EXCELing with Mathematical Modeling

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Lecture – 50 (Two species competition model)

Hello, welcome to the course EXCELing with Mathematical Modelling.

Today we will be discussing about a two species competition model in discrete case.

So, let us consider that in a jungle there are two species say lion and tiger and they compete for food space. So, their common food may be deer or any other animal.

So, how this model will look like in a discrete case.

So, if we consider say B to be the number of lions and G to be the number of tigers in $(n+1)^{th}$ time. So,

$$B_{n+1} = B_n + \alpha_1 B_n - \beta_1 B_n G_n - \gamma_1 B_n^2$$

In the similar manner,

$$G_{n+1} = G_n + \alpha_2 G_n - \beta_2 B_n G_n - \gamma_2 G_n^2$$

Now, if you recall we have something called intraspecific competition and inter specific competition. So, intra specific competition means that they are competing among each other for food.

So, in this case, this term is the intra specific competition, this term is the intra specific competition and they are happening at the rate of γ_1 and γ_2.

And interspecific competition is when they are competing for food with different species.

So, in this case the lion is competing with tiger and tiger is competing with lion that is interspecific competition and they are happening at a rate of β_1 and β_2.

These are the growth rates for the lion, this is the growth rate for the tiger.

So, once this model is formed obviously we will go and look for its equilibrium solution.

So, the equilibrium solution will be given by no change from n to $(n+1)^{th}$ generation or time.

So,

$$B_{n+1} = B_n = B^*$$

$$G_{n+1} = G_n = G^*$$

You substitute them here and you get

$$B^* = B^* + \alpha_1 B^* - \beta_1 B^* G^* - \gamma_1 {B^*}^2$$
$$\Rightarrow B^*(\alpha_1 - \beta_1 G^* - \gamma_1 B^*) = 0 \quad \text{——— (1)}$$

From the second equation, similarly I will get

$$G^* = G^* + \alpha_2 G^* - \beta_2 B^* G^* - \gamma_2 {G^*}^2$$
$$\Rightarrow G^*(\alpha_2 - \beta_2 B^* - \gamma_2 G^*) = 0 \quad \text{——— (2)}$$

So, from equations (1) and (2) we have to find the equilibrium solution.

From (1) we get, $B^* = 0$ and $\alpha_1 - \beta_1 G^* - \gamma_1 B^* = 0$. Now,

$$B^* = 0 \Rightarrow \quad G^* = 0, \qquad \alpha_2 - \gamma_2 G^* = 0$$
$$\Rightarrow G^* = 0, \qquad G^* = \frac{\alpha_2}{\gamma_2}$$

So, combining this, we have our equilibrium solution as

$$(B^*, G^*) = (0,0), \qquad \left(0, \frac{\alpha_2}{\gamma_2}\right)$$

Also, we have $\alpha_1 - \beta_1 G^* - \gamma_1 B^* = 0$ ——— (3)

Now, from (2) we get,

$$G^* = 0, \qquad \alpha_2 - \beta_2 B^* - \gamma_2 G^* = 0$$

Now. $G^* = 0 \Rightarrow B^*(\alpha_1 - \gamma_1 B^*) = 0$

$$\Rightarrow B^* = 0, \qquad B^* = \frac{\alpha_1}{\gamma_1}$$

So, combining this, we have our equilibrium solution as

$$(B^*, G^*) = (0,0), \qquad \left(\frac{\alpha_1}{\gamma_1}, 0\right),$$

Also, we have

$$\alpha_2 - \beta_2 B^* - \gamma_2 G^* = 0 \quad \text{——— (4)}$$

So, right now with this you have three equilibrium solutions as

$$(0,0), \qquad \left(\frac{\alpha_1}{\gamma_1}, 0\right), \qquad \left(0, \frac{\alpha_2}{\gamma_2}\right).$$

The final one will come from solving (3) and (4). So, we rewrite them as

$$\gamma_1 B^* + \beta_1 G^* - \alpha_1 = 0,$$

$$\beta_2 B^* + \gamma_2 G^* - \alpha_2 = 0$$

Using the method of cross multiplication, we get,

$$\frac{B^*}{\alpha_1\gamma_2 - \alpha_2\beta_1} = \frac{G^*}{\alpha_2\gamma_1 - \alpha_1\beta_2} = \frac{1}{\gamma_1\gamma_2 - \beta_1\beta_2}$$

$$B^* = \frac{\alpha_1\gamma_2 - \alpha_2\beta_1}{\gamma_1\gamma_2 - \beta_1\beta_2}, \qquad G^* = \frac{\alpha_2\gamma_1 - \alpha_1\beta_2}{\gamma_1\gamma_2 - \beta_1\beta_2}$$

So, you have four points of equilibria.

For the existence of these points of equilibria, $\left(\frac{\alpha_1}{\gamma_1}, 0\right)$ is clearly positive, $\left(0, \frac{\alpha_2}{\gamma_2}\right)$ is also clearly positive. But for B^*,

either $\alpha_1\gamma_2 - \alpha_2\beta_1 > 0, \gamma_1\gamma_2 - \beta_1\beta_2 > 0$ or $\alpha_1\gamma_2 - \alpha_2\beta_1 < 0, \gamma_1\gamma_2 - \beta_1\beta_2 < 0$.

Similarly, for G^*,

either $\alpha_2\gamma_1 - \alpha_1\beta_2 > 0, \gamma_1\gamma_2 - \beta_1\beta_2 > 0$ or $\alpha_2\gamma_1 - \alpha_1\beta_2 < 0, \gamma_1\gamma_2 - \beta_1\beta_2 < 0$.

So, after getting the equilibrium solution we go for the stability analysis.

So, $(0,0)$, $\left(\frac{\alpha_1}{\gamma_1}, 0\right)$, $\left(0, \frac{\alpha_2}{\gamma_2}\right)$ and (B^*, G^*) are four equilibrium solutions.

Let,

$$f_1 = B + \alpha_1 B - \beta_1 BG - \gamma_1 B^2,$$

and

$$f_2 = G + \alpha_2 G - \beta_2 BG - \gamma_2 G^2$$

So, this is the coefficient matrix at the equilibrium point (B^*, G^*),

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial B} & \frac{\partial f_1}{\partial G} \\ \frac{\partial f_2}{\partial B} & \frac{\partial f_2}{\partial G} \end{pmatrix} = \begin{pmatrix} 1 + \alpha_1 - \beta_1 G^* - 2\gamma_1 B^* & -\beta_1 B^* \\ -\beta_2 G^* & 1 + \alpha_2 - \beta_2 B^* - 2\gamma_2 G^* \end{pmatrix}$$

At (0,0),

$$A = \begin{pmatrix} 1+\alpha_1 & 0 \\ 0 & 1+\alpha_2 \end{pmatrix}$$

and clearly the eigenvalues are $\lambda_1 = 1 + \alpha_1$ and $\lambda_2 = 1 + \alpha_2$

Clearly,

$$\lambda_1 > 1, \qquad \lambda_2 > 1$$

So, the system is unstable at (0,0).

So, (0,0) means both the species dies and nobody wants an extinction of the species. That is why, it is obvious and as the model suggests the system is unstable.

At $\left(\frac{\alpha_1}{\gamma_1}, 0\right)$,

$$A = \begin{pmatrix} 1+\alpha_1 - \beta_1 G^* - 2\gamma_1 B^* & -\beta_1 B^* \\ -\beta_2 G^* & 1+\alpha_2 - \beta_2 B^* - 2\gamma_2 G^* \end{pmatrix}$$

$$= \begin{pmatrix} 1-\alpha_1 - 2\gamma_1 \dfrac{\alpha_1}{\gamma_1} & -\beta_1 \dfrac{\alpha_1}{\gamma_1} \\ 0 & 1+\alpha_2 - \beta_2 \dfrac{\alpha_1}{\gamma_1} \end{pmatrix} = \begin{pmatrix} 1-\alpha_1 & -\dfrac{\alpha_1\beta_1}{\gamma_1} \\ 0 & 1+\alpha_2 - \dfrac{\alpha_1\beta_2}{\gamma_1} \end{pmatrix}$$

So the eigenvalues are

$$\lambda_1 = 1 - \alpha_1, \qquad \lambda_2 = 1 + \alpha_2 - \frac{\alpha_1\beta_2}{\gamma_1}$$

For the system to be stable you must have

$$|\lambda_1| = |1-\alpha_1| < 1, \qquad |\lambda_2| = \left|1 + \alpha_2 - \frac{\alpha_1\beta_1}{\gamma_1}\right| < 1$$

So, if this condition is satisfied we can see that in this case the lion species survives whereas the tiger species goes to extinction. So this equilibrium point is stable provided these two conditions holds.

In the similar manner, for $\left(0, \frac{\alpha_2}{\gamma_2}\right)$,

$$A = \begin{pmatrix} 1+\alpha_1 - \dfrac{\alpha_2\beta_1}{\gamma_2} & 0 \\ -\dfrac{\alpha_2\beta_1}{\gamma_2} & 1-\alpha_2 \end{pmatrix}$$

and the eigenvalues are

$$\lambda_1 = 1 + \alpha_1 - \frac{\alpha_2\beta_1}{\gamma_2}, \qquad \lambda_2 = 1 - \alpha_2$$

And this system will be stable if

$$|\lambda_1| = \left|1 + \alpha_1 - \frac{\alpha_2\beta_1}{\gamma_2}\right| < 1, \qquad |\lambda_2| = |1 - \alpha_2| < 1$$

And the equilibrium point (B^*, G^*), you follow similar steps.

So, these are quite complicated values, you can do of your own, you substitute them in place of (B^*, G^*) and whatever the eigenvalues you get, the modulus of that must be less than 1.

This I leave it to you for your calculations.

Now, let us look into the numerical solution of this particular module.

So how the model behaves, what is the dynamics of the model for various values of alphas, betas and gammas.

Okay, I have this spreadsheet of Microsoft Excel, quickly I write it here, you have b n plus 1 which is equal to Bn plus alpha 1 into Bn minus beta 1 into Bn into Gn and minus gamma 1 into b n square.

So, you have this equation and similarly I can get the equation for g n also.

So, this is your b n plus 1 and this is your g n plus 1.

Let us quickly write what is alpha 1, beta 1, gamma 1, alpha 2, beta 2, gamma 2.

So, alpha 1 is 1.9, this is 0.0003 this is 0.03, 0.03, alpha 2 is 1.1, beta 2 is 0.0001 and gamma 2 is 0.001.

I highlight them a little and change the font that is easy to see put them in the middle okay.

So, I get the initial values of both the species to be same say to be 200 and then calculate this.

So, this is equal to Bn which is this plus alpha 1 which is 1.9 and it is a constant multiplied by BN again 200 minus beta 1 again it is a constant multiplied by BN multiplied by Gn minus gamma 1 which is again a constant.

So, I put a dollar multiplied by Bn square.

To calculate this value, I put this is equal to Gn plus alpha 2 which is a constant multiplied by Bn into Gn minus beta 2 again a constant multiplied by Bn into Gn and minus gamma 2 which is again a constant multiplied by G square.

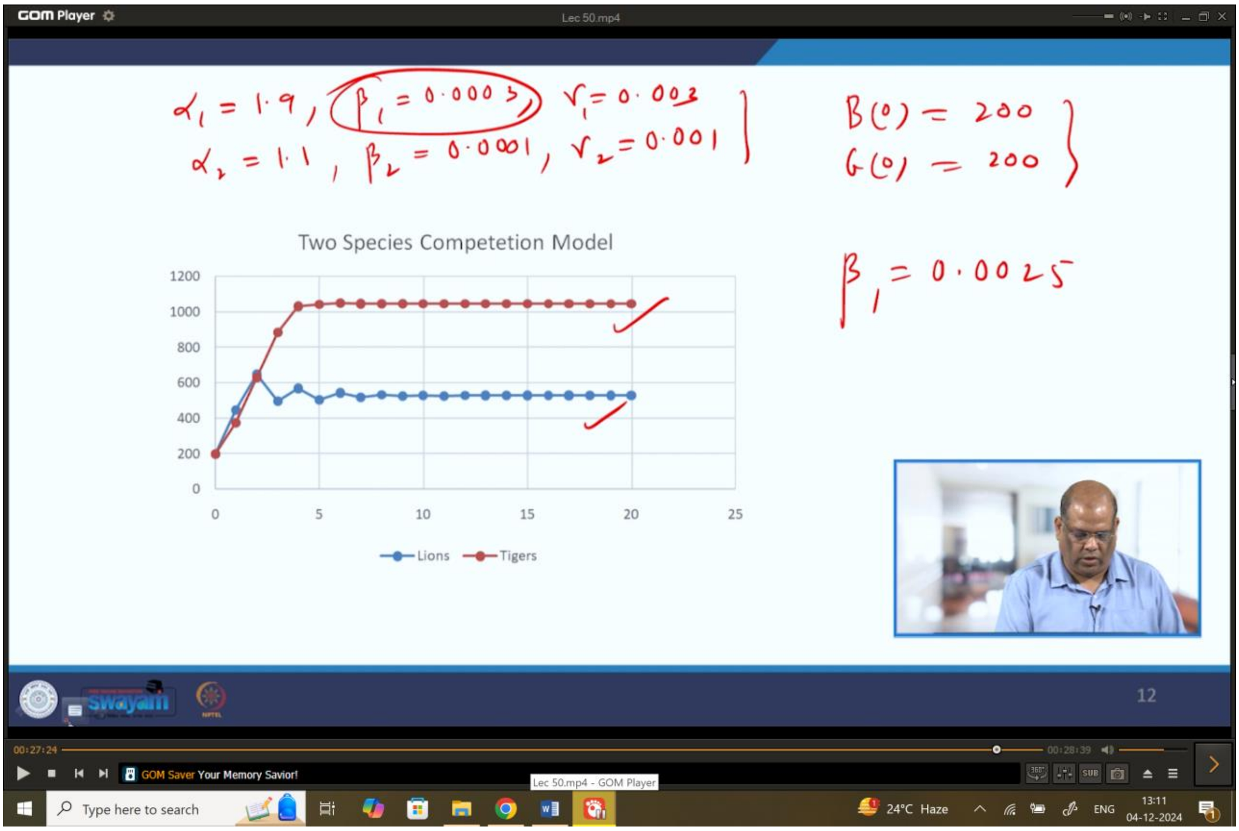
So, these two values have been calculated.

So, if I want to solve them, go to insert, go to the chart and draw this. so what you get here is two species competition model select it go to series one edit Series 1 is lions and series 2 is tigers.

So, from the graph you observe that for this particular set of parameters both the species survives though they have a competition in between them.

Now if you want to change the value say if I change this value to 0.0025 so all you have to do is just click outside and everything changes automatically and you just see the number of lions they just go into extinct whereas the number of tigers they survive.

Similarly, you can play with other parameter values and obtain different other dynamics.



Let us now go back to our slides.

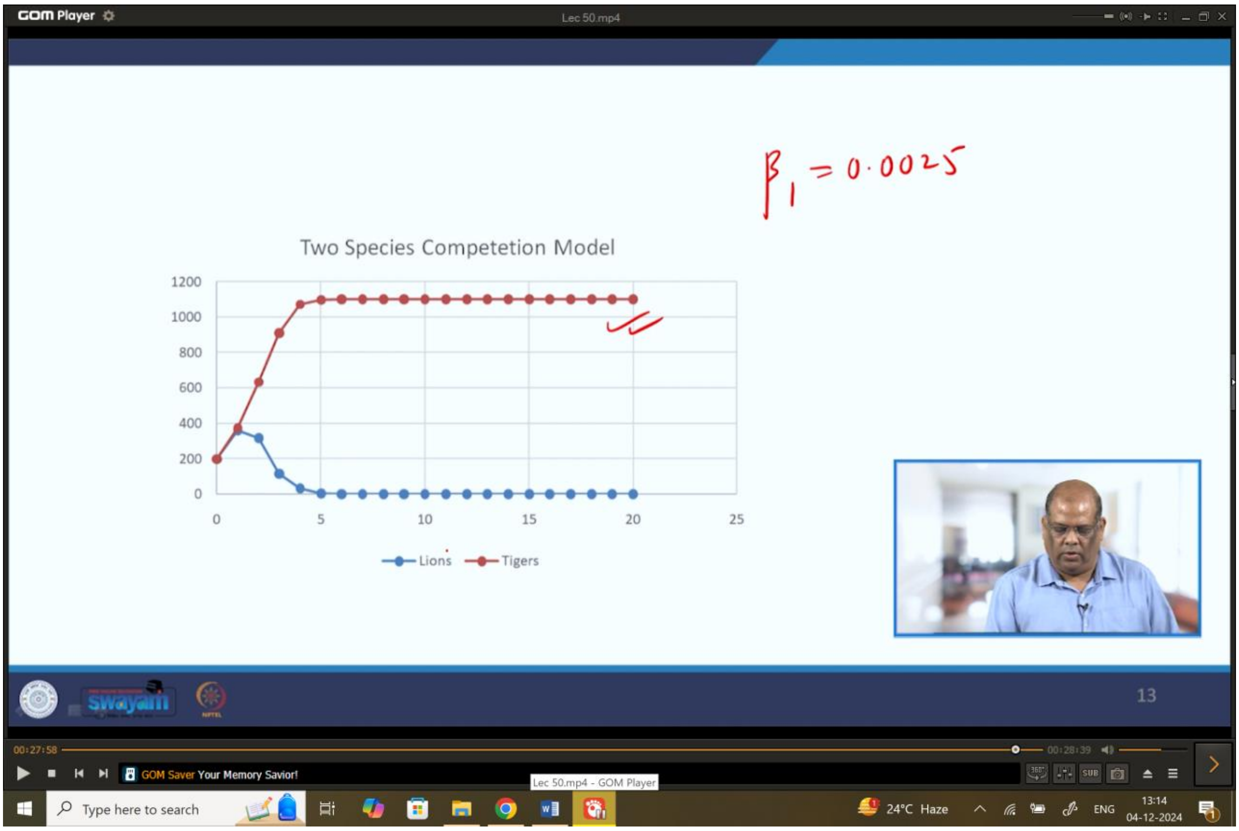
We now look into the numerical solution that we just generated and here the value of alpha 1 is 1.9, beta 1 is 0.0003, gamma 1 is 0.003, alpha 2 is equal to 1.1, beta 2 is 0.0001 and gamma 2 is 0.001.

The initial values of both lions and tigers are 200 and 200.

So, you see that with this parameter values, both the species survives and they reach to some steady state.

Now, if we change the value of beta 1, say I put beta 1 equal to 0.0025, keeping all the other parameter values same, keeping all initial conditions also same and we see that the tiger species goes to extinction sorry the lion species goes to extinction whereas the tigers survive with beta 1 equal to 0.0025.

Similarly, with another set you can show that the tiger species they go to extinction where the lion species they survive.



However, we always maintain that both the species to survive as seen in our real life scenario in the real jungle.

So, with that we come to the end of this lecture.

In my next lecture, we will be talking about disease infection model in discrete case.

Till then, bye bye.