EXCELing with Mathematical Modeling
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Week – 02
Lecture – 09 (Linear Stability Analysis-I)

Hello, welcome to the course EXCELing with Mathematical Modeling.

Today, we will be discussing about the stability of a system.

We will start with the equilibrium solution and then we will move up to the stability of the system.

What do you mean by an equilibrium solution or a steady state solution?

So, if we consider a system of n nonlinear autonomous differential equation, by autonomous we mean that your differential equation

$$\frac{dx}{dt} = f(\bar{x}),$$

where $\bar{x} = (x_{1,},\ x_2, x_3 \dots \dots, x_n)^T$, and $f(\bar{x}) = (f_1(\bar{x}), f_2(\bar{x}), f_3(\bar{x}), \dots., f_n(\bar{x}))^T$.

this side will not contain explicitly the function of time.

That means it will be

$$\frac{dx}{dt} = f(\bar{x}),$$

but I cannot put any function of time here.

So that is it is an autonomous system.

So we consider an autonomous system and we define that the steady state solution or equilibrium solution or critical point, there is another name fixed point, by all these we mean that the solution of the system when the value of this x does not change with time, and what does that mean? If your x does not change with time, obviously your

$$\frac{dx}{dt} = 0 = f(x^*).$$

So we put the right hand side equal to zero and we calculate our steady state solution or equilibrium solution or critical point or fixed point.

So if you have

$$\frac{dx}{dt} = f(x),$$

because it is an autonomous system to find the steady state solution, all you have to do is put

$$\frac{dx}{dt} = f(x) = 0$$

and calculate that value of x for which this equation gives a zero solution.

If that value of $x = x^*$, which we called a steady state solution or fixed point or equilibrium point, then at this point $f(x^*) = 0$ and this happens because from the definition where there is no change in the solution as you vary x. Let us take an example.

So, if you want to find the steady state solution of

$$\frac{dx}{dt} = x^2 - 9.$$

So, as told before all you have to do is, the steady state solution is obtained when your

$$\frac{dx}{dt} = 0 \Longrightarrow x^2 - 9 \Longrightarrow x = \pm 3.$$

So,

$$x = -3$$

and

$$x = 3$$

are your steady state solutions or fixed points or critical points.

We move to our second example.

You have to find the equilibrium solution of

$$\frac{dx}{dt} = -x^3 + 3x^2 - 2x.$$

.

As we have previously done, we take

$$\frac{dx}{dt} = 0 \ \Longrightarrow -x^3 + 3x^2 - 2x = 0$$

$$\Longrightarrow x\,(-x^2 + 3x - 2) = 0$$

So,

$$x = 0 \text{ and } -x^2 + 3x - 2 = 0$$

and if you factorize them, it is

$$(x-1)(x-2) = 0$$

.

So you have your equilibrium solutions to be $x = 0, 1, 2.$

So I guess it is very much clear when we have a single equation and you have to find the steady state solution or equilibrium solution, you put

$$\frac{dx}{dt} = 0,$$

and whatever the expression in the right hand side, you take that equal to zero, solve the equation and whatever roots you get, those are the equilibrium solutions or steady state solutions or fixed points.

Let us take an example where there are no numerical, but we have the parameters like r, k, E, these are the parameters here. So, to obtain the critical point of this particular equation,

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - Ex,$$

it is the same, we now put

$$\frac{dx}{dt} = 0 \implies rx\left(1 - \frac{x}{k}\right) - Ex = 0$$

So,

$$x(r\left(1 - \frac{x}{k}\right) - E) = 0.$$

One of the solution is

$$x = 0,$$

and to solve the other one, you just take

$$r\left(1 - \frac{x}{k}\right) - E = 0 \implies 1 - \frac{x}{k} = \frac{E}{r}$$

$$\implies \frac{x}{k} = 1 - \frac{E}{r} \implies x = k\left(\frac{r - E}{r}\right)$$

Sometimes in the book you will see that the notation is x^* because we denote the equilibrium solutions with x^*. Actually, it is the value of x for which you are getting the equilibrium solution and the notation is x^*.

Now, if your equation, this equation, right now it is just a differential equation, but when you will be doing the modeling, this will have some specific meaning, this may represent a population.

So, if it represents a population, then we have always the positive values. So, either it is a zero value or we have positive value, we cannot have a negative value.

In that case you have to take care of this equilibrium solution that so that this becomes positive.

If such is the case where your differential equation represents some cells or some population, in that case your equilibrium solution has to be positive and for that you must have

$$k\left(\frac{r - E}{r}\right) > 0 \implies r - E > 0 \implies r > E.$$

So, this is the condition which you have to mention such that your critical point or your equilibrium solution is positive, and this happens only when you know that your model will give you a positive solution only and it cannot give a negative solution.

Let us now take two equations. So, if you want to find the equilibrium solution of two equations,

$$\frac{dx}{dt} = x - y, \qquad \frac{dy}{dt} = 1 - xy^2,$$

So, let me put in the form (x^*, y^*). So, if (x^*, y^*) is the equilibrium solution and to find them you have to put

$$\frac{dx}{dt} = 0, \qquad \frac{dy}{dt} = 0$$

Now, if

$$\frac{dx}{dt} = 0 \Rightarrow x - y = 0,$$

and

$$\frac{dy}{dt} = 0 \Rightarrow 1 - xy^2 = 0.$$

$$\Rightarrow x = y \text{ and } x\,y^2 = 1$$

And since $x = y$ or $y = x$, you substitute it here $xy^2 = 1$, So, I put $y = x$ and this will imply

$$xx^2 = 1 \Rightarrow x^3 = 1$$

So, one of the solution is

$$x = 1,$$

which you can already see, otherwise you take it to the right hand side put them equal to zero, factorize them

$$(x - 1)(x^2 + x + 1) = 0$$

and this will give you

$$x = 1.$$

And if you put

$$(x^2 + x + 1) = 0,$$

and try to find the solution $(x^2 + x + 1) = 0$, we use

$$x = -\frac{1}{2} \pm \frac{\sqrt{1-4}}{2} \Rightarrow x = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}.$$

So, this will give you an imaginary solution and hence $x = 1$, is the only real solution here and since $x = 1$ and $x = y$, we have $y = 1$.

So, (1,1) is your steady state solution here or equilibrium solution.

We take another example

$$\frac{dx}{dt} = a\,x - bxy, \qquad \frac{dy}{dt} = -cy + dxy,$$

where we have instead of numerical we have put x and y, sorry, the parameters a, b, c and d. As defined before our equilibrium solution will be

$$\frac{dx}{dt} = 0$$

which will give us

$$a\,x - bxy = 0,$$

which gives us

$$x(a\ - by) = 0,$$

which gives

$$x = 0 \;\text{ and }\; y = \frac{a}{b}.$$

In the similar manner we put

$$\frac{dy}{dt} = 0,$$

which gives

$$-cy + dxy = 0 \;\Rightarrow\; y(-c + dx) = 0$$

$$\Rightarrow\; y = 0 \;\text{ and }\; x = \frac{c}{d}$$

Now, how to choose the combinations here? Let me explain it in two ways.

The very first is here $x = 0$. So, you put it in the second equation like this one and see what you get. So, if I put x = 0 in $-cy + dxy = 0$, I get

$$-cy + 0 = 0 \Longrightarrow y = 0.$$

So, I have, when $x = 0$ from this, I got from the first equation, I put it in the second equation, and I calculate the value of y which is y = 0.

So, clearly (0,0) is one of the fixed point or steady state solution or equilibrium solution.

Now, you put $y = \frac{a}{b}$, you can see here the computation is $x = \frac{c}{d}$. So, another solution is

$$\left(\frac{c}{d}, \frac{a}{b}\right)$$

Well how do you get that?

So if I put

$$y = \frac{a}{b} \quad \text{in} \quad y(-c + dx) = 0,$$

I get

$$\frac{a}{b}(-c+dx)=0, \qquad \text{as} \quad \frac{a}{b}\neq 0, \qquad \Rightarrow \ x=\frac{c}{d}\ .$$

So,

$$(x^*,y^*)=\left(\frac{c}{d},\frac{a}{b}\right).$$

Now, there is another way to check. There is a common mistake that people take it as a combination like (0, 0), (x, 0), (0, y) that $\left(\frac{c}{d},\frac{a}{b}\right)$. So, all four combinations, they try to take.

So, you have (0, 0), you have $(x^*,0)$, you have $(0,y^*)$ and you have (x^*,y^*). These are all possible computations of the equilibrium solution that you make it.

So, another way to check is that you just put this value here and see whether it is satisfying or not.

So if I put this value (0,0) in

$$\frac{dx}{dt}=a\,x-bxy, \qquad \frac{dy}{dt}=-cy+dxy,$$

this satisfies. So clearly this is a steady state solution.

Now, I put $(x^*,0)$. So if I put $(x^*,0)$, here what I will get $ax^*-bx^*\times 0=0$, so, I get $ax^*=0$, since $a\neq 0$, my only conclusion is $x^*=0$. Since $x^*=0$, this becomes actually (0,0). So, there is does not exist any non-zero x^* such that I get the solution to be $(x^*,0)$. Hence this cannot be a solution because we already have (0,0).

Similarly, you put (0, y^*). So, if I put (0, y^*) here, this gives me zero, but from here I get

$$-cy^*+d\times 0\times y^*=0 \ \Longrightarrow\ -cy^*=0 \Longrightarrow\ y^*=0.$$

So, again there does not exist any non-zero y^* for which we have (0, y^*) as a solution and hence this would not be the solution.

And the next thing is you put (x^*,y^*) where both x^* and y^* are non-zeros and you will be getting this particular solution,

$$(x^*,y^*)=\left(\frac{c}{d},\frac{a}{b}\right).$$

So, there is another way of checking that how many equilibrium solutions there will be and you get the solution.

We now look into the stability of the system. So, what do you mean by the stability?

So, in a very layman language, if there is an object and then you give a small push, if it comes back to its original position, we say the system is stable.

If it does not come back, we say the system is unstable.

Let us look at this small animation.

So, you can see this plane is moving and there is turbulence.

So if the plane it comes back to its original position then we say that the system is stable whereas if it flips back and does not comes back then it is unstable.

Let us look in the another example.

As you can see this ant, it gives a small push to this board attached and it just oscillates a little, and then comes back to its original position.

The same thing happens with the ball, it gives a small push, it oscillates and comes back to the original position. So, this is what we call a stable equilibrium.

There is another called neutral equilibrium that the system was already in equilibrium, it is given a small push, it moves back to a new position, but then still it is stable.

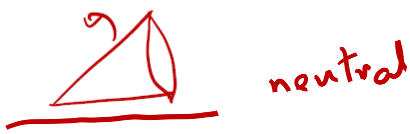
You can see in this ball, if it gives a small push it moves back but in that position also it is stable. So that is called a neutral equilibrium.

And the unstable one, when the ant gives a push, so from the original position it moves back to another position, it cannot come back to its original position, the same happens with the ball, it is pushed, it moves back.

Another common example is that you can consider a cone which is an example of stable equilibrium. If you give a small push, it will come back to its original position. This is a stable equilibrium.

If I consider an inverted cone, and if I give a small push obviously it will fall down and it will not come back to its original position. So this is an unstable equilibrium.

And, the neutral one is you consider the cone like this. So if you just roll it, it will move to another position somewhere here, but it is still in the same position but in a different place. So, this is called a neutral stability.



Let us now look it into the definition of stability mathematically. So, I consider the system

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y).$$

So, I assume that (0,0) is the fixed point and let C be the path that is found by this differential equation.

I also define let $x = f(t)$ and $y = g(t)$ be the parametric solution of this differential equation, and I define

$$D(t) = \sqrt{(f(t))^2 + (g(t))^2}$$

which means that the distance of $(f(t), g(t))$ from the origin.

Now let us see the definition. It says that the critical point (0,0) is stable, if for every $\varepsilon > 0$, there exist a $\delta > 0$, such that every path C for which

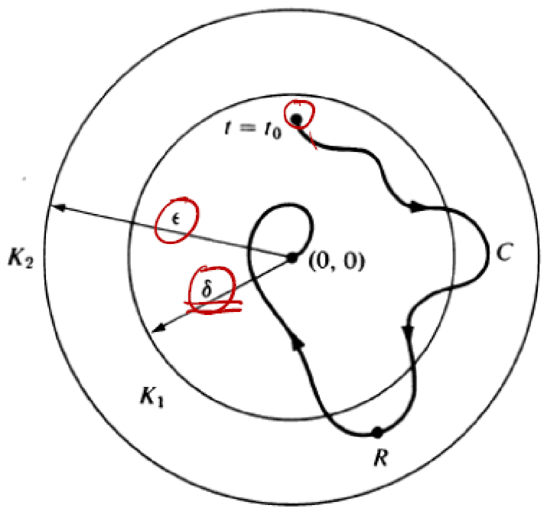
$$D(t_0) < \delta$$

for some t_0,

$$D(t) < \varepsilon, \text{ for } t \geq t_0.$$

Now this definition somewhat reminds you of the ε- δ definition of limit.

Let us look into the figure for further clarification.



So, what it says is that there are two circles.

One is this one $D(t_0) < \delta$, circle of radius δ and $D(t) < \varepsilon$, which is circle of radius ε.

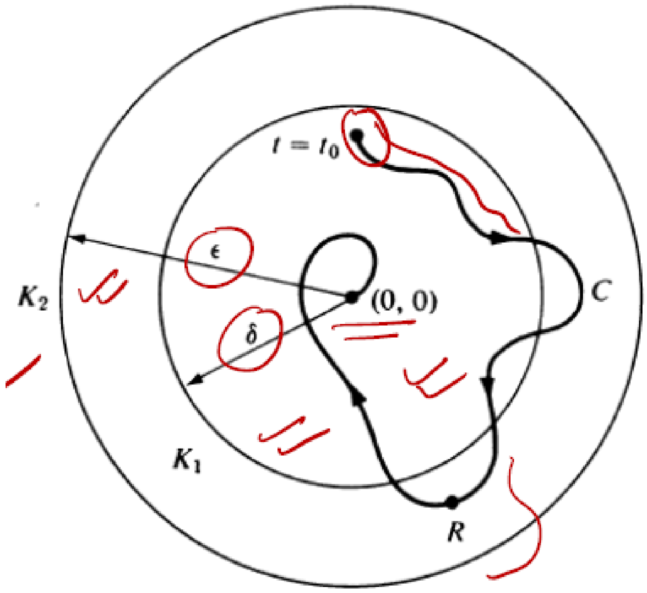
You start from $t = t_0$, some point which lies inside the circle of radius δ and then this is the path.

So, what it says is, that if your system is stable then this path C that must lie within this radius ε.

So, if you can find such an ε and δ for which this will hold, then you say your system is stable.

Let me rephrase that again. So, it says that if (0,0) is stable, then every path C, which lies inside the circle K_1. So, this is your circle K_1 you have started for t_0.

So, every path C which lies inside this circle K_1 of some radius which is δ here at time $t = t_0$, will remain inside the circle of radius K_2 if it is stable.



If it is not stable it may go out of the circle of radius K_2.

So if you can find such δ and such ε for this what which this will hold, we say your system is stable.

There is another word which you will be finding that is asymptotically stable.

Now what is that mean and how it differs from the stability of the system?

So, by asymptotically stable, it is much stronger, it is stronger than the stability, it means that the system has to be stable, that is, all the properties that is here need to be satisfied and along with that you must have

$$\lim_{t\to\infty} f(t) = 0 \text{ and } \lim_{t\to\infty} g(t) = 0.$$

If you recall this $f(t)$ and $g(t)$ they are the parametric equation of the differential equation of the solution of the differential equation.

It means that the tiny difference that we have between the stability and the asymptotically stable, one is this mathematical that it has to be stable, and along with that you have to satisfy these two conditions.

Then your system is asymptotically stable.

And in stability, it means that the path which remains close to (0, 0) will remain close to (0,0) when $t \geq t_0$.

And here along with it remains close to (0,0), the path will approach (0,0) and this mathematical condition guarantees that it will approach to (0,0).

So that is the basic difference between the stability of the system and the asymptotically stability of the system.

Let us now move to the problems.

So how do you check that given a system or given a differential equation, how do you check its stability?

So, we have this dynamical system of the form

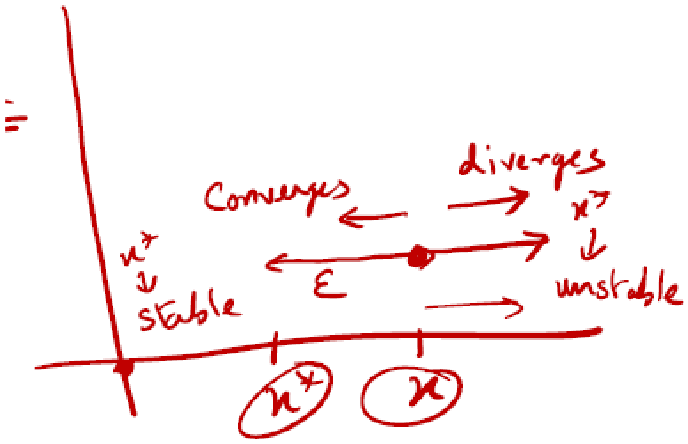
$$\frac{dx}{dt} = f(x).$$

So, we will be deriving here the condition for which this system is stable.

So, you have some x here and you have the equilibrium solution x^* here and the difference between them, let it be say ε.

So, you give a small push if it moves back towards the equilibrium position, we say it converges and if moves back from the equilibrium solution, we say it diverges.

So, in this case your equilibrium solution x^* is stable, and in this case your x^* is unstable.



So, you have

$$\frac{dx}{dt} = f(x)$$

and you take this $\varepsilon = x - x^*$.

So, it is calculated from here and this will imply $d\varepsilon$.

You can also substitute basically $x = x^* + \varepsilon$.

You are giving a small push about the equilibrium point and from there you can calculate

$$\varepsilon = x - x^*.$$

If I differentiate both sides

$$\frac{d\varepsilon}{dt} = \frac{dx}{dt} + \frac{dx^*}{dt}$$

and since x^* is a constant this will be

$$\frac{dx^*}{dt} = 0.$$

So, now I substitute back here $\frac{dx}{dt}$ is replaced by $\frac{d\varepsilon}{dt}$, f(x) is replaced by $f(x^* + \varepsilon)$ and I will be using the Taylor series expansion here. So,

$$\frac{d\varepsilon}{dt} = f(x^* + \varepsilon) = f(x^*) + \varepsilon f'(x^*) + \frac{\varepsilon^2}{2!} f''(x^*) + \cdots$$

Since we are doing the linear stability analysis, we will ignore $\frac{\varepsilon^2}{2!} f''(x^*)$ and higher order terms.

So,

$$\frac{d\varepsilon}{dt} \approx f(x^*) + \varepsilon f'(x^*).$$

Now since x^* is the equilibrium solution, it is going to satisfy $f(x^*) = 0$ and we are left with

$$\frac{d\varepsilon}{dt} \approx \varepsilon f'(x^*).$$

This is a simple differential equation and the solution will be

$$\varepsilon = \varepsilon_0 e^{f'(x^*)}$$

where ε_0 is the arbitrary constant.

Now, if your $f'(x^*) < 0$, then you can see that your as t becomes large your ε will move to zero and if your ε moves to zero your x will coincide with x^*. So in that case your x^* will converge and your system is stable.

So the condition that the system is stable is $f'(x^*) < 0$.

So once again if $f'(x^*) < 0$ then what happens?

From here, you can see, this part is less than zero. So, as t becomes large then your ε goes to zero.

So the moment ε goes to zero this point and this point coincides. So you have given a small push but it comes back to its original position and hence the system is stable.

Whereas if your $f'(x^*) > 0$, then obviously, as t becomes larger values of ε and this moves away from the equilibrium solution and you have an unstable equilibrium.

If $f'(x^*) = 0$, then we do not have any conclusion we have to do further higher analysis.

So, in this linear stability analysis for a single equation, the condition that the system will be stable is $f'(x^*) < 0$ and the system is unstable if $f'(x^*) > 0$.

If $f'(x^*) = 0$, there is no conclusion.

Now let us take some examples.

So we take the same example,

$$\frac{dx}{dt} = -x^3 + 3x^2 - 2x$$

for which we have found the fixed point and because you are going to find the stability of the equilibrium point or about the equilibrium point.

So, if you just recall we have the equilibrium points by taking

$$\frac{dx}{dt} = 0,$$

which will mean

$$-x^3 + 3x^2 - 2x = 0$$

and if you solve this, you will be getting

$$x = 0, 1, 2.$$

Now if you want to check the stability at $x = 0$, $x = 1$ and $x = 2$, you have to take this as f(x) and you have to find what is your

$$f'(x) = -3x^2 + 6x - 2$$

Now you have to consider for each x so for $x = 0$, we find,

$$f'(0) = -30^2 + 6.0 - 2 = -2 < 0,$$

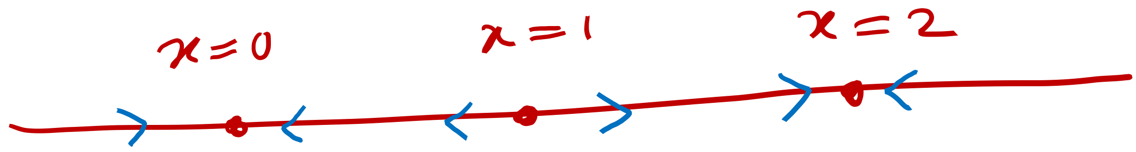
implies the system is stable about $x = 0$.

For $x = 1$, $f'(1) = -3.1^2 + 6.1 - 2 = 1 > 0$, implies the system is unstable about $x = 1$.

And, in the similar manner, if you find for $x = 2$, $f'(2) = -3.2^2 + 6.2 - 2 = -2 < 0$,

which implies that the system is stable about $x = 2$.

Now, if you want this diagram, which we will be coming later also but for this problem, you draw a line, this is your 0, this is your 1, this is your 2.



So, for 0, it is stable, so your arrow will be like this.

For 1, it is unstable, so your arrow is like this, that is it is moving out of the solution.

For 2, it is again stable, so your arrow will be like this.

So, in this lecture, we learned about the equilibrium point and about the stability of the system.

In our next lecture, I will be improvising them for a system of equations, mainly, two equations for the stability of the system.

Till then, bye bye.