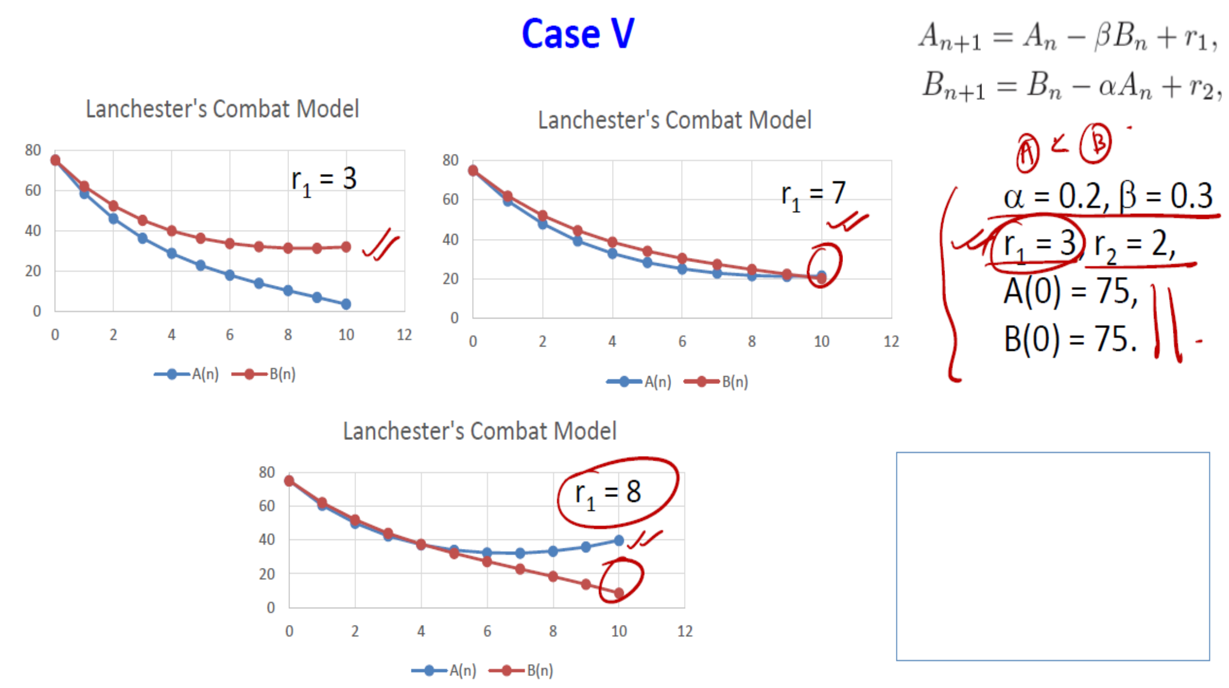
Case 4: $\boldsymbol{\alpha = 0.2, \beta = 0.3, r_1 = 3, r_2 = 2, A(0) = 75, B(0) = 75.}$

Here team B has better fighting effectiveness with $\beta = 0.3$.

However, team A has more reinforcement both the numbers we have kept the same and in this case because of the effectiveness of fighting effect effectiveness of team B team B won the battle and team A lose the battle somewhere here so here team B wins the battle.



Case 5: $\boldsymbol{\alpha = 0.2, \beta = 0.3, r_2 = 2, A(0) = 75, B(0) = 75,\ r_1 = 3, 7, 8.}$



So, what we do is that we keep the number of both the armies same, the fighting effectiveness of team A is less than team B, the reinforcement of team A is more than team B. So, if the reinforcement is 3, then we see that here your team B wins.

If the reinforcement becomes more for team A, you see when it is 7, it is almost they have the equal chance of winning the bar, both the teams have the equal chance.

However, when the reinforcement becomes 8, in that particular case you see that team A wins and team B going to lose.

So, if this is the scenario, the team A will try to give more reinforcement to its team and the team B, their strategy is to restrict the reinforcement from team A.

So this covers many of the scenarios of this particular combat model.

You can play with this model with various values of the parameters and come to your own conclusion.

So with this, we come to an end of this particular lecture.

In my next lecture, we will be talking about a two species competition model in discrete case.

Till then, bye-bye.