EXCELing with Mathematical Modeling

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Week – 11

Lecture – 53 (Price and Demand model)

Hello, welcome to the course EXCELing with Mathematical Modelling.

Today we will be talking about an interesting model, namely, the price and demand model.

So, let me start with two equations of price and demand model, which are in the form

$$P_{n+1} = P_n + a_1 D_n - b_1 {P_n}^2 + h_1$$

$$D_{n+1} = D_n + \frac{c_1}{P_n} - k_1$$

Here, P_n and D_n are the price and demand for a commodity at time n.

You have seen, say, during Diwali, your demand for firecrackers or the candles is huge and at the same time it depends at what hour you are going to buy, based on the demand, the price for the particular candle or the firecrackers also increase, so, more the demand more the price.

But at the same time if there is more supply, then obviously the wholesalers have to reduce the prices.

So this is basically known as the supply function.

First there is demand, there is a rise in price but then there is a supply, an additional supply and hence the person who is selling they have to reduce the price so that their stocks are cleared and this is a constant increase in the price.

Now when it comes to demand, D_n is the demand at time n, D_{n+1} is the demand at time $(n + 1)$ and as the price increases the demand falls and vice versa, hence it is inversely proportional.

So, as you see that the price of a certain commodity which you want, but the price is quite beyond your reach or price is quite high, your demand becomes less, okay, let us buy one of them not seven of them.

So, that is why this demand is inversely proportional to the price.

And then there is a constant decrease in the demand as your constant increase in the price.

Here your parameters a_1, b_1, c_1 , k_1 are all positive, however h_1 can take any sign.

So, now you have the idea of this demand and price model. Let us see the analysis of this.

So, we start with the equilibrium solution. So, for the equilibrium solution we have

$$P_{n+1} = P_n = P^*$$

$$D_{n+1} = D_n = D^*$$

and you substitute these in the model equations

$$P_{n+1} = P_n + a_1 D_n - b_1 {P_n}^2 + h_1$$

$$D_{n+1} = D_n + \frac{c_1}{P_n} - k_1$$

and we get

$$P^* = P^* + a_1 D^* - b_1 P^{*2} + h_1$$

$$D^* = D^* + \frac{c_1}{P^*} - k_1 \Rightarrow \frac{c_1}{P^*} = k_1$$

$$\Longrightarrow P^* = \frac{c_1}{k_1}$$

and

$$a_1 D^* - b_1 P^{*2} + h_1 = 0$$

$$\Longrightarrow a_1 D^* = b_1 P^{*2} - h_1 = b_1 \frac{c_1^2}{k_1^2} - h_1$$

$$\Longrightarrow D^* = \frac{b_1 c_1^2 - h_1 k_1^2}{a_1 k_1^2}$$

So, your equilibrium point is

$$(P^*, D^*) = \left(\frac{c_1}{k_1}, \frac{b_1 c_1^2 - h_1 k_1^2}{a_1 k_1^2}\right).$$

So, once you get the equilibrium point, you now have to do the stability analysis. Let

$$f_1 = P + a_1 D - b_1 P^2 + h_1$$

$$f_2 = D + \frac{c_1}{P} - k_1$$

Then,

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial P} & \frac{\partial f_1}{\partial D} \\ \frac{\partial f_2}{\partial P} & \frac{\partial f_2}{\partial D} \end{pmatrix}_{(P^*, D^*)} = \begin{pmatrix} 1 - 2b_1 P & a_1 \\ -\frac{c_1}{P^2} & 1 \end{pmatrix}_{(P^*, D^*)}$$

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial P} & \frac{\partial f_1}{\partial D} \\ \frac{\partial f_2}{\partial P} & \frac{\partial f_2}{\partial D} \end{pmatrix}_{(P^*,D^*)} = \begin{pmatrix} 1 - 2b_1 \frac{c_1}{k_1} & a_1 \\ -c_1 \frac{k_1^2}{c_1^2} & 1 \end{pmatrix} = \begin{pmatrix} 1 - 2\frac{b_1 c_1}{k_1} & a_1 \\ -\frac{k_1^2}{c_1} & 1 \end{pmatrix}$$

Next, we have to calculate the eigenvalues and for that

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1 - 2\frac{b_1 c_1}{k_1} - \lambda & a_1 \\ -\frac{k_1^2}{c_1} & 1 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (\lambda - 1)\left[\lambda - \left(1 - 2\frac{b_1 c_1}{k_1}\right)\right] + \frac{a_1 k_1^2}{c_1} = 0$$

$$\Rightarrow \lambda^2 - \left(2 - \frac{2b_1 c_1}{k_1}\right)\lambda + \left(1 - \frac{2b_1 c_1}{k_1} + \frac{a_1 k_1^2}{c_1}\right) = 0$$

And then you have to solve for the λ's, which will be λ_1 and λ_2. If

$$|\lambda_1| < 1, |\lambda_2| < 1,$$

we will say that the system is stable.

Now let us look into this numerically.

So, we assign the values $a_1 = 2.5$, $b_1 = 0.1, h_1 = -1$, $c_1 = 5$, and $k_1 = 1$.

And if you put those values in A, we get

$$A = \begin{pmatrix} 1 - \frac{2b_1 c_1}{k_1} & a_1 \\ -\frac{k_1^2}{c_1} & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2.5 \\ -0.2 & 1 \end{pmatrix}$$

And if you find the eigenvalues, this will be

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 0 - \lambda & 2.5 \\ -0.2 & 1 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda(\lambda - 1) + 0.5 = 0$$

$$\Rightarrow \lambda^2 - \lambda + 0.5 = 0$$

$$\Rightarrow \lambda = \frac{1 \pm \sqrt{1 - 4 \times 1 \times 0.5}}{2} = \frac{1 \pm \sqrt{-1}}{2} = \frac{1}{2} \pm i\frac{1}{2} = 0.5 \pm 0.5\,i$$

Now,

$$|0.5 \pm 0.5i| = \sqrt{(0.5)^2 + (\pm 0.5)^2} = 0.7071 < 1$$

So the system is stable, namely, a stable spiral or a stable focus.

Now, let us look into the numerical solutions and visualize the graph with the help of Microsoft Excel spreadsheet.

So I already have the spreadsheet open.

Now this is one small technique which I like you to show that if you want to calculate eigenvalues with the help of Microsoft Excel, so this is what you have to do.

So the very first thing is I have taken an arbitrary matrix, that is 3, 2.5, 0, 1 and I can easily see that eigenvalue of this is 3 and 1. So I know the result of this eigenvalue.

Then I write the identity matrix.

Then I put the lambda and I put a minus lambda I and then determinant A minus lambda I. So basically I need to solve determinant $|A - \lambda I| = 0$.

So, for that, first I calculate $A - \lambda I$ and to calculate this what you do is you highlight this cell and put an '='.

Before putting an '=', you know that it will be $A - \lambda I$. So it will be a 2x2 matrix.

So first highlight this 2x2.

So what you do is you first highlight this 4 cells because it is a 2x2 matrix and when you calculate $A - \lambda I$, it is going to give you a 2x2 matrix.

So, you highlight this 4 cells and put an '=' and then select this A minus put an arbitrary value of lambda which is 0 multiplied by this I and then what you will do is you will press control shift and then enter and only then this four values will simultaneously come.

So, one more time, to calculate this A minus lambda I as you can see you have to highlight this four cells first, put an '=', select this A minus lambda multiplied by this I and then press shift control at the sme time and then press enter then this 4 values will come.

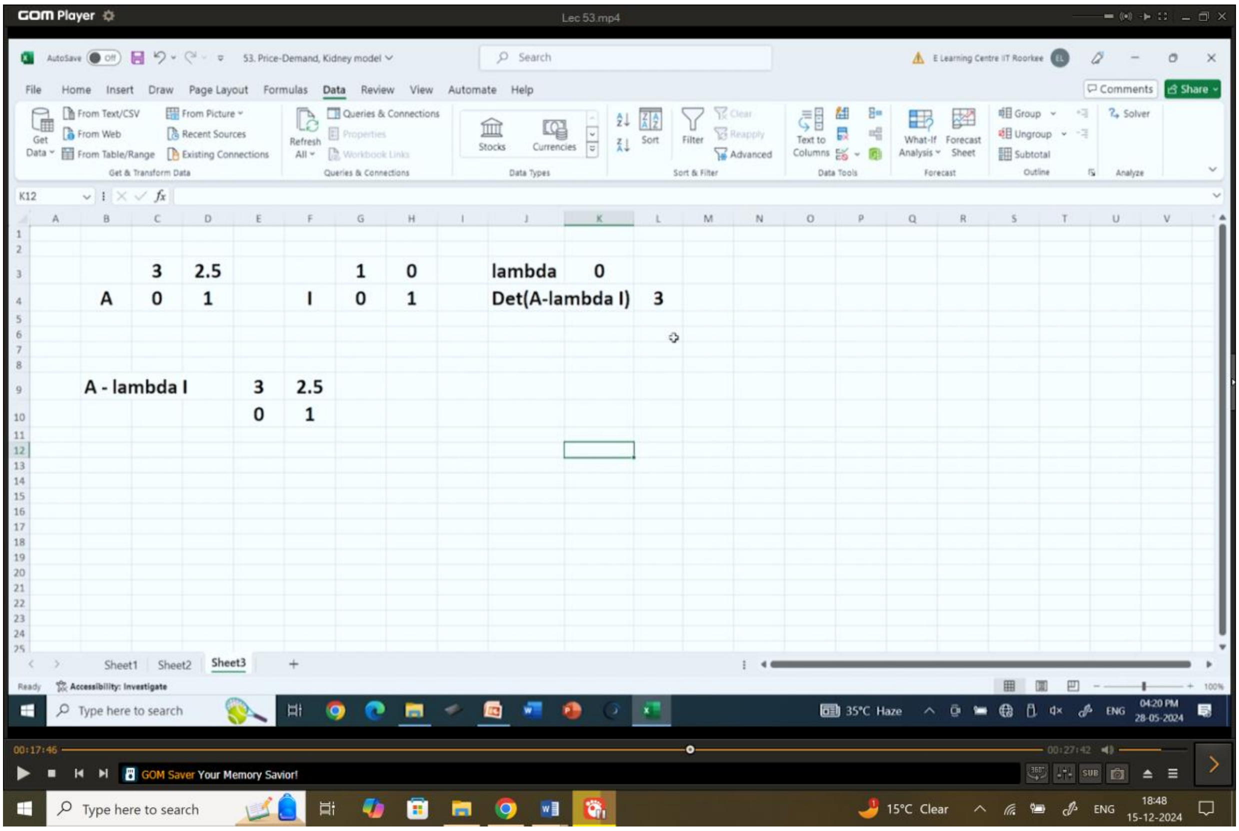
Next you calculate this determinant $|A - \lambda I|$, which is equal to, now for determinant, the formula is you type MDET. So, this gives the matrix determinant.

So, MDTERM, click this.

So, this gives the matrix determinant of this value $A - \lambda I$. You close it and enter.

So the reason I chose this matrix so you can check the result at each step.

So the determinant, $|A - \lambda I|$ is this and if you know this determinant, you expect this is 3, so this value came to be 3, so now you know it.



Now the final step is that you go to this data then go to this "what if" analysis and click goal seek.

Okay, so what is your goal?

The goal is that this $|A - \lambda I|$will be equal to 0.

So, I click this cell.

So, this cell value is 3, 2 value 0 and how will you do that?

By adjusting the values of lambda.

So, you click this and click okay.

So, you can see that one of the value it gave lambda to be 1.

Now here is the problem.

The problem is here I can see that values is 3 and 1.

So it will give a value, one at a time and you have to be a bit close to the actual eigenvalue.

So that is one deficiency of this particular thing.

But at least once you get one value you can easily calculate the other one with the help of hand or otherwise you try you have to just guess the initial value.

So here if I now put the value to be, say, 2.5 because I know that the value is 3.

So again I will go to if analysis goal seek, then I again choose this determinant this value has to be 0 by changing the cell lambda and let me see okay.

So, now it comes to 3.

So, this is a little disadvantage but some sort of it works quite well.

So, if you play it with a little with few iterations you will get all the eigenvalues.

So, this is technique how you can calculate eigenvalues using Microsoft Excel.

Now, let us look into the model. So, quickly I write it here

$$P_{n+1} = P_n + a_1 D_n - b_1 {P_n}^2 + h_1$$
$$D_{n+1} = D_n + \frac{c_1}{P_n} - k_1$$

this and the values of parameters are $a_1 = 2.5,\ b_1 = 0.1, h_1 = -1\ ,\ c_1 = 5\ ,$ and $k_1 = 1$.

Let me make them a bit bigger.

Okay, so here is my $(n + 1)$ and here is my D_{n+1}.

So, I put $n = 0$, this is equal to 0 plus 1 and drag it say 30, 40 values say up to 50 values.

So, this is the initial value, let us take as 4 and 2.

So, this is equal to P_n which is this value plus a_1, which is 2.5 times D_n, which is this value minus b_1, which is 0.1 times ${P_n}^2$ plus h_1, which is minus 1.

And this is equal to D_n which is this value, plus c_1, which is 5, divided by P_n which is this.

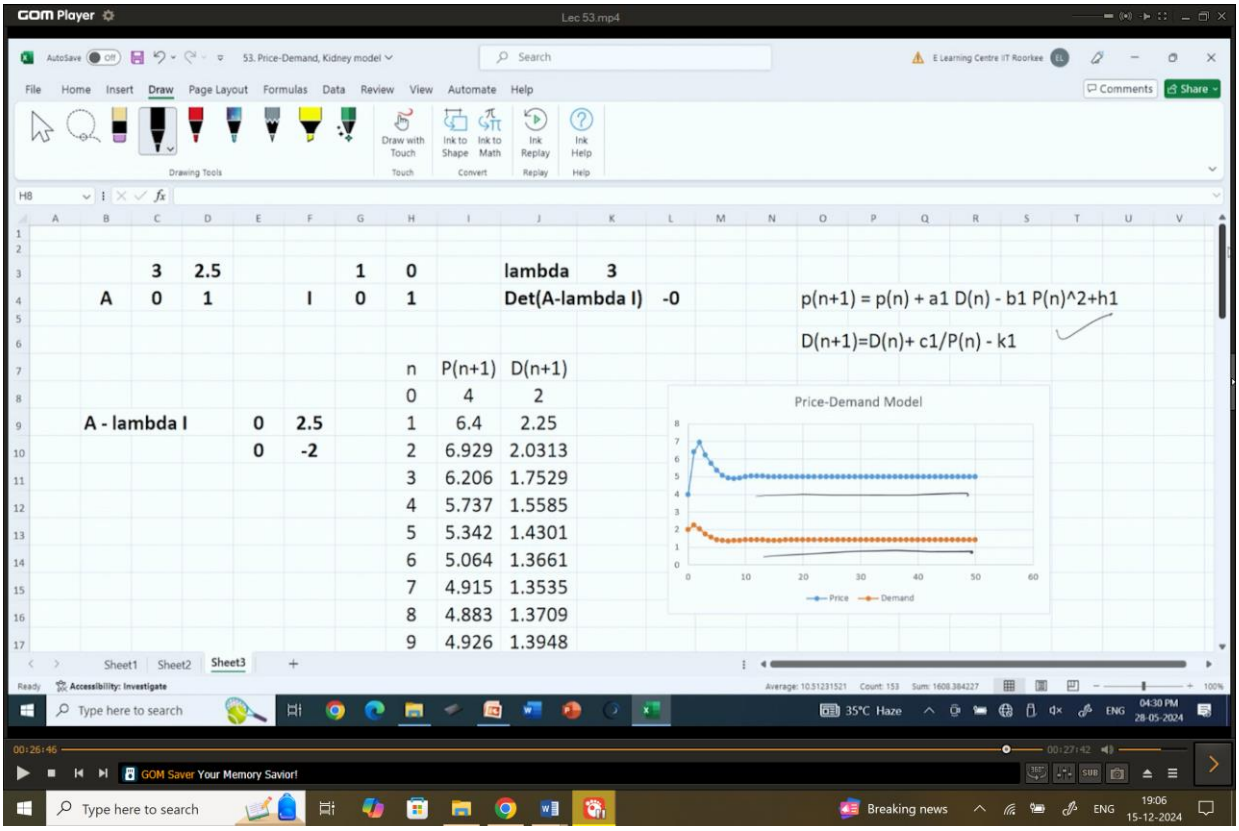
Let us put this in a bracket, minus k_1, which is 1.

Now, let us drag this value up to 50.

So, if I now plot this, I go to insert, I go to this chart and I plot this.

I can write price demand model. The series I go to select data series 1 edit I just write price and series 2 edit I just write demand so you get this curve and sometimes you want to copy this curve so right click copy, go to your slides and paste.

So, in this particular case I have to close this graph, let us keep it like this okay.



So, what do you just see that we have numerically drawn the figure of this model which we were describing, the price and the demand model which is this.

And what you see that initially there is an increase of price, increase of demand and ultimately both price and demand, they come to a steady value, the steady value being 5 and 1.4.

So with this we come to an end of this interesting model of price and demand and what you can do is in this particular model you can change the values of the parameters, to see that how this curve behaves by changing the parameters a_1, b_1, h_1 , c_1, and k_1.

In my next lecture we will be talking about two discrete models, namely, paper tower model and calorie burning model.

Till then, bye-bye.