Theory of Composite Shells

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Week – 03

Lecture – 03

special cases of Shell governing equations

Dear learners welcome to week- 03, lecture- 03. In this lecture, I will explain the Special cases of Shells. We have developed a generalized governing equation, we have to work on cylindrical shell, spherical shell, conical shell, or ellipsoidal shell.

We can convert this general differential equation for special cases. I will explain with the help of an example starting from the plate, how do we convert this partial differential equation as per the requirements.

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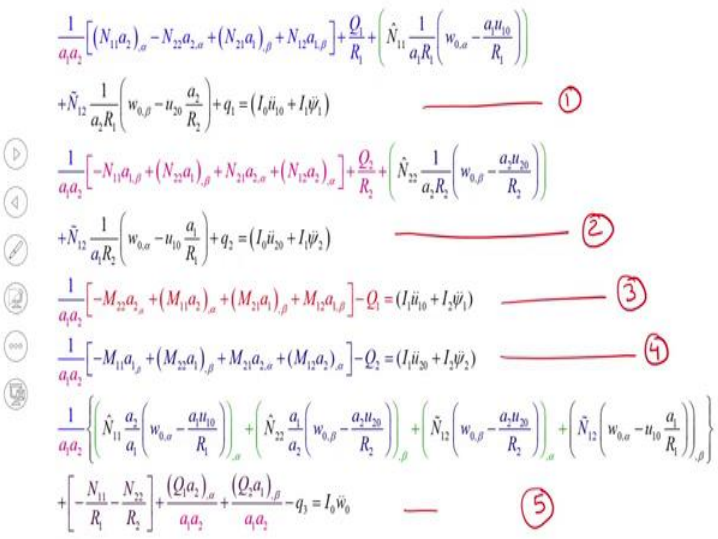

Following are the five differential equations:

$$\frac{1}{a_1a_2}\left[\left(N_{11}a_2\right)_{,\alpha} - N_{22}a_{2,\alpha} + \left(N_{21}a_1\right)_{,\beta} + N_{12}a_{1,\beta}\right] + \frac{Q_1}{R_1} + \left(N_{11}\frac{1}{a_1R_1}\left(w_{0,\alpha} - \frac{a_1u_{10}}{R_1}\right)\right)$$

$$+N_{12}\frac{1}{a_2R_1}\left(w_{0,\beta} - u_{20}\frac{a_2}{R_2}\right) + q_1 = \left(I_0\ddot{u}_{10} + I_1\ddot{\psi}_1\right) \quad equation(1)$$

$$\frac{1}{a_1a_2}\left[-N_{11}a_{1,\beta} + \left(N_{22}a_1\right)_{,\beta} + N_{21}a_{2,\alpha} + \left(N_{12}a_2\right)_{,\alpha}\right] + \frac{Q_2}{R_2} + \left(N_{22}\frac{1}{a_2R_2}\left(w_{0,\beta} - \frac{a_2u_{20}}{R_2}\right)\right)$$

$$+N_{12}\frac{1}{a_1R_2}\left(w_{0,\beta} - u_{10}\frac{a_1}{R_1}\right) + q_2 = \left(I_0\ddot{u}_{20} + I_1\ddot{\psi}_2\right) \quad equation(2)$$

$$\frac{1}{a_1a_2}\left[-M_{22}a_{2,\alpha} + \left(M_{11}a_2\right)_{,\alpha} + \left(M_{21}a_1\right)_{,\beta} + M_{12}a_{1,\beta}\right] - Q_1 = \left(I_1\ddot{u}_{10} + I_2\ddot{\psi}_1\right) \quad equation(3)$$

$$\frac{1}{a_1a_2}\left[-M_{11}a_{1,\beta} + \left(M_{22}a_1\right)_{,\beta} + M_{21}a_{2,\alpha} + \left(M_{12}a_2\right)_{,\alpha}\right] - Q_2 = \left(I_1\ddot{u}_{20} + I_2\ddot{\psi}_2\right) \quad equation(4)$$

$$\frac{1}{a_1a_2}\left[\left(N_{11}\frac{a_2}{a_1}\left(w_{0,\alpha} - \frac{a_1u_{10}}{R_1}\right)\right)_{,\alpha} + \left(N_{22}\frac{a_1}{a_2}\left(w_{0,\beta} - \frac{a_2u_{20}}{R_2}\right)\right)_{,\beta} + \left(N_{12}\left(w_{0,\beta} - u_{20}\frac{a_2}{R_2}\right)\right)_{,\alpha} + \left(N_{12}\left(w_{0,\alpha} - u_{10}\frac{a_1}{R_1}\right)\right)_{,\beta}\right]$$

$$+\left(-\frac{N_{11}}{R_1} - \frac{N_{22}}{R_2}\right) + \frac{\left(Q_1a_2\right)_{,\alpha}}{a_1a_2} + \frac{\left(Q_2a_1\right)_{,\beta}}{a_1a_2} = I_0\ddot{w}_0 \quad equation(5)$$

In some of the books, this numbering may differ.

The most important part is that if you want to solve a non-linear problem for a shell, then first you need a solution of a linear differential equation. Then it will be acting as an initial guess and we can solve a non-linear problem. The very first approach is to go for a linear partial differential equation for the shell.

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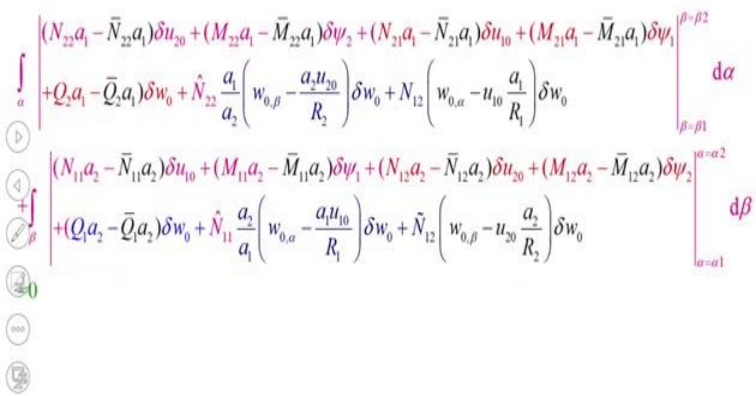

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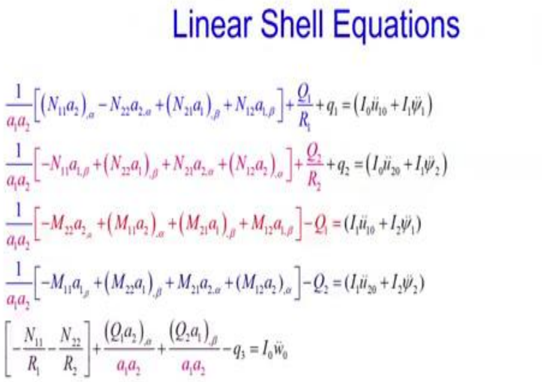

If you remove the nonlinear terms it will reduce to a first-order linear shell partial differential equation in which α and β are two orthogonal curvilinear parameters.

$$\frac{1}{a_1 a_2}\left[\left(N_{11}a_2\right)_{,\alpha} - N_{22}a_{2,\alpha} + \left(N_{21}a_1\right)_{,\beta} + N_{12}a_{1,\beta}\right] + \frac{Q_1}{R_1} + q_1 = \left(I_0\ddot{u}_{10} + I_1\ddot{\psi}_1\right)$$

$$\frac{1}{a_1 a_2}\left[-N_{11}a_{1,\beta} + \left(N_{22}a_1\right)_{,\beta} + N_{21}a_{2,\alpha} + \left(N_{12}a_2\right)_{,\alpha}\right] + \frac{Q_2}{R_2} + q_2 = \left(I_0\ddot{u}_{20} + I_1\ddot{\psi}_2\right)$$

$$\frac{1}{a_1 a_2}\left[-M_{22}a_{2,\alpha} + \left(M_{11}a_2\right)_{,\alpha} + \left(M_{21}a_1\right)_{,\beta} + M_{12}a_{1,\beta}\right] - Q_1 = \left(I_1\ddot{u}_{10} + I_2\ddot{\psi}_1\right)$$

$$\frac{1}{a_1 a_2}\left[-M_{11}a_{1,\beta} + \left(M_{22}a_1\right)_{,\beta} + M_{21}a_{2,\alpha} + \left(M_{12}a_2\right)_{,\alpha}\right] - Q_2 = \left(I_1\ddot{u}_{20} + I_2\ddot{\psi}_2\right)$$

$$\left(-\frac{N_{11}}{R_1} - \frac{N_{22}}{R_2}\right) + \frac{\left(Q_1 a_2\right)_{,\alpha}}{a_1 a_2} + \frac{\left(Q_2 a_1\right)_{,\beta}}{a_1 a_2} - q_3 = I_0\ddot{w}_0$$

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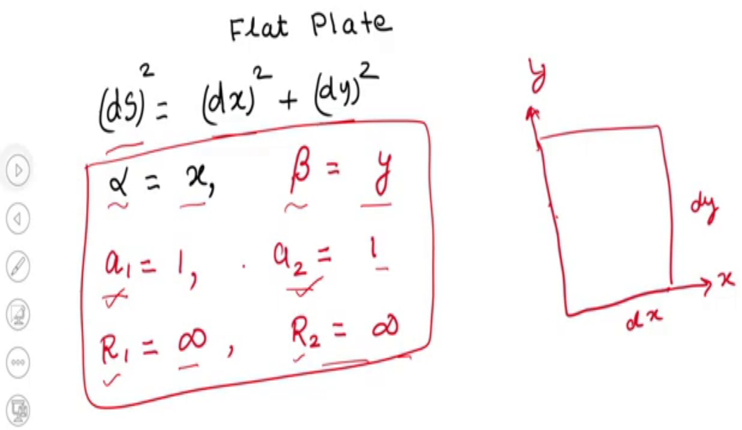

Let us say, we want to find the equation for a flat plate, can we deduce the plate equation from the shell governing equation? For the case of a plate the parameter

α = x and β = y,

$(dS)^2 = (dx)^2 + (dy)^2$,

Lame's parameters $a_1 = a_2 = 1$, and

Radius of curvature $R_1 = R_2 = \infty$.

For making a special case, the most important part is to assign curvilinear parameters i.e., which direction you assign as α and which direction you assign as β it is very important. Then, finding the lame's parameters for that body or geometry and finding the radii R_1 and R_2.

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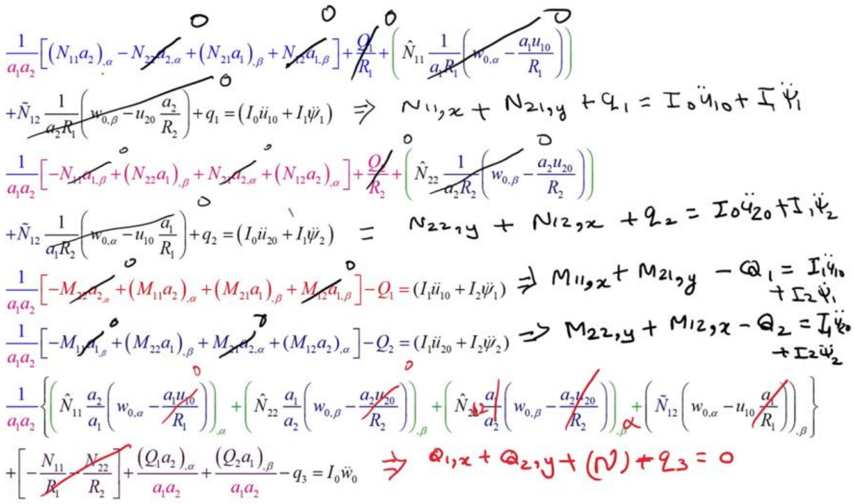

If we have these parameters, this is the main governing equation from which I am going to explain equation (1):

The first term $\left(N_{11}a_2\right)_{,\alpha} = N_{11,x}$ (a_2 = 1 and $,\alpha$ = $,x$).

Now, we have second term minus $N_{22}a_{2,\alpha}$ here, you have to take derivative of lame's parameter with respect to first coordinate, a_2 = 1 and if we take derivative, $N_{22}a_{2,\alpha}$ will going to be 0.

The term, $\left(N_{21}a_1\right)_{,\beta} = N_{21,y}$

The term $N_{12}a_{1,\beta}$ will not contribute because $a_{1,\beta}$, the derivative of the first Lame's parameter with respect to y = 0.

$\frac{Q_1}{R_1}$ will not contribute as $R_1 = \infty$, and $\frac{1}{\infty} = 0$.

From the linear term $-N_{22}a_{2,\alpha}$ and $N_{12}a_{1,\beta}$ will not contribute.

In these non-linear terms:

$$\left(N_{11}\frac{1}{a_1R_1}\left(w_{0},_{\alpha}-\frac{a_1u_{10}}{R_1}\right)\right) + N_{12}\frac{1}{a_2R_1}\left(w_{0},_{\beta}-u_{20}\frac{a_2}{R_2}\right):\ \frac{1}{R_1} \text{ is here and } R_1 = \infty,$$

these terms will not contribute. Loading term and dynamic inertia remain the same.

1 = x, if you write N_{11} it is fine. If you want to write in terms of N_{xx} that is also fine. The equation (1) will be:

$$N_{11,x} + N_{21,y} + q_1 = I_0 \ddot{u}_{10} + I_1 \ddot{\psi}_1$$

If you are interested to write in terms of x then it will be:

$$N_{xx,x} + N_{yx,y} + q_1 = I_0 \ddot{u}_{10} + I_1 \ddot{\psi}_1 .$$

Here, I_1 = 0. The reason behind that is for the case of shell, the definition of I_1 is:

$$I_1 = \int_{\frac{-h}{2}}^{\frac{h}{2}} \rho \varsigma \left(1 + \frac{\varsigma}{R_1}\right)\left(1 + \frac{\varsigma}{R_2}\right) d\varsigma .$$

Here, R_1 and $R_2 = \infty$. Therefore, for a symmetric plate $I_1 \ddot{\psi}_1$ will not exist, there will be just $I_0 \ddot{u}_{10}$.

In the second equation:

The first term $-N_{11} a_{1,\beta}$ will not contribute because a_1 is constant, the second term $\left(N_{22} a_1\right)_{,\beta}$ will contribute, the third term $N_{21} a_{2,\alpha}$ will not contribute, and the fourth term $\left(N_{12} a_2\right)_{,\alpha}$ will contribute and $\frac{Q_2}{R_2}$ will not contribute the same way the non-linear terms $\left(N_{22} \frac{1}{a_2 R_2}\left(w_{0,\beta} - \frac{a_2 u_{20}}{R_2}\right)\right) + N_{12} \frac{1}{a_1 R_2}\left(w_{0,\beta} - u_{10} \frac{a_1}{R_1}\right)$ will vanish. We will get equation (2) like this:

$$N_{22,y} + N_{12,x} + q_2 = I_0 \ddot{u}_{20} + I_1 \ddot{\psi}_2 .$$

Then, come to the third equation:

The first term $-M_{22} a_{2,\alpha}$ will not contribute, the second term $\left(M_{11} a_2\right)_{,\alpha}$ will contribute $\left(M_{21} a_1\right)_{,\beta}$ will contribute and $M_{12} a_{1,\beta}$ will not contribute and Q_1 will remain. Here, in this shell equation, there are no non-linear terms. This equation (3) becomes like this:

$$M_{11,x} + M_{21,y} - Q_1 = I_1 \ddot{u}_{10} + I_2 \ddot{\psi}_1 .$$

In the 4th equation:

$-M_{11}a_{1,\beta}$ will not contribute, $\left(M_{22}a_1\right)_{,\beta}$ will contribute, $M_{21}a_{2,\alpha}$ will not contribute, and $\left(M_{12}a_2\right)_{,\alpha}$ will contribute. This equation (4) becomes like this:

$M_{22,y} + M_{12,x} - Q_2 = I_1\ddot{u}_{20} + I_2\ddot{\psi}_2$.

In the fifth equation, first we have put the non-linear terms then the linear contribution. In the non-linear contribution the terms $\frac{a_1 u_{10}}{R_1}$, $\frac{a_2 u_{20}}{R_2}$, $u_{20}\frac{a_2}{R_2}$, and $u_{10}\frac{a_1}{R_1}$ will not contribute. Then, this $\left(-\frac{N_{11}}{R_1} - \frac{N_{22}}{R_2}\right)$ will not contribute and $\frac{\left(Q_1 a_2\right)_{,\alpha}}{a_1 a_2} + \frac{\left(Q_2 a_1\right)_{,\beta}}{a_1 a_2}$ will remain. You will see that in the 5th equation we will have some non-linear contribution for the case of a rectangular plate. Equation (5) will be:

$Q_{1,x} + Q_{2,y} + (N) - q_3 = I_0\ddot{w}_0$

Here, (N) = non-linear terms

If you want to say that we do not want a dynamic term then you can deduce the dynamic term also.

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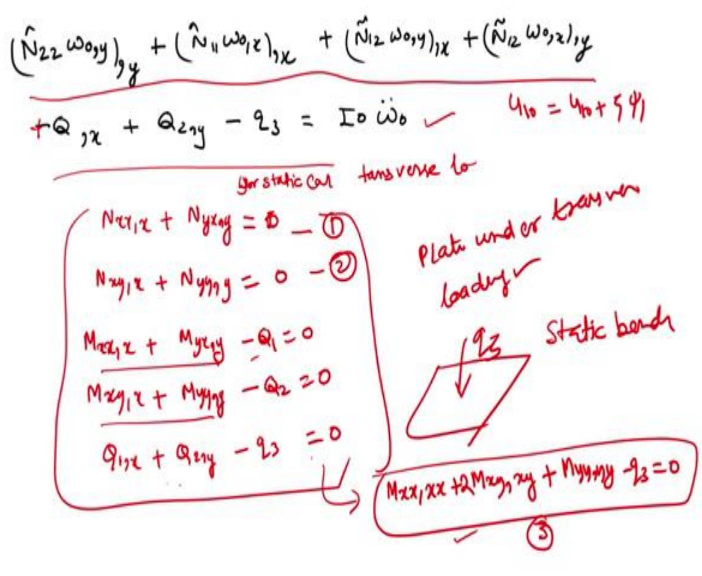

The non-linear contributions will be:

$$\left(\hat{N}_{22}w_0,_y\right)_{,y}+\left(\hat{N}_{11}w_0,_x\right)_{,x}+\left(N_{12}w_0,_y\right)_{,x}+\left(N_{12}w_0,_x\right)_{,y}.$$

These are the same equations of partial differential equations for the case of a rectangular plate under FSDT (first-order shear deformation theory)

You will get a similar set of equations:

$$N_{xx,x}+N_{yx,y}=0$$
$$N_{xy,x}+N_{yy,y}=0$$
$$M_{xx,x}+M_{yx,y}-Q_1=0$$
$$M_{xy,x}+M_{yy,y}-Q_2=0$$
$$Q_{1,x}+Q_{2,y}-q_3=0$$

The first equation is:

$N_{xx,x}+N_{yx,y}=0$, only for a simple static case and only transverse loading.

Generally, in most of the books, they consider only plate under transverse loading bending case then this equation will be equal to 0. The second equation:

$$N_{xy,x}+N_{yy,y}=0.$$

Then, the third equation:

$$M_{xx,x}+M_{yx,y}-Q_1=0;$$

Fourth equation:

$$M_{xy,x}+M_{yy,y}-Q_2=0$$

And the fifth equation:

$$Q_{1,x}+Q_{2,y}-q_3=0.$$

For the case of a plate, under transverse loading static bending case, you can find these five sets of governing equations. You can verify

$Q_1=M_{xx,x}+M_{yx,y}$ and $Q_2=M_{xy,x}+M_{yy,y}$.

If you substitute it into this equation, it reduces to a classical plate theory and will become:

$$M_{xx,xx} + 2M_{xy,xy} + M_{yy,yy} - q_3 = 0 \quad equation(3)$$

This will be the third equation for the case of CLT and following are the first and second equations:

$$N_{xx,x} + N_{yx,y} = 0 \quad equation(1)$$

$$N_{xy,x} + N_{yy,y} = 0 \quad equation(2)\,.$$

For the case of the plate, we can able to deduce the set of governing equations and for the case of an FSDT, five equations are there. If you want to know for a classical shell then you have to use these two equations and substitute in the 5th equation and the final equation can be obtained.

From this, we have developed a generalized partial differential equation for a doubly curved shell. Now, depending upon our requirement or the application, we can deduce and we can get the governing equations. Now, the question comes here that we have derived by taking the assumptions of first-order shear deformation shell theories, we have assumed the displacement field as:

$$u_1 = u_{10} + \psi_1 \varsigma\,.$$

If you want to do for a third-order or a higher-order theory, then first you have to develop a higher-order model and then, later on, the lower-order model can be obtained as a special case. For example, in this case, the first-order shear deformation theory we have developed. We can get a special case like classical shell theories or classical plate theories the lower order can be obtained by putting some index.

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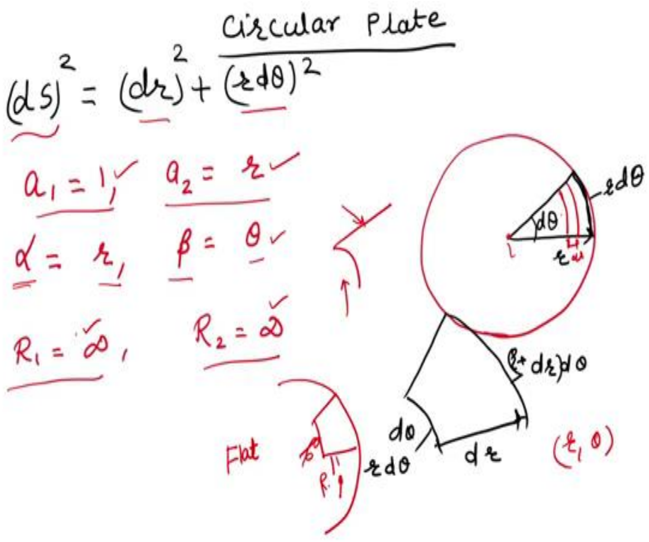

Now, I am going to explain to derive or deduce that set of equations for a circular plate. If you take an example of a circular plate, take a small element at distance r and that distance will be dr and this will be dθ. In that case:

$$(dS)^2 = (dr)^2 + (rd\theta)^2.$$

Here $a_1 = 1$, $a_2 = \mathrm{r}$ and the curvilinear parameters $\alpha = \mathrm{r}$ and $\beta = \theta$, in polar coordinate, we are talking about r and θ.

For the case of a circular plate the radius of curvature $R_1 = R_2 = \infty$ because this is flat only sometimes you may say it is curved in that direction, but we are not taking that as a curvature. it is basically in a longitudinal sense.

If we have a cylindrical shell, then you can see that from the bottom the shell is cylindrical, but for the case of the plate if you see from the bottom, it is a flat piece. $R_1 = R_2 = \infty$ for the case of a circular plate.

If we have a cylindrical shell, then you can see that from the bottom the shell is cylindrical, but for the case of the plate if you see from the bottom, it is a flat piece. $R_1 = R_2 = \infty$ for the case of a circular plate.

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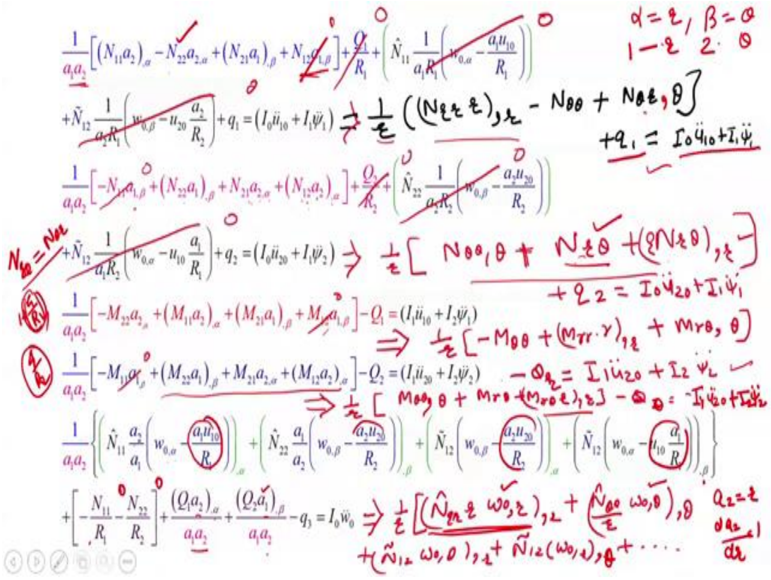

In equation (1):

$R_1 = R_2 = \infty$, therefore, $\frac{Q_1}{R_1}$ will not contribute similarly these non-linear terms:

$\left(N_{11}\frac{1}{a_1R_1}\left(w_0,_\alpha - \frac{a_1u_{10}}{R_1}\right)\right) + N_{12}\frac{1}{a_2R_1}\left(w_0,_\beta - u_{20}\frac{a_2}{R_2}\right)$ will also not contribute.

In the linear term $\frac{1}{a_1a_2} = \frac{1}{r}$, because in this case, $a_1 = 1$ and $a_2 = $ r.

In the first term:

$\left(N_{11}a_2\right)_{,\alpha}$; α = r and $\beta = \theta$, we can say that 1 = r and 2 = θ

If you take derivative of a_2 with respect to α :

$\left(N_{11}a_2\right)_{,\alpha} = \left(N_{rr}r\right)_{,r}$

This term $N_{22}a_{2,\alpha} = N_{\theta\theta}$ because, here $N_{22} = N_{\theta\theta}$ and $a_{2,\alpha}$ =1.

Then, $\left(N_{21}a_1\right)_{,\beta} = N_{\theta r,\theta}$ because, $a_1 = 1$ and derivative β means derivative θ. In the term $N_{12}a_{1,\beta}$, a_1 is 1, if you take derivative with respect to θ it is going to give you 0.

Three terms will contribute and 1st equation will be:

$$\frac{1}{r}\left[\left(N_{rr}r\right)_{,r}-N_{\theta\theta}+N_{\theta r,\theta}\right]+q_1=\left(I_0\ddot{u}_{10}+I_1\ddot{\psi}_1\right) .$$

Similarly, in the 2nd equation:

The first term $-N_{11}a_{1,\beta}$ will not contribute, the second term $\left(N_{22}a_1\right)_{,\beta}$ will contribute, the third term $N_{21}a_{2,\alpha}$ will contribute and the fourth term $\left(N_{12}a_2\right)_{,\alpha}$ will also contribute.

The term $\left(N_{22}a_1\right)_{,\beta}$ = $N_{\theta\theta,\theta}$,

This term $N_{21}a_{2,\alpha}$ = $N_{r\theta}$, because $a_{2,\alpha}$ = 1, and the term $\left(N_{12}a_2\right)_{,\alpha}$ = $\left(rN_{r\theta}\right)_{,r}$

Here, $N_{r\theta}$ = $N_{\theta r}$ because the only difference is due to $1+\frac{\varsigma}{R_1}$ *and* $1+\frac{\varsigma}{R_2}$ terms. Here, $R_1=R_2=\infty$. So, they will not create a difference. Hence, $N_{r\theta}$ and $N_{\theta r}$ are same for the case of a circular plate.

You can look at any plate and shell book in which the circular plate is discussed. The 2nd equation will be:

$$\frac{1}{r}\left[N_{\theta\theta,\theta}+N_{r\theta}+\left(rN_{r\theta}\right)_{,r}\right]+q_2=\left(I_0\ddot{u}_{20}+I_1\ddot{\psi}_2\right).$$

Now, come to the 3rd equation:

The term $-M_{22}a_{2,\alpha}$ = $-M_{\theta\theta}$ because $a_{2,\alpha}$ = 1,

The term $\left(M_{11}a_2\right)_{,\alpha}$ = $\left(M_{rr}.r\right)_{,r}$,

The term $\left(M_{21}a_1\right)_{,\beta}$ will not contribute, the term $M_{12}a_{1,\beta}$ $=M_{r\theta,\theta}$. Three terms will contribute. We get equation (3) equals to

$$\frac{1}{r}\left[-M_{\theta\theta}+\left(M_{rr}.r\right)_{,r}+M_{r\theta,\theta}\right]-Q_r=\left(I_1\ddot{u}_{10}+I_2\ddot{\psi}_1\right)$$

In the 4th equation: the first term $-M_{11}a_{1,\beta}$ will not contribute, these terms $\left(M_{22}a_1\right)_{,\beta}+M_{21}a_{2,\alpha}+\left(M_{12}a_2\right)_{,\alpha}$ will contribute.

We get $\left(M_{22}a_1\right)_{,\beta} = M_{\theta\theta,\theta}$,

$M_{21}a_{2,\alpha} = M_{r\theta}$, and

$\left(M_{12}a_1\right)_{,\alpha} = \left(M_{r\theta}r\right)_{,r}$.

The 4th equation will be:

$$\frac{1}{r}\left[M_{\theta\theta,\theta} + M_{r\theta} + \left(M_{r\theta}r\right)_{,r}\right] - Q_\theta = \left(I_1\ddot{u}_{20} + I_2\ddot{\psi}_2\right).$$

Now, the 5th equation:

Here again these terms $\frac{a_1u_{10}}{R_1}$, $\frac{a_2u_{20}}{R_2}$, $u_{20}\frac{a_2}{R_2}$, and $u_{10}\frac{a_1}{R_1}$ will not contribute.

The equation will be:

$$\frac{1}{r}\left(\hat{N}_{rr}r\left(w_{0},_{r}\right)\right)_{,r} + \left(\frac{\hat{N}_{\theta\theta}}{r}\left(w_{0},_{\theta}\right)\right)_{,\theta} + \left(N_{r\theta}\left(w_{0},_{\theta}\right)\right)_{,r} + \left(N_{r\theta}\left(w_{0},_{r}\right)\right)_{,\theta} + Q_{1,x} + Q_{2,y} - q_3 = I_0\ddot{w}_0$$

These terms $\left(-\frac{N_{11}}{R_1} - \frac{N_{22}}{R_2}\right)$ will not contribute.

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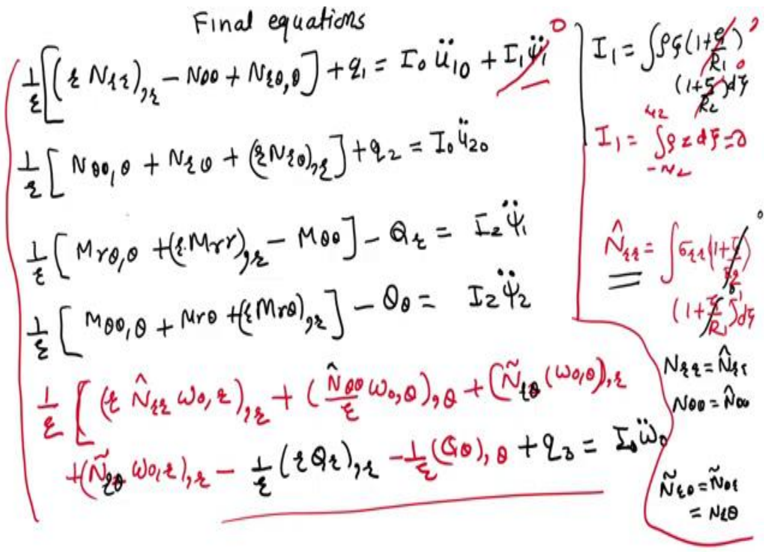

We get the final set of governing equations that we have deduced from the shell that will look like this.

$$\frac{1}{r}\left[\left(r.N_{rr}\right)_{,r}-N_{\theta\theta}+N_{r\theta,\theta}\right]+q_1=I_0\ddot{u}_{10}$$

$$\frac{1}{r}\left[N_{\theta\theta,\theta}+N_{r\theta}+\left(r.N_{r\theta}\right)_{,r}\right]+q_2=I_0\ddot{u}_{20}$$

$$\frac{1}{r}\left[M_{r\theta,\theta}+\left(r.M_{rr}\right)_{,r}-M_{\theta\theta}\right]-Q_r=I_2\ddot{\psi}_1$$

$$\frac{1}{r}\left[M_{\theta\theta,\theta}+M_{r\theta}+\left(r.M_{r\theta}\right)_{,r}\right]-Q_\theta=I_2\ddot{\psi}_2$$

$$\frac{1}{r}\left(r.\hat{N}_{rr}w_{0,r}\right)_{,r}+\left(\frac{\hat{N}_{\theta\theta}}{r}\left(w_{0,\theta}\right)\right)_{,\theta}+\left(N_{r\theta}w_{0,\theta}\right)_{,r}+\left(N_{r\theta}w_{0,r}\right)_{,\theta}-\frac{1}{r}\left(r.Q_r\right)_{,r}-\frac{1}{r}\left(Q_\theta\right)_{,\theta}+q_3=I_0\ddot{w}_0$$

For this case also I_1 contribution will not be there. And these are the same governing equations.

The purpose of explaining this is once you get a governing differential equation for a generalized shell element, then you can also derive the governing equations for the lower order whether you talk about a plate, a circular plate, or a cylinder.

Sometimes, for the case of development, it will be a special case to get the solution for a circular plate. First, you have to convert the basic equations into a circular equation and then you can solve it.

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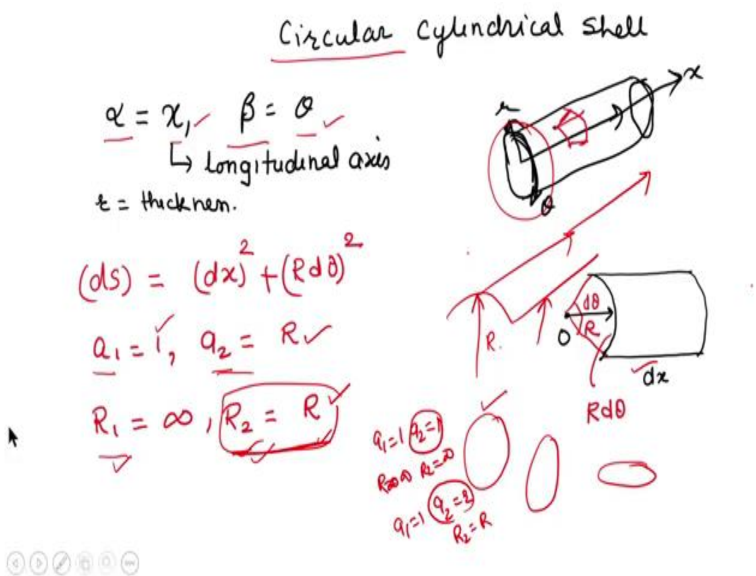

The next most important case is the circular cylindrical shell. The circular plate and the rectangular plate are easy to derive, but definitely that in the cylindrical coordinate system or circular cylindrical shell.

I have written the circular cylindrical shell because in this shell theories you may have a number of cylinders having different faces like it may be circular, it may be oval or it may be elliptical. You will have different a_1 and a_2 for that.

The simplest case is the circular cylindrical shell and in most of the literature or books, these have been covered. I would like to say that the most simplified one is the circular cylindrical shell and most of the analytical solutions are available for this case.

If you want to go for a typical or a very different kind of geometry one can, go ahead depending upon the requirements. In 90% of cases, in our day-to-day life or in the structural application, the circular cylindrical shells are used. Odd shapes one can try.

For that you have to obtain a_1, a_2, R_1, and R_2. For this case, $\alpha = x$ and $\beta = \theta$, x is the length along the longitudinal direction and β is along the circumferential θ direction. If you cut a cylindrical element and if you see from the bottom, you will find a radius is there in that direction. And, the longitudinal direction the lengthwise there is in the second direction ∞.

a_1 = 1 and $a_2 = R$. In the case of a circular plate, we represent this with r and here $R_1 = \infty$ and $R_2 = R$. From the mathematical point of view, from a circular plate to a cylindrical shell the difference is that radius in the second direction R_2 = R, this is only one parameter changed now.

Previously, for a rectangular plate a_1 = 1, a_2 = 1, $R_1 = R_2 = \infty$.

For a circular plate R_2 = R and a_2 = R, the governing equations will look entirely different when compared to a flat rectangular plate. If you include this effect of the radius then all the set of governing equations will be entirely different and will not look like a plate.

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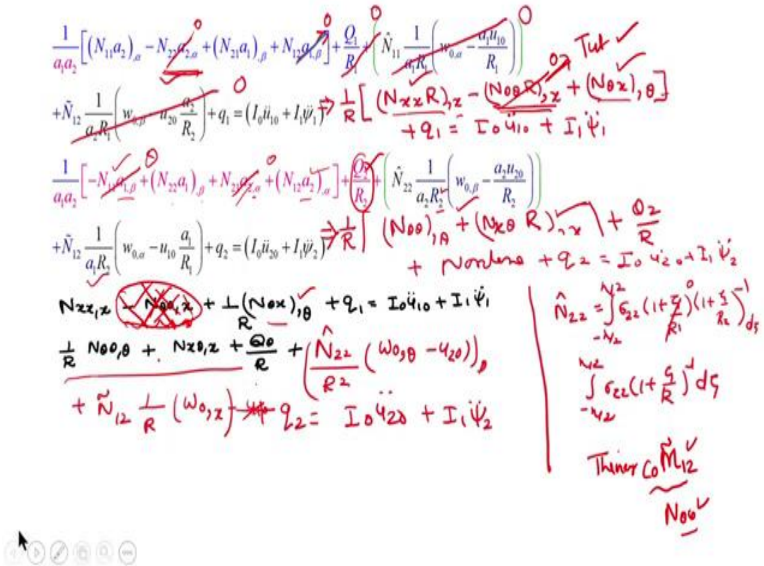

In first equation, instead of $\frac{1}{a_1 a_2}$ we will write $\frac{1}{R}$ and the term $\left(N_{11}a_2\right)_{,\alpha} = \left(N_{xx}R\right)_{,x}$. The term $-N_{22}a_{2,\alpha}$ will not contribute because $a_{2,\alpha}$ = R, x = R is not a function of x. Term $\left(N_{21}a_1\right)_{,\beta} = \left(N_{\theta x}\right)_{,\theta}$ and this term $N_{12}a_{1,\beta}$ will not contribute.

$a_{2,\alpha}$ means a_2 with respect to x will not contribute.

This term $-N_{22}a_{2,\alpha}$ will not be there, I will have to check this equation will clarify in the tutorial part or the next lecture. Equation (1) will be:

$$N_{xx,x} + \frac{1}{R}\left(N_{\theta x}\right)_{,\theta} + q_1 = I_0 \ddot{u}_{10} + I_1 \ddot{\psi}_1$$

Then we have 2nd equation:

The term $-N_{11}a_{1,\beta}$ will not contribute,

The term $\left(N_{22}a_1\right)_{,\beta} = \left(N_{\theta\theta}\right)_{,\theta}$

The term $N_{21}a_{2,\alpha}$ will also not contribute,

The term $\left(N_{12}a_2\right)_{,\alpha} = \left(N_{x\theta}R\right)_{,x}$.

In equation (1):

This term $-N_{22}a_{2,\alpha}$ = 0 and it will not contribute.

In some of the books on the concept of thin cylindrical shell, we have an extra term $C_0 a_1 \tilde{M}_{12,\beta}$ or $-C_0 a_2 \tilde{M}_{12,\alpha}$

Here, $\tilde{M}_{12}$ having the contribution of $N_{\theta\theta}$ and so on. For a thick shell, this term will not contribute and then we have this 2nd equation will be:

$$\frac{1}{R}N_{\theta\theta,\theta}+N_{x\theta,x}+\frac{Q_\theta}{R}+\left(\frac{N_{22}}{R^2}\left(w_0,_\theta-u_{20}\right)\right)+N_{12}\frac{1}{R}\left(w_0,_x\right)+q_2=\left(I_0\ddot{u}_{20}+I_1\ddot{\psi}_2\right)$$

In the 1st equation $\frac{Q_1}{R_1}$ will not come

But in the 2nd equation:

$\frac{Q_2}{R_2}$ will be there because R_2 = R. It will contribute and some non-linear term will also contribute because R_2 is there.

In the 1st equation, there will be no non-linear terms, but in the 2nd equation, non-linear term will also contribute because R_2 = nonzero.

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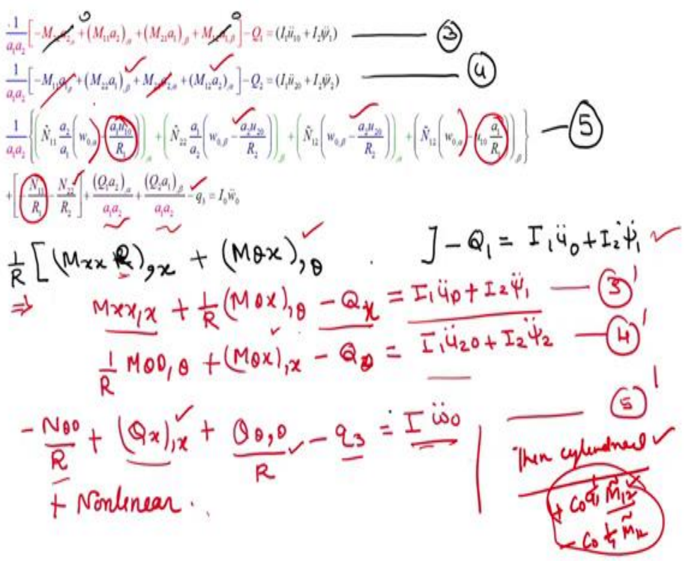

In the 3rd equation:

This term $-M_{22}a_{2,\alpha}$ will not contribute only these $\left(M_{11}a_2\right)_{,\alpha}+\left(M_{21}a_1\right)_{,\beta}$ two terms will contribute.

The equation (3) will be:

$$\frac{1}{R}\left[\left(M_{xx}R\right)_{,x}+\left(M_{x\theta}\right)_{,\theta}\right]-Q_1=I_1\ddot{u}_{10}+I_2\ddot{\psi}_1$$

Here you see that R is not a function of x, so, you can take common. Ultimately, it will get cancelled from the outside and the equation becomes like this:

$$M_{xx,x}+\frac{1}{R}\left(M_{\theta x}\right)_{,\theta}-Q_x=I_1\ddot{u}_{10}+I_2\ddot{\psi}_1.$$

Then come to the 4th equation:

$-M_{11}a_{1,\beta}$ will not contribute, $M_{21}a_{2,\alpha}$ also will not contribute.

Only these $\left(M_{22}a_1\right)_{,\beta}$ and $\left(M_{12}a_2\right)_{,\alpha}$ terms will contribute.

$\left(M_{22}a_1\right)_{,\beta}=M_{\theta\theta,\theta}$ and $\left(M_{12}a_1\right)_{,\alpha}=\left(M_{\theta x}\right)_{,x}$

And equation (4) will become:

$$\frac{1}{R}M_{\theta\theta,\theta}+\left(M_{\theta x}\right)_{,x}-Q_\theta=I_1\ddot{u}_{20}+I_2\ddot{\psi}_2.$$

In the 5th equation, some non-linear terms will not contribute, $\frac{a_1u_{10}}{R_1}$ and $u_{10}\frac{a_1}{R_1}$ terms will not contribute. Partial terms are here this term $-\frac{N_{11}}{R_1}$ will not contribute but this term $-\frac{N_{22}}{R_2}$ will contribute. The equation (5) will become:

$$-\frac{N_{\theta\theta}}{R}+\left(Q_x\right)_{,x}+\frac{Q_{\theta,\theta}}{R}+nonlinear-q_3=I_0\ddot{w}_0.$$

In this way, we have deduced the governing equations for a cylindrical shell. In some of the books of the thin cylindrical shells these equations are deduced and problems are solved.

But they are very special ones. Already I have given the treatment for a thin cylindrical shell in the 1st and 2nd equation we will add a term $C_0 a_1 \tilde{M}_{12,\beta}$ or $-C_0 a_2 \tilde{M}_{12,\alpha}$.

In the 1st equation we add and in the 2nd equation we subtract it. If we consider these terms, then in this equation will be valid for a thin cylindrical shell.

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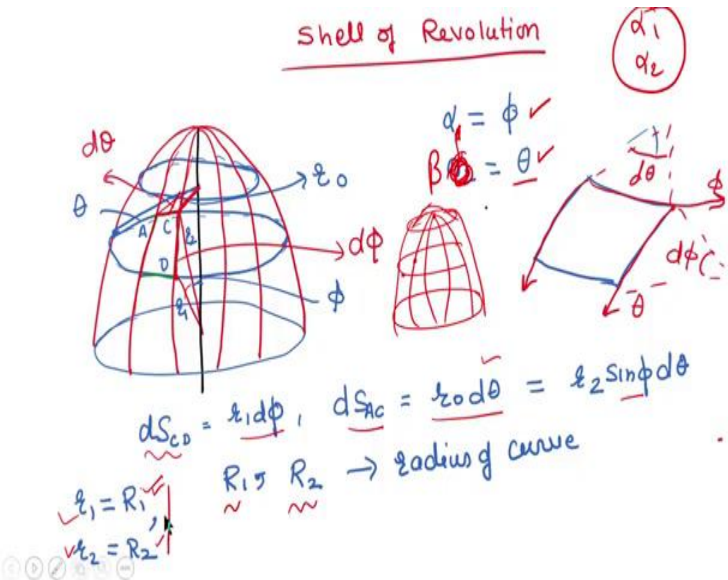

Now, the shell of revolution: this is the most generalized form, we have already classified the shell. The first classification is by the revolution of the surfaces i.e., we can generate the surfaces by translation through a parametric line.

The shell surfaces can be generated through the revolution about the meridian axis or the central axis. In this case, the doubly curved shell will have curvature in both directions. let us say $\alpha = \phi$ and because the coordinate system is β, here $\beta = \theta$.

In some of the books the coordinates are α_1 and α_2 but, in this course, we have taken α and β. If you see the surface of revolution and you see these parametric lines, we will have horizontal circles.

It is going like this in parametric lines. We will have two radius of curvature - one along this and one along this. If you take a shell element and if you see from the bottom then one radius of curvature will be along this horizontal one, and one will be for a vertical one. If you take a shell element like this then the length of the shell element DS_{CD} along this will be $r_1 d\phi$ because this angle is $d\phi$.

This angle is $d\theta$, which is making on a horizontal one.

$DS_{AC} = r_0 d\theta = r_2 \sin\phi d\theta$.

r_0 can be expressed in terms of $r_2 \sin\phi$. Here, R_1 and R_2 are radii of curvature, generally, the formulation is done in terms of r_1 and r_2, but at the end of the day the radius of curvature is related to R_1 and R_2.

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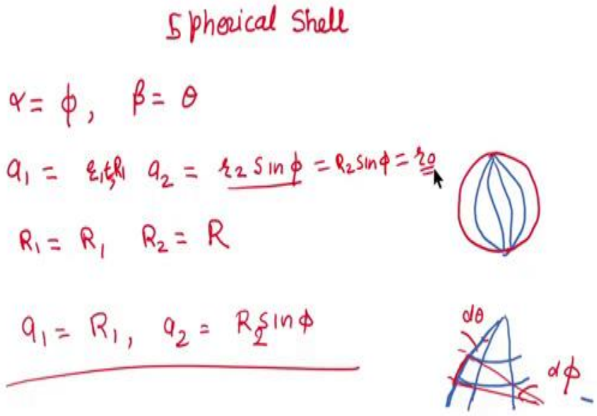

From here $\alpha = \phi$, $\beta = \theta$, $a_1 = r_1 = R_1$, and $a_2 = r_2 \sin\phi = R_2 \sin\phi = r_0$.

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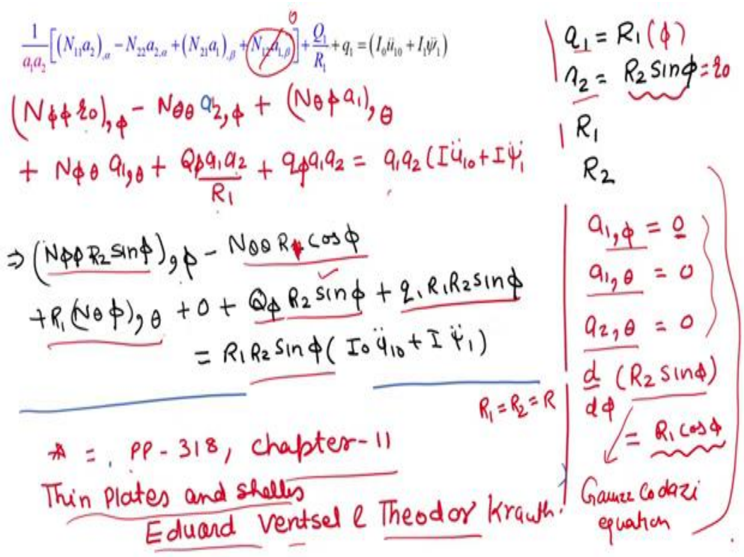

With these parameters, I will explain to work with this 1st equation. Till now, it was very comfortable to work with $\frac{1}{a_1 a_2}$ in 1st equation but now, $a_1 = R_1$ and $a_2 = R_2 \sin\phi$, in the bottom coordinate $R_2 \sin\phi$ comes into the picture.

If this is the situation, we do not want some trigonometric terms at the bottom, maximum we can have only the radius one. We have to first convert it into the normal form so that we do not have $a_1 a_2$. We will multiply $a_1 a_2$ with this $\frac{Q_1}{R_1}$.

It will go to the multiplication. If you remember initially this $a_1 a_2$ were with Q_1 and q_1. Initially, whatever we have obtained from that weak form of integrals $a_1 a_2$, we have to put it back and we can start working with this.

The important point is if you want to know $a_{1,\phi}$, $a_{1,\theta}$, or $a_{2,\theta}$, they will contribute to 0. But in $a_{2,\phi}$, $a_2 = R_2 \sin\phi$, using the concept of Gauss-Codazzi equations $a_{2,\phi} = R_1 \cos\phi$.

The detailed derivations of a doubly curved system can be seen in the book of Thin Plate and Shells by Eduard Ventsel and Theodor the page number is 318 chapter- 11. This complete derivation is given in chapter 16 or chapter 14.

We can say in equation (1):

This term $\left(N_{11} a_2\right)_{,\alpha} = \left(N_{\phi\phi} R_2 \sin\phi\right)_{,\phi}$ because $a_2 = r_0 = R_2 \sin\phi$ and $\alpha = \phi$.

The term $-N_{22}a_{2,\alpha}$ = $-N_{\theta\theta}R_1\cos\phi$, and the term $\left(N_{21}a_1\right)_{,\beta}$ = $R_1\left(N_{\theta\phi}\right)_{,\theta}$.

This term $N_{12}a_{1,\beta}$ will not contribute, this will be 0.

The term $\dfrac{Q_1}{R_1}$ = $Q_\phi R_2\sin\phi$ because, a_1a_2 goes up here, $a_1 = R_1$ and $a_2 = R_2\sin\phi$ and R_1 gets cancelled. Plus $q_1R_1R_2\sin\phi$.

Equation (1) will be:

$$\left(N_{\phi\phi}R_2\sin\phi\right)_{,\phi} - N_{\theta\theta}R_1\cos\phi + R_1\left(N_{\theta\phi}\right)_{,\theta} + 0 + Q_\phi R_2\sin\phi + q_1R_1R_2\sin\phi = R_1R_2\sin\phi\left(I_0\ddot{u}_{10} + I_1\ddot{\psi}_1\right)$$

This is the form of the differential equation for the case of a doubly curved shell which is having the surface of revolution. If R_1 = R_2 = R, then we can get these equations for the case of a spherical shell, otherwise, the shell may have elliptical or some different shell surfaces.

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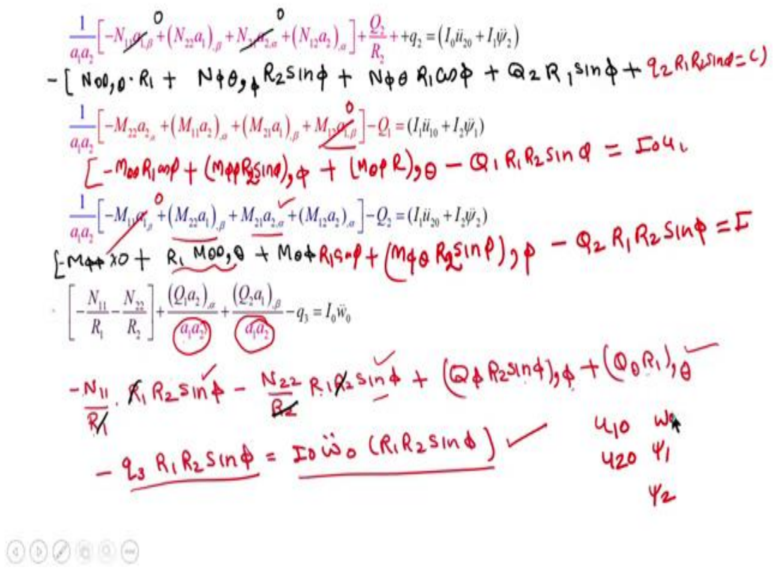

In the same way, we can get the set of equations in the second direction, third direction, and fourth direction also.

Equation (2) will be:

$$N_{\theta\theta,\theta}.R_1 + N_{\phi\theta}R_2 \sin\phi + N_{\phi\theta}R_1 \cos\phi + Q_2 R_2 \sin\phi + q_2 R_1 R_2 \sin\phi = R_1 R_2 \sin\phi\left(I_0 \ddot{u}_{20} + I_1 \ddot{\psi}_2\right).$$

Equation (3) will be:

$$-M_{\theta\theta}R_1 \cos\phi + \left(M_{\phi\phi}R_2 \sin\phi\right)_{,\phi} + \left(M_{\theta\phi}R\right)_{,\theta} - Q_1 R_1 R_2 \sin\phi = R_1 R_2 \sin\phi\left(I_1 \ddot{u}_{10} + I_2 \ddot{\psi}_1\right)$$

Then, in equation (4), this term $-M_{11}a_{1,\beta}$ will not contribute, the terms $\left(M_{22}a_1\right)_{,\beta}$,

$M_{21}a_{2,\alpha}$ and $\left(M_{12}a_2\right)_{,\alpha}$ will contribute. It will be:

$$R_1 M_{\theta\theta,\theta} + M_{\theta\phi}R_1 \sin\phi + \left(M_{\phi\theta}R_2 \sin\phi\right)_{,\phi} - Q_2 R_1 R_2 \sin\phi = R_1 R_2 \sin\phi\left(I_1 \ddot{u}_{20} + I_2 \ddot{\psi}_2\right)$$

In the 5th equation, we have to do some mathematical simplification because $a_1 a_2$ is coming at the bottom, we have to multiply it. Here, R_1 will get cancelled.

Equation (5) will be:

$$-N_{11}R_2 \sin\phi - N_{22}R_1 \sin\phi + \left(Q_\phi R_2 \sin\phi\right)_{,\phi} + \left(Q_\theta R_1\right)_{,\theta} - q_3 R_1 R_2 \sin\phi = R_1 R_2 \sin\phi\left(I_0 \ddot{w}_0\right).$$

Once we get this set of equations, now the questions come, can we solve the problems? Can we get the displacements? We aim to find the displacement stress movements. Can we directly work with these equations? We cannot directly work with these equations because they are in stress resultant form.

Whether you talk about a cylindrical shell, a complete shell or any kind of shell we have to first convert it into a primary displacement form. Primary displacements are $u_{10},\ u_{20},\ w_0,\ \psi_1,\ and\ \psi_2$.

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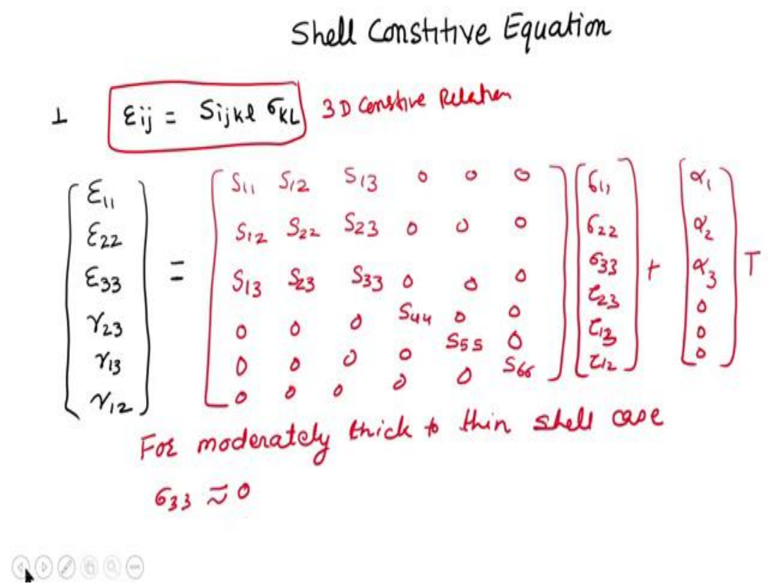

For this, the very first approach is to find the shell constitutive equations like $\varepsilon_{ij} = S_{ijkl}\sigma_{kl}$ is known as generalized Hooke's law or 3-dimensional constitutive equations relations, they are for a 3-dimensional body. Now, we have to find it for the case of the shell. In the starting for a moderately thick to thick shell we assume a concept of plane stress, under that head σ_{33} is considered 0.

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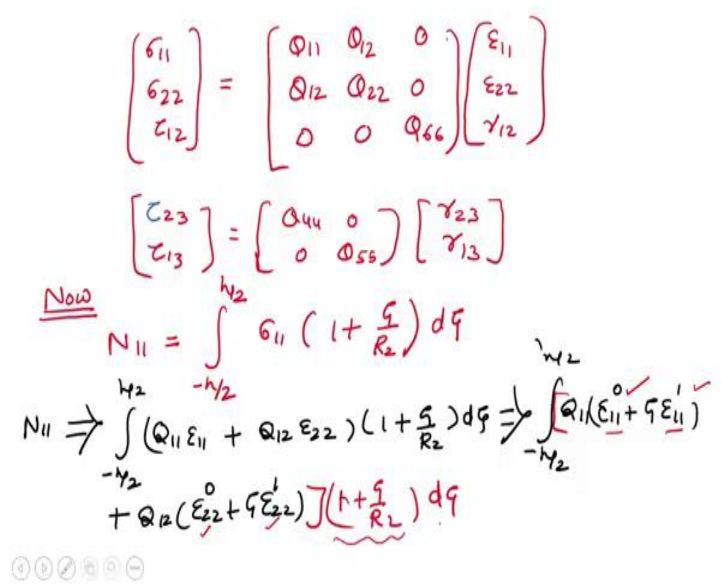

If you choose like that then the stresses σ_{11}, σ_{22} and τ_{12} can be expressed in terms of a matrix Q and strains, where Q_{11}, Q_{12}, Q_{22}, and $Q_{66,}$ are known as reduced stiffness coefficients and in the case of shear τ_{23} and τ_{13} are expressed with the help of Q_{44} and Q_{55}. Now, how do you get a constitutive relation for the case of a shell?

The definition of N_{11} stress resultant in the first direction is:

$$\int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{11}\left(1+\frac{\varsigma}{R_2}\right)d\varsigma .$$

We know that using the constitutive relation of a material we get:

$$\int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{11}\varepsilon_{11} + Q_{12}\varepsilon_{22}\left(1+\frac{\varsigma}{R_2}\right)d\varsigma .$$

In the plate, it is very simple, but for the case of shell it is slightly complex. ε_{11} has two components. ε_{11}^{0} component is causing stretching in the membrane and ε_{11}^{1} component is causing the bending in the shell, then similarly ε_{22}.

Now, we can say N_{11} will be:

$$\int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{11}\left(\varepsilon_{11}^{0}+\varsigma\varepsilon_{11}^{1}\right)+Q_{12}\left(\varepsilon_{22}^{0}+\varsigma\varepsilon_{22}^{1}\right)\left(1+\frac{\varsigma}{R_2}\right)d\varsigma .$$

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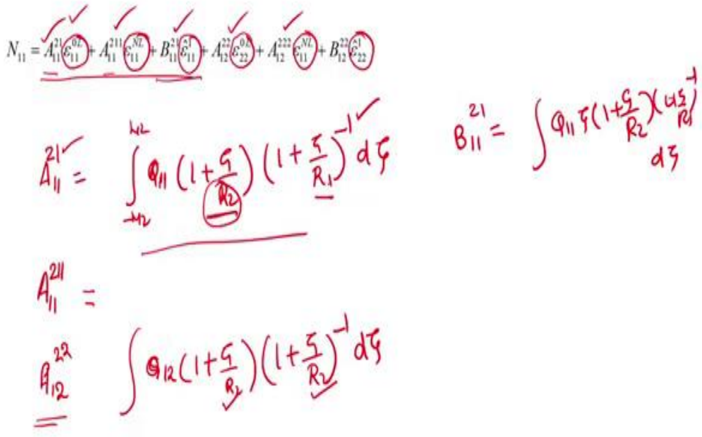

The rest of the equations, I will cover in the next lecture.

Thank you very much.