EXCELing with Mathematical Modeling

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Week – 02

Lecture – 08 (Linear, Quadratic, Cubic Models)

Hello, welcome to the course EXCELing with Mathematical Modeling.

Today we will be learning about some very very basic models, namely, the linear model, a quadratic model and a cubic model and where you can use them.

Let us go for the linear models and we start with an example.

Suppose salesperson selling some product, in this case it is a detergent, it can be Surf Excel, it can be Ariel.

So, they put it in a small pouches and the salesperson goes from home to home and sells them.

So let us take that the price of each pouch, say, it is 30 rupees and he has target to sell x numbers of them.

So the simple mathematical model is

$$y(x) = 30x, \;\; x \geq 0,$$

where x is the number of products or number of packets that he will be selling.

So obviously if your $x = 0,$ then there is no sale and he takes from 0 rupees and as your x increases his sale increases and we get a very simple model,

$$y(x) = 30x.$$

Let us take another quite simple model. Say, you borrow some money from a friend, and you want to repay him.

So, you tell that okay, I will give you 1000 rupees or say 200 rupees every month.

So, you have borrowed 1000 rupees. You want to repay him 200 rupees each month.

So, I multiply it by the time. I add them. But, after 1 month, your 200 rupees will decrease so obviously this will be minus.

So now you have $y(t) = -200t + 1000$. So, obviously, if you put the first month, t = 1, your

$$y(1) = -200 + 1000 = 800.$$

And, it continuously decreases every month till your balance is equal to 0 and you can see that if you solve this equation

$$y(t) = -200t + 1000 = 0,$$

your answer is $t = 5$ months.

So, in 5 months you can repay him back.

So, this is one of the simplest example of linear models.

So, where you use this linear models, so basically you will be using this linear model in such a situation that there is a constant change whether it can be increased or whether it decreases.

So, in the first example the constant change is, this 30. So, as your number of product increases, it increases by the multiple of 30, where, here, in the second example there is a constant decrease of 200, and hence it comes with a negative sign.

So, these two are the examples where your linear functions can be used.

So, let us take the example of a quadratic model.

Before I go into this model, let me start by saying that, we consider the case say a particle is moving with a constant acceleration. If I want to write this, in the form of the differential equation, so I know that, if f is my acceleration, so

$$\frac{dv}{dt} = f, \qquad \text{is some constant}$$

So, this is

$$\frac{dv}{dt}$$

because rate of change of velocity, that is some constant acceleration f. So, my next step is to integrate it

$$\int dv = \int f\, dt$$

and I will get

$$v = f\,t + constant.$$

If I want to find the value of constant I need an initial condition and, say, at time

$$t = 0, \qquad v(0) = u.$$

You substitute it here

$$v = f\,t + \text{constant},$$

and you get

$$u = 0 + \text{constant},$$

$$u = \text{constant}.$$

So, you get your equation

$$v = u + f\,t.$$

v is the rate of change of distance with time. So, it is

$$\frac{dx}{dt} = u + ft$$

If I integrate,

$$\int dx = \int (u + ft)\, dt$$

I get

$$x = ut + \frac{1}{2} f t^2 + \text{constant.}$$

Again, I apply the initial condition say at $t = 0$, you have $x = 0$.

So, if I substitute

$$0 = u \times 0 + \frac{1}{2} f 0^2 + \text{constant,}$$

I will get my

$$\text{constant} = 0$$

and you get the formula

$$x = ut + \frac{1}{2} f t^2.$$

You are all familiar with this formula you have learned this in physics that

$$v = u + ft$$

and

$$x = ut + \frac{1}{2} f t^2$$

Suppose I change this initial condition and I write that at $t = 0$, I have $x(0) = x_0$, then your value of the constant is going to be

$$x_0 = u \times 0 + \frac{1}{2} f 0^2 + \text{constant,}$$

$$x_0 = constant$$

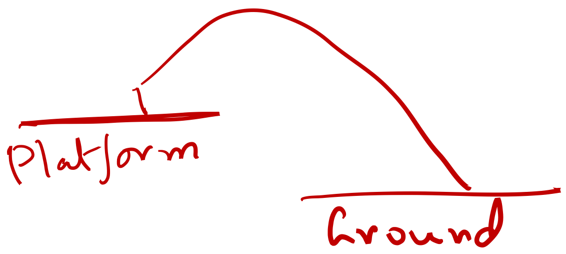
So, you have the formula that

$$x = x_0 + ut + \frac{1}{2} f t^2.$$

Now let us come to the problem.

So you have a person standing on a platform which is 20 meter off the ground. The person throws the ball in the air with some initial velocity 5m/s and so that it lands on the ground and not on the platform.

So basically the person is standing on this platform, throws the ball and it goes down and the ground. So this is your platform and this is your ground.



So, there can be many questions.

So, one of them is how high the ball will go and when it will, what time it reaches maximum height?

Now, if we take say v_0 to be the initial velocity. So, it is exactly the same formula

$$H(t) = -gt^2 + v_0 t + h_0,$$

the formula which we derived that gives

$$x = x_0 + ut + \frac{1}{2} f t^2.$$

Now, if I compare this, this x is the distance, so which is H(t) here, x_0 is the initial condition, so it is the initial height h_0, f is the acceleration, here g is the acceleration due to gravity and since you are moving, throwing the ball up, it is negative, so $-gt^2$, and u is the initial velocity, where v_0 is the initial velocity.

So basically it is exactly this equation $x = x_0 + ut + \frac{1}{2} f t^2$, only written in different notations.

In this particular problem we have we take our value of $g = 10 m/s^2$. The v_0 is the initial velocity, which is $v_0 = 5 m/s$, and the initial height the platform is $h_0 = 20m$.

So, you put all these values here

$$H(t) = -gt^2 + v_0 t + h_0,$$

and you get it as some function of H,

$$H(t) = -10t^2 + 5t + 20,$$

which is also a function of time.

Now we know that it is this quadratic equation, a quadratic expression and if we plot them we will be getting a parabola. So, we put it in the parabolic form, in the parabolic form of the equation.

So, what we do is we just make this a whole square.

So, you take $10t^2 - 5t = -H(t) + 20 \Rightarrow 10(t^2 - 0.5t) = -(H(t) - 20)$

$$\Rightarrow t^2 - 2 \times t \times 0.25t + 0.25^2 - 0.25^2 = -0.1(H(t) - 20),$$

$$\Rightarrow (t - 0.25)^2 = -0.1\, H(t) + 2 + 0.0625$$

$$\Rightarrow (t - 0.25)^2 = -0.1\, (H(t) - 20.0625)$$

So after simplification, this equation $H(t) = -10t^2 + 5t + 20$, you put it in this form, $(t - 0.25)^2 = -0.1\, (H(t) - 20.0625)$, and then you have to plot.

So, this is the equation if I now plot it I am going to get a parabola something like this. So, what is the top part? This is the top part and if I, so approximately this is going to be 0.25.

So, this is the time and this is the height H(t) and the maximum height reached is this which I can tell from here it is 20.625.

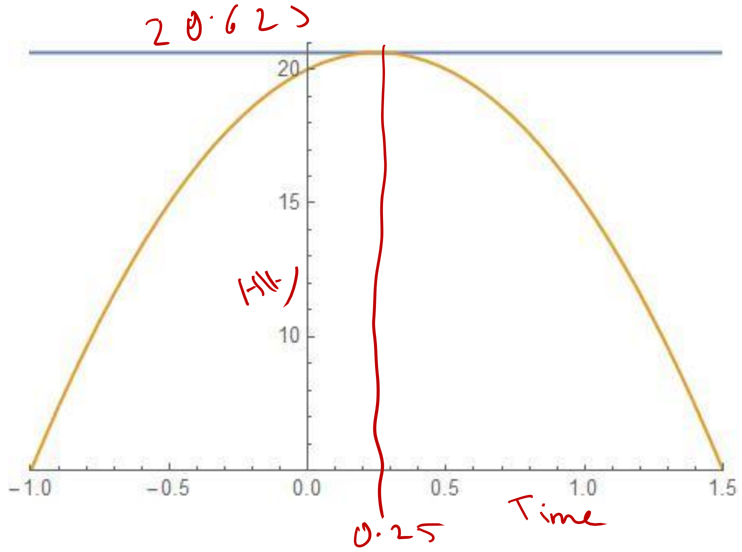
So, what is your conclusion?

So, in this case this quadratic function the vertex is a global maximum. So, every time you will reach a maximum at this point vertex and this can be used to model this falling object.

So, the conclusion is from this model that the maximum height that will be reached is 20.625, which is 0.25 seconds after the ball is thrown to the air.

And, in similar situation some quantity which say decreases and then increases means if you get something like this, you can reach a global minimum at this particular point.

So, then also this quadratic function is used where you want a global maximum or a global minimum.



Let us move to another interesting example, say, somebody wants to open a company, a startup company, of some product, I take, say, aftershaves.

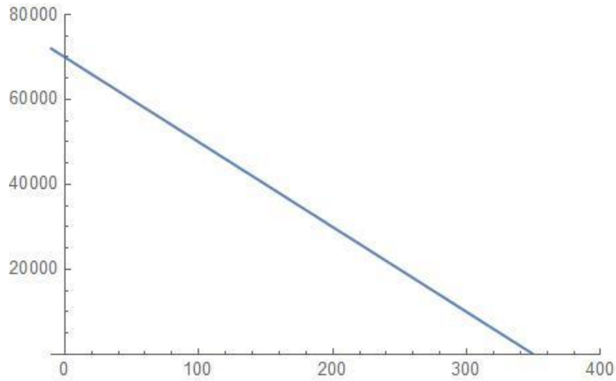
So, you have to make an initial investment obviously for the setup, which is 7 lakhs. And the cost of manufacturing each of these aftershave is Rs. 110.

Fine.

So, what other things we need to know?

So, he need to know the market. So, obviously, there will be some demand and it will follow some function, some curve.

So, he does the market survey and come up with this curve. So, this is your demand curve.



On the basis of this, he creates a function, say, the demand function for the unit sale and this function is

$$\text{Demand (D)} = 70000 - 200\text{x}, \qquad \text{(for the unit sales)}$$

which is this line.

So, with some data he got this plot and he believes that its demand will depend on this function, and x is the price.

So, if your $x = 0$, clearly you end up giving up everything free because it is only 70000.

If your $x = 350$, then you get your demand curve is zero, that is, there is no sale basically.

So, you have to find this value of x, so, that is the whole idea of this problem that, what is that x such that the startup company shows a profit and can run.

For example, if I put x=300 (say), then it will be your

$$D = 70000 - 200 \times 300 = 10000.$$

So, you have a demand for 10000.

But I cannot just decide this value just like that. So I have to use some mathematical modeling to get an approximate idea.

So, what is that optimal value?

So, what is that value of x that I should put here so that my company can make the profit?

So, this is how it will work.

So, this is some function demand. So, number of units needs to be made.

So, unit sell, that is, x. So

$$\text{Total sell} = (70000 - 200\,x)\,x,$$

and I will sell with at a price of x, which I have to determine. So I multiply this by x. So this is my total selling price of all the products.

Now let us go for the cost price.

The cost price, you have a setup, which costs you 7 lakh. So, please note this 7 lakh has nothing to do with 70,000, okay. So, it is just another number.

So, this $$\text{Cost Price} = 700000 + 110(70000 - 200\,x)$$

And if I simplify this, I will get

$$\text{Cost Price} = 8400000 - 22000\,x$$

So, this is my cost price, $8400000 - 22000\,x$, for all the products, this is my selling price, $(70000 - 200\,x)\,x$, and my profit is

$$\text{Profit} = (70000 - 200x)x - (8400000 - 22000x)$$

So, let me rewrite it, the profit is going to be

$$\text{Profit} = 70000x - 200x^2 - (8400000 - 22000\,x),$$

and if this is simplified,

$$\text{Profit} = -200x^2 + 92000x - 8400000$$

Now, suppose I want to see for what value x, my profit is a zero. So, you just put this equal to zero,

$$\text{Profit} = -200x^2 + 92000x - 8400000 = 0,$$

and you solve the equation

$$200x^2 - 92000x + 8400000 = 0$$

You can just use any method, but straight away you can use the formula

$$x = \frac{92000 \pm \sqrt{(92000)^2 - 4 \times (200) \times (84000)}}{2 \times (200)}.$$

And, if you solve this, you will be getting

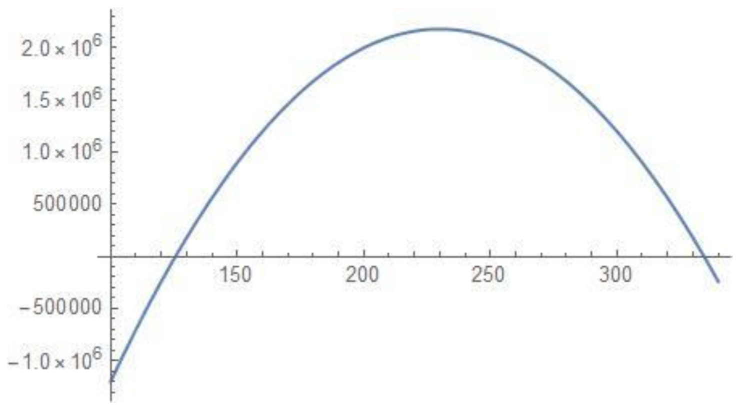
$$x = 230 \pm 104$$

and the values are 126 and 334.

So, for x equal to 126 or 334, your profit will be 0 and hence you cannot choose these values.

So, what do you do?

You plot the graph; you plot the profit graph. So, if you plot it you will get something like this.



So, again the global maximum, so this is your price and this is your profit.

So, this is the maximum value of x you can choose, at the vertex, you just drop it and you will get this value.

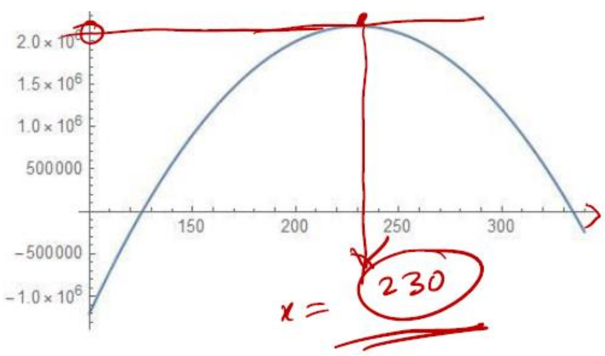
In this case it is 230. So, if you choose your x to be 230, then your unit sell, which is the demand graph

$$\text{unit sell} = 70000 - 200 \times 230 = 24000.$$

So, this is the number which will bring you the maximum profit and then

$$\text{sales} = 24000 \times 230,$$

which you have determined, which is maximum at the vertex,



and this gives $\text{sales} = 24000 \times 230 = 5520000.$

Your $$\text{Cost price} = 700000 + 24000 \times 110 = 3340000.$$

So, your $$\text{Profit} = 5520000 - 3340000 = 218000,$$

which you can get from here also, this gives you the profit.

So, the number is quite encouraging and so one can go with this startup company for aftershaves and can make better or higher profit in the near future.

So, that is another example where you use this quadratic modeling to solve the problem.

Let us take another example. So, in this particular example, you have a steel sheet.

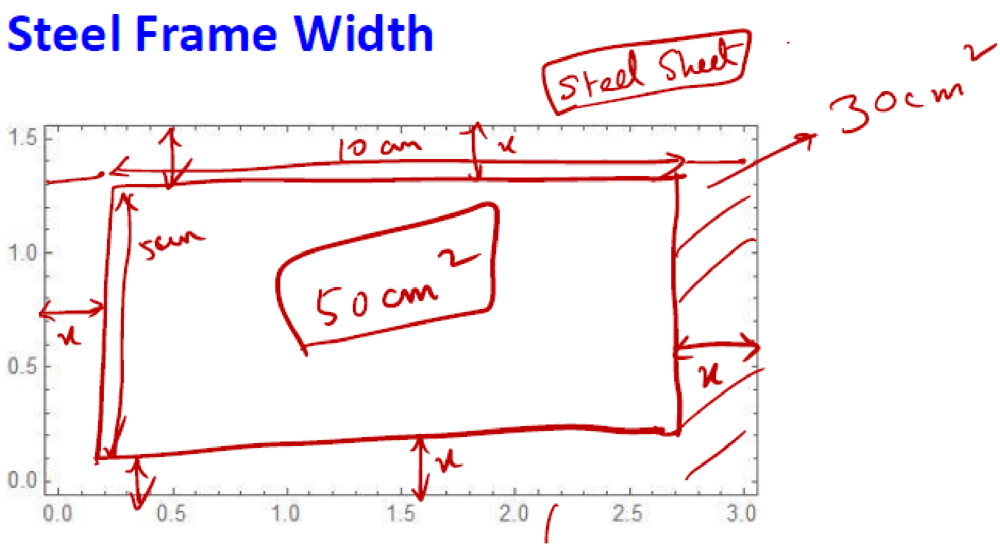
So, this the rectangular frame which you see is some steel sheet.

I ignore the thickness of the sheet. Now, you have been asked that you have to build or to make a steel frame.

So, you were told, okay, you cut up inside, take this rectangular sheet out, which is again given that it has to be 5 by 10, say, I put centimeter.

So, $50\ \text{cm}^2$ of the steel sheet has to be taken out with the dimension 5 by 10 or 10 by 5.

And so, what is going to be this width, which you take as x, such that the area of this frame is $30\ \text{cm}^2$ So, that is your problem.



So, again you have to calculate, say, what is the area of the steel sheet before cutting.

So, if this is your 10 and this is your x and this is your x,

$$\text{Area (Before cutitng)} = (10 + 2x)(5 + 2x)\text{cm}^2.$$

If I multiply, this is

$$\text{Area (Before cutitng)} = 50 + 20x + 10x + 4x^2.$$

After simplification,

$$\text{Area (Before cutitng)} = 4x^2 + 30x + 50$$

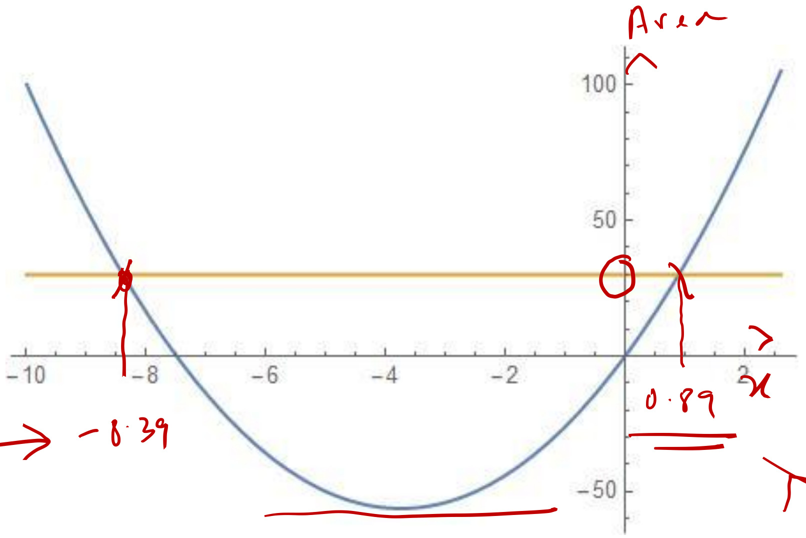
Now, you have taken a sheet out, which is 50cm^2, so,

$$\text{Area (after cutitng)} = 4x^2 + 30x + 50 - 50.$$

So, you are left with

$$4x^2 + 30\,x.$$

So, again, you have to plot this curve and get the global minimum in this case.



So, if I plot it, and get the curve like this, but this value $(4x^2 + 30\,x)$ has to be 30cm^2.

So, this is the area and this is the x, this is your x and this is your area.

So, I draw a line where this value is 30 and I draw a line parallel to the x axis and it has intersected the curve at here and at here.

So, basically my x value will be this and my x value will be this.

So, if I solve this equation, if I solve the equation $4x^2 + 30x = 30$, using the formula,

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

So, my value will be $x = -8.39, 0.89$. So, you can see that this value which is approximately minus 8.39 and this is 0.89.

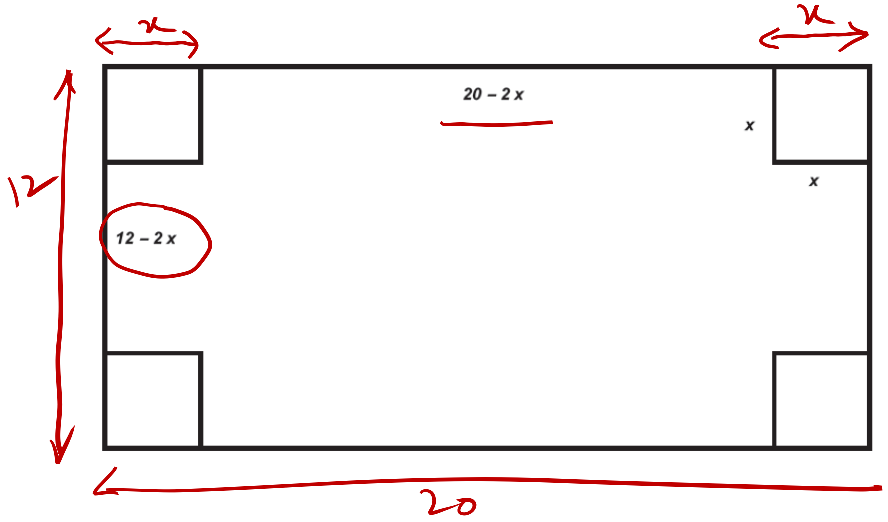
And since these are dimensions, it cannot be negative. So, you take your x to be 0.89 and your goal is achieved. So, that is all.

In this particular case, though there is a global minimum, but we need this particular values to solve the problem. So, this is again another example of this quadratic modeling.

Let us now move into the cubic modeling.

So, in this particular case, as you can see, you have rectangular sheet and you have to make it a box by cutting the edges, where the dimension of the edge is x. So, you have to find what is that value of x for which this can be done.

So, first we have to find the volume, for any cubic model it is going to be, I mean if there is a problem of the volume, it is the cubic model that we use it.



So, if this is x and again this point is x this value is x. So,

$$\text{total lenght} = 20 - 2\,\text{x}.$$

With the similar argument it is

$$\text{Breadth} = 12 - 2\,\text{x}$$

Your dimension was 20 by 12. So, this whole thing was 20, this thing was 12.

So, your

$$\text{Volume} = (20 - 2\text{x})(12 - 2\text{x})\text{x}$$

So, if you simplify this, it is going to be

$$\text{Volume} = 4\text{x}^3 - 64\text{x}^2 + 240\text{x}$$

Now, if you put $x = 2$, I can calculate that this value going to be

$$\text{Volume} = 4\text{x}^3 - 64\text{x}^2 + 240\text{x} = 256\ (\text{inch})^3.$$

Now, if somebody tells that, no, I want the value, say, the volume of this rectangular bus has to be 100. So, then you have this volume formula and you write

$$\text{Volume} = 4\text{x}^3 - 64\text{x}^2 + 240\text{x} = 100,$$

and you solve this equation.

If you solve this, you will get this value to be $x = 0.4751$.

So, the question is how you will solve this, you have already now done Microsoft Excel, where we have seen how to find the root of this algebraic equation.

So, you can use that method to get the roots of this particular algebraic equation.

If you do that you will get the values to be $x = 0.471, 5,$ and 10.53.

Now if I substitute all these values here, I will get them, for 5, it is directly equal to zero, for the other two values, up to two places of decimal, the values are zeros.

So, approximately you can choose any one of the value for x, for which your volume will be 100, and as per convenience of the shape of the box or what kind of product you are sending in that particular box, you choose your x values.

But, it will give you the same constant volume 100.

So, this is an example where you can use the cubic model and when we involve the volumes, we use this cubic models.

So, in this particular lecture you get an idea, how a very simple kind of problem can be tackled by using this linear model, this quadratic model and the cubic model.

We will take up another part of the equation where we will be using the growth models in our next lecture.

Till then, bye-bye.