# Theory of Composite Shells

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**Week – 04**
**Lecture – 02**
**Membrane shell theory and moment**

Dear learners welcome to lecture -02 of the week- 04. In this lecture, I will explain the Membrane shell theory and the moment shell theory. In the previous lectures, we have completed the basic formulation and the special cases for the spherical shell, cylindrical shell, circular plate, and rectangular plate.

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## Linear Shell Equations

$$\frac{1}{a_1a_2}\left[\left(N_{11}a_2\right)_{,\alpha}-N_{22}a_{2,\alpha}+\left(N_{21}a_1\right)_{,\beta}+N_{12}a_{1,\beta}\right]+\frac{Q_1}{R_1}+q_1=\left(I_0\ddot{u}_{10}+I_1\ddot{\psi}_1\right)$$

$$\frac{1}{a_1a_2}\left[-N_{11}a_{1,\beta}+\left(N_{22}a_1\right)_{,\beta}+N_{21}a_{2,\alpha}+\left(N_{12}a_2\right)_{,\alpha}\right]+\frac{Q_2}{R_2}+q_2=\left(I_0\ddot{u}_{20}+I_1\ddot{\psi}_2\right)$$

$$\frac{1}{a_1a_2}\left[-M_{22}a_{2_{,\alpha}}+\left(M_{11}a_2\right)_{,\alpha}+\left(M_{21}a_1\right)_{,\beta}+M_{12}a_{1,\beta}\right]-Q_1=\left(I_1\ddot{u}_{10}+I_2\ddot{\psi}_1\right)$$

$$\frac{1}{a_1a_2}\left[-M_{11}a_{1_{,\beta}}+\left(M_{22}a_1\right)_{,\beta}+M_{21}a_{2,\alpha}+\left(M_{12}a_2\right)_{,\alpha}\right]-Q_2=\left(I_1\ddot{u}_{20}+I_2\ddot{\psi}_2\right)$$

$$\left[-\frac{N_{11}}{R_1}-\frac{N_{22}}{R_2}\right]+\frac{\left(Q_1a_2\right)_{,\alpha}}{a_1a_2}+\frac{\left(Q_2a_1\right)_{,\beta}}{a_1a_2}-q_3=I_0\ddot{w}_0$$

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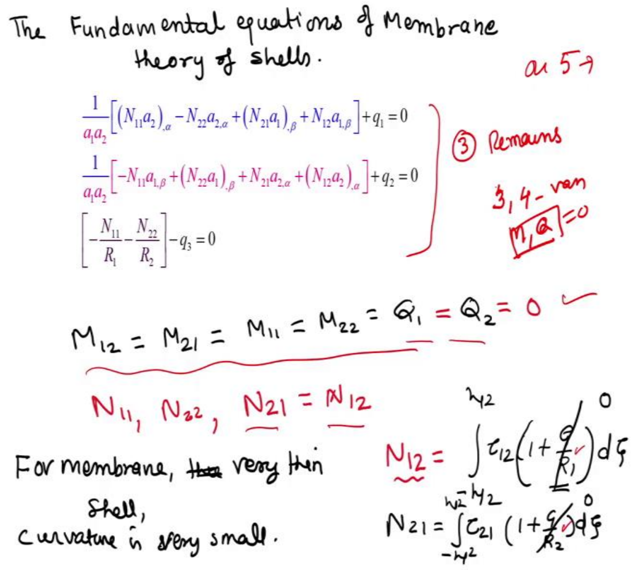


Before a generalized solution of a complete shell, there are two basic theories one is membrane shell theory and another is moment shell theory.

Already, in lecture- 01, I have covered the membrane shell theory applies to thin shells and can take only the membrane loading means, it cannot sustain the bending stresses or bending moments, as the shell is thin and subject to only in-plane stretching cases.

There are numerous examples in which the membrane theory of shells is applied. Since, it cannot take any bending stress, the moments $M_{11}$, $M_{22}$, the twisting moment $M_{21}$, and the shear forces $Q_1$, $Q_2$ will be 0.

Now, we are saying that the shell is thin, therefore, $\frac{\varsigma}{R_1}$ term can be neglected. If, we do so, then, $N_{12} = N_{21}$. Out of the five governing equations:

$$\frac{1}{a_1a_2}\left[\left(N_{11}a_2\right)_{,\alpha} - N_{22}a_{2,\alpha} + \left(N_{21}a_1\right)_{,\beta} + N_{12}a_{1,\beta}\right] + \frac{Q_1}{R_1} + q_1 = \left(I_0\ddot{u}_{10} + I_1\ddot{\psi}_1\right) \quad equation(1)$$

$$\frac{1}{a_1a_2}\left[-N_{11}a_{1,\beta} + \left(N_{22}a_1\right)_{,\beta} + N_{21}a_{2,\alpha} + \left(N_{12}a_2\right)_{,\alpha}\right] + \frac{Q_2}{R_2} + q_2 = \left(I_0\ddot{u}_{20} + I_1\ddot{\psi}_2\right) \quad equation(2)$$

$$\frac{1}{a_1a_2}\left[-M_{22}a_{2,\alpha} + \left(M_{11}a_2\right)_{,\alpha} + \left(M_{21}a_1\right)_{,\beta} + M_{12}a_{1,\beta}\right] - Q_1 = \left(I_1\ddot{u}_{10} + I_2\ddot{\psi}_1\right) \quad equation(3)$$

$$\frac{1}{a_1a_2}\left[-M_{11}a_{1,\beta} + \left(M_{22}a_1\right)_{,\beta} + M_{21}a_{2,\alpha} + \left(M_{12}a_2\right)_{,\alpha}\right] - Q_2 = \left(I_1\ddot{u}_{20} + I_2\ddot{\psi}_2\right) \quad equation(4)$$

$$\left(-\frac{N_{11}}{R_1} - \frac{N_{22}}{R_2}\right) + \frac{\left(Q_1a_2\right)_{,\alpha}}{a_1a_2} + \frac{\left(Q_2a_1\right)_{,\beta}}{a_1a_2} - q_3 = I_0\ddot{w}_0 \quad equation(5)$$

The following three equations remain for the case of membrane theory of shells.

$$\frac{1}{a_1a_2}\left[\left(N_{11}a_2\right)_{,\alpha} - N_{22}a_{2,\alpha} + \left(N_{21}a_1\right)_{,\beta} + N_{12}a_{1,\beta}\right] + q_1 = 0 \quad equation\,(1)$$

$$\frac{1}{a_1a_2}\left[-N_{11}a_{1,\beta} + \left(N_{22}a_1\right)_{,\beta} + N_{21}a_{2,\alpha} + \left(N_{12}a_2\right)_{,\alpha} + q_2\right] = 0 \quad equation\,(2)$$

$$\left(-\frac{N_{11}}{R_1} - \frac{N_{22}}{R_2}\right) - q_3 = 0 \quad equation\,(3)$$

The equation numbers 3rd and 4th of the five governing equations get vanish because these contain moment and Q terms. Later derived equation (1), equation (2), and equation (3), are valid for a doubly curved shell. And, you can find special cases like the surface of revolution for a cylindrical shell.

Substituting the corresponding parameters, $a_1$, $a_2$, $R_1$, and $R_2$, you can get that set of equations for the membrane theory of shells.

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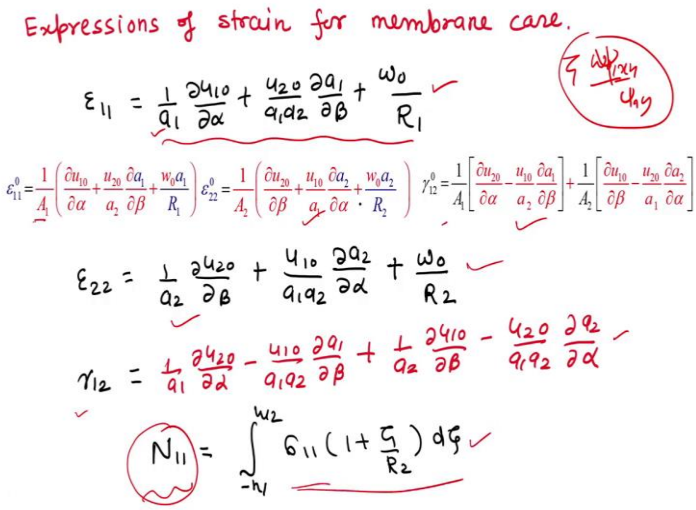


Now, what is the expression of strains for the membrane case? For the membrane case, there is no curvature, no bending, and no rotations, these are considered 0, only the linear membrane stretching part is considered, $\psi_{,x}$ or $\psi_{,y}$ are not considered.

Because the shell is thin, $\frac{\varsigma}{R_2}$ is neglected, therefore, $\frac{1}{A_1}$ can be replaced by $\frac{1}{a_1}$.

$$\varepsilon_{11} = \frac{1}{a_1}\frac{\partial u_{10}}{\partial \alpha} + \frac{u_{20}}{a_1 a_2}\frac{\partial a_1}{\partial \beta} + \frac{w_0}{R_1}.$$

$$\varepsilon_{22} = \frac{1}{a_2}\frac{\partial u_{20}}{\partial \beta} + \frac{u_{10}}{a_1 a_2}\frac{\partial a_2}{\partial \alpha} + \frac{w_0}{R_2}$$

$$\gamma_{12} = \frac{1}{a_1}\frac{\partial u_{20}}{\partial \alpha} - \frac{u_{10}}{a_1 a_2}\frac{\partial a_1}{\partial \beta} + \frac{1}{a_2}\frac{\partial u_{10}}{\partial \beta} - \frac{u_{20}}{a_1 a_2}\frac{\partial a_2}{\partial \alpha}.$$

These are the expressions of strain for $\varepsilon_{11}$, $\varepsilon_{22}$, and $\gamma_{12}$. Now, the general expression for $N_{11} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{11}\left(1 + \frac{\varsigma}{R_2}\right) d\varsigma$.

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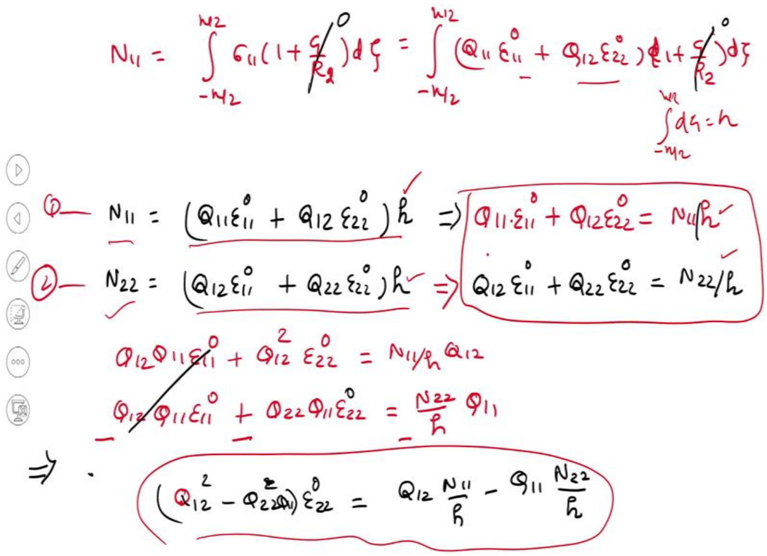

Now, if we substitute the constitutive relations:

$$N_{11} = \int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{11}\varepsilon_{11}^{0} + Q_{12}\varepsilon_{22}^{0}\left(1+\frac{\varsigma}{R_2}\right)d\varsigma \ ,$$

Then $N_{11} = \left(Q_{11}\varepsilon_{11}^{0} + Q_{12}\varepsilon_{22}^{0}\right)h$,

Because these are the only function of $\alpha$ and $\beta$ not the function of thickness. If you take the integration $\int_{\frac{-h}{2}}^{\frac{h}{2}} d\varsigma = h$. So, this h comes from here, the same way the expression for $N_{22} = \left(Q_{12}\varepsilon_{11}^{0} + Q_{22}\varepsilon_{22}^{0}\right)h$.

If we further mathematically simplify,

$$N_{11} = \left(Q_{11}\varepsilon_{11}^{0} + Q_{12}\varepsilon_{22}^{0}\right)h \Rightarrow Q_{11}\varepsilon_{11}^{0} + Q_{12}\varepsilon_{22}^{0} = \frac{N_{11}}{h}$$

$$N_{22} = \left(Q_{12}\varepsilon_{11}^{0} + Q_{22}\varepsilon_{22}^{0}\right)h \Rightarrow Q_{12}\varepsilon_{11}^{0} + Q_{22}\varepsilon_{22}^{0} = \frac{N_{22}}{h}$$

From here, we can express the value of $\varepsilon_{11}$ or $\varepsilon_{22}$ in terms of $\frac{N_{11}}{h}$ or $\frac{N_{22}}{h}$.

By multiplying this equation $Q_{11}\varepsilon_{11}^{0}+Q_{12}\varepsilon_{22}^{0}=\dfrac{N_{11}}{h}$ with $Q_{12}$ , we get:

$$Q_{12}Q_{11}\varepsilon_{11}^{0}+Q_{12}^{2}\varepsilon_{22}^{0}=Q_{12}\frac{N_{11}}{h}$$

And by multiplying $Q_{12}\varepsilon_{11}^{0}+Q_{22}\varepsilon_{22}^{0}=\dfrac{N_{22}}{h}$ equation with $Q_{11}$, we get

$$Q_{11}Q_{12}\varepsilon_{11}^{0}+Q_{11}Q_{22}\varepsilon_{22}^{0}=Q_{11}\frac{N_{22}}{h}\,.$$

Now, we subtract the 2nd equation from the 1st then, we get:

$$\left(Q_{12}^{2}-Q_{11}Q_{22}\right)\varepsilon_{22}^{0}=Q_{12}\frac{N_{11}}{h}-Q_{11}\frac{N_{22}}{h}\ .$$

From here, the expression of $\varepsilon_{22}$ in terms of stresses can be expressed.

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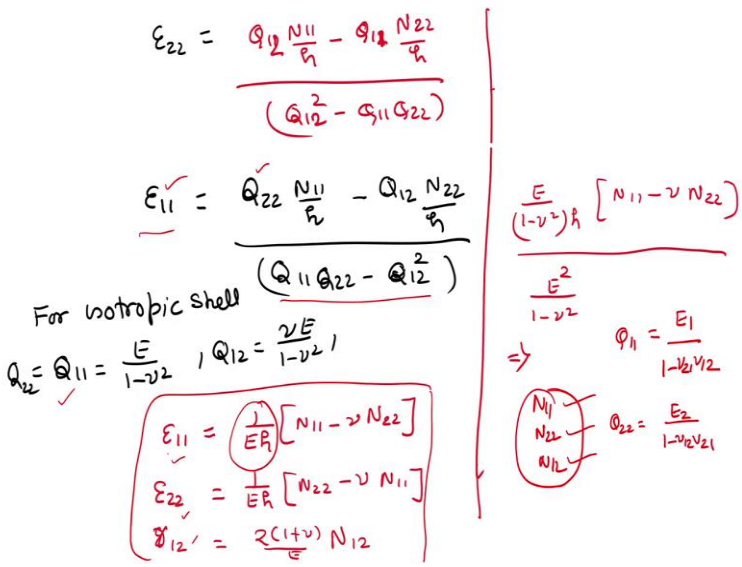


The expression of $\varepsilon_{22}$ will be:

$$\varepsilon_{22}=\frac{Q_{12}\dfrac{N_{11}}{h}-Q_{11}\dfrac{N_{22}}{h}}{\left(Q_{12}^{2}-Q_{11}Q_{22}\right)}$$

Similarly, the expression for $\varepsilon_{11}$ will be:

$$\varepsilon_{11} = \frac{Q_{22}\frac{N_{11}}{h} - Q_{12}\frac{N_{22}}{h}}{\left(Q_{11}Q_{22} - Q_{12}^2\right)}$$

If you substitute for an isotropic case:

$$Q_{22} = Q_{11} = \frac{E}{1-\mu^2} \text{ and } Q_{12} = \frac{\mu E}{1-\mu^2}.$$

If, you substitute that, that reduces to only $\frac{1}{Eh}$.

We can say that $\varepsilon_{11}$ is:

$$\varepsilon_{11} = \frac{1}{Eh}\left(N_{11} - \mu N_{22}\right)$$

$$\varepsilon_{22} = \frac{1}{Eh}\left(N_{22} - \mu N_{11}\right)$$

$$\gamma_{12} = \frac{2\left(1+\mu\right)}{E} N_{12}$$

You may be thinking, why are we interested to find $\varepsilon_{11}$, $\varepsilon_{22}$, and $\gamma_{12}$? Because our ultimate aim is to find the deflection or the strains in the body and then the stresses.

Because the governing equations are in terms of $N_{11}$, $N_{22}$, and $N_{12}$, for this case, if we somehow know the value of stress resultant, then we can tell that how much strain will be developed in that body. Therefore, we have derived the expression of strains in terms of stress resultants.

For the case of a composite plate:

$$Q_{11} = \frac{E_1}{1-\mu_{21}\mu_{12}}$$

$$Q_{22} = \frac{E_2}{1-\mu_{12}\mu_{21}}$$

The expression will be slightly complex, if you substitute those values one can obtain the expression.

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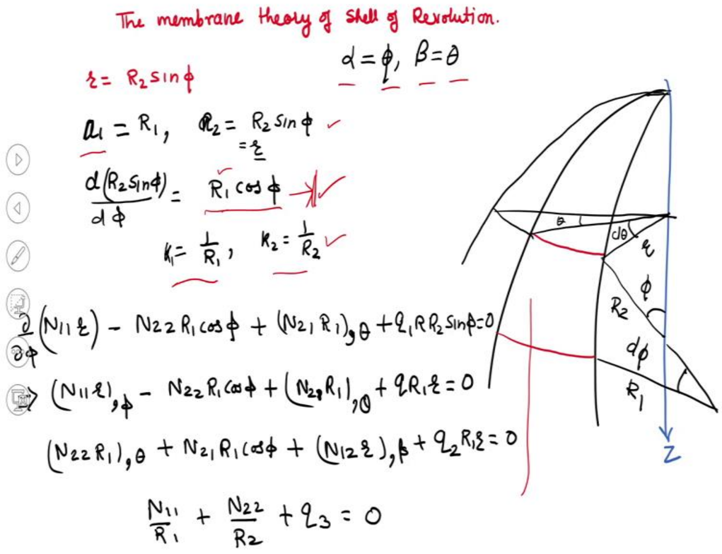


Now, we will express those 3 equations depending upon the special cases, because ultimately, we are going to solve the problem. The very first case is the shell of revolution in which the generator may be a curved line.

For that case $a_1 = R_1$ and the Lame's parameter $a_2 = r = R_2 \sin\phi$

If, we have to take the derivative with respect to $\phi$ , $\dfrac{d\left(R_2 \sin\phi\right)}{d\phi} = R_1 \cos\phi$

Already, in lecture 01 of week 04, I have explained that how it will be $R_1 \cos\phi$ by using some terminologies and curvatures: $K_1 = \dfrac{1}{R_1}$ and $K_2 = \dfrac{1}{R_2}$ .

If you substitute these parameters taking $\alpha = \phi$ and $\beta = \theta$ , these equations will look like this:

$$\left(N_{11} r\right)_{,\phi} - N_{22} R_1 \cos\phi + \left(N_{21} R_1\right)_{,\theta} + q_1 R_1 r = 0 \quad equation(1)$$
$$\left(N_{22} R_1\right)_{,\theta} + N_{21} R_1 \cos\phi + \left(N_{12} r\right)_{,\phi} + q_2 R_1 r = 0 \quad equation(2)$$
$$\frac{N_{11}}{R_1} + \frac{N_{22}}{R_2} + q_3 = 0 \quad equation(3)$$

Instead of $R_2 \sin\phi$, we are using $r$ in the equations, so that the equations look clean. But later on, when we are going to solve the problem then, $r$ needs to be replaced by capital

$R_2 \sin\phi$.

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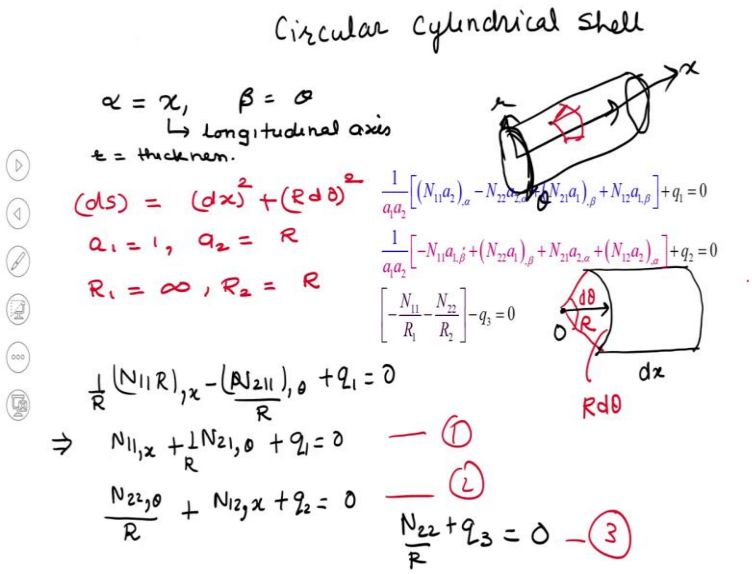


Similarly, in the governing equations for a circular cylindrical shell, $a_1$ = 1, $a_2$ = R, $R_1$= $\infty$, and $R_2$ = R. If you substitute this expression, these three equations we will obtain.

$$\left(N_{11}r\right)_{,\phi} - N_{22}R_1\cos\phi + \left(N_{21}R_1\right)_{,\theta} + q_1R_1r = 0 \quad equation(1)$$
$$\left(N_{22}R_1\right)_{,\theta} + N_{21}R_1\cos\phi + \left(N_{12}r\right)_{,\phi} + q_2R_1r = 0 \quad equation(2).$$
$$\frac{N_{11}}{R_1} + \frac{N_{22}}{R_2} + q_3 = 0 \quad equation(3)$$

The solution of the circular cylindrical shell is very easy and from the structural point of view, that most of the surfaces are made of circular cylindrical shell, it is a well-researched area in which the structures are developed and the solution is easy.

One can get the solution for the generalized shell of revolution, But, for an example point of view, I will explain that to solve this equation. Here, the 3rd equation says that $N_{\theta\theta}$ or $N_{22}$ = R $q_3$. And, if you substitute in the second equation, then $N_{12}$ can be obtained and if you substitute in the first equation, then $N_{11}$ can be obtained.

In this way, the 1st, 2nd, and 3rd equations are solved and during that process, if some constants come, they will be found.

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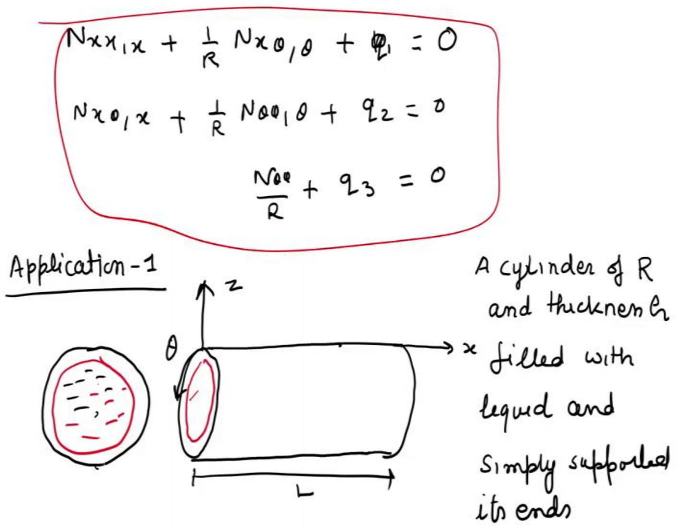


All three equations are written here in a proper way:

$$N_{xx,x} + \frac{1}{R} N_{x\theta,\theta} + q_1 = 0 \quad \text{equation (1)}$$

$$\frac{N_{\theta\theta,\theta}}{R} + N_{x\theta,x} + q_2 = 0 \quad \text{equation (2)}$$

$$\frac{N_{\theta\theta}}{R} + q_3 = 0 \quad \text{equation (3)}$$

The very first application, for example, a closed cylindrical shell under liquid pressure, or a liquid storage tank. If you say that there are cylindrical shells in which the LPG, some oil, or water is stored inside that tank and the cylinder filled with liquid is simply supported from the ends.

Then, we can directly utilize this set of equations. Initially, after developing the governing equations, we have to convert those equations into the basic primary variables, then we solve them. But, these types of cases of a membrane shell theory can be solved in terms of stresses themselves.

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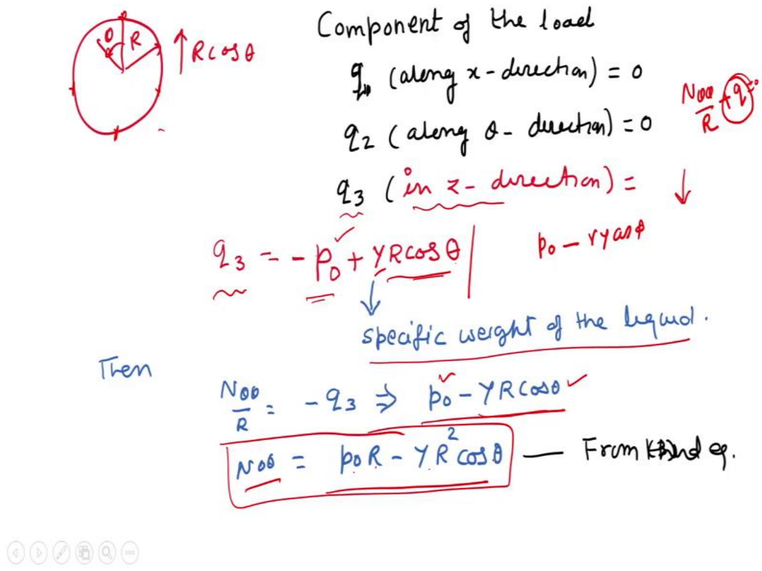


For the case of liquid pressure, you can say that Z is going upward, but the pressure is acting downwards. Then, $q_3$ can be written as:

$$q_3 = -p_0 + \gamma R \cos\theta$$

We have taken that pressure is acting in the negative surface normal of that.

From there, $\frac{N_{\theta\theta}}{R} = -q_3 = p_0 - \gamma R \cos\theta$ , in some of the books it is $p_0 - \gamma R \cos\theta$, we assume that liquid pressure will be in opposite direction always.

Here, $\gamma$ is known as the specific weight of the liquid and this is the component that, when $\theta$ = 0, it will be equal to $\gamma$ $R_1$; when, $\theta$ = 90˚, then it will be 0, there will be no pressure at the top of the liquid. If you substitute in the 3rd equation it becomes $p_0 - \gamma R \cos\theta$.

Depending upon the initial assumptions, in the 3rd equation, some are putting $\frac{N_{\theta\theta}}{R}$ - $q_3$ or some are putting $\frac{N_{\theta\theta}}{R}$ $+q_3$ . Because of that, there is a slight change in the surface normal. From here, $N_{\theta\theta}$ can be found, it is varying, circumferential stress resultant $N_{\theta\theta}$ will be:

$N_{\theta\theta} = p_0 R - \gamma R^2 \cos\theta$

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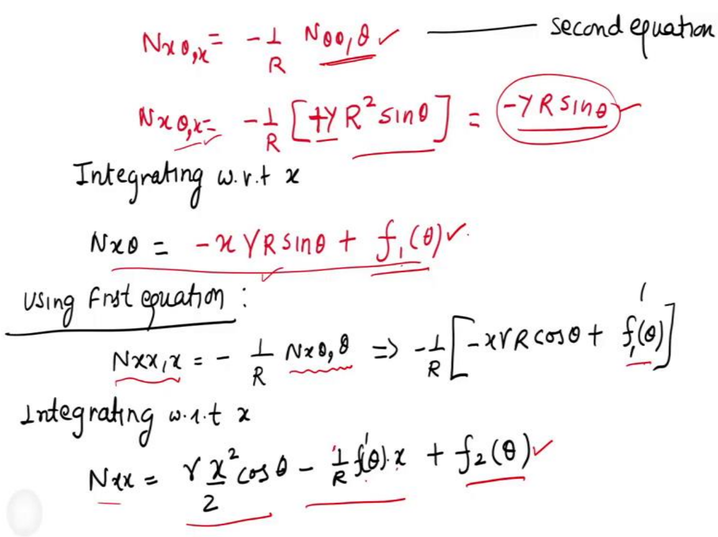


Now from this equation, if we substitute in the 2nd equation:

$$N_{x\theta,\theta} = -\frac{1}{R} N_{\theta\theta,\theta} \Rightarrow -\frac{1}{R} \gamma R^2 \sin\theta$$

Ultimately, it will be: $-\gamma R \sin\theta$.

Now, we have to integrate with respect to $\theta$. If, we integrate with respect to $\theta$, then $-\gamma R \sin\theta$ is acting as a constant and $N_{x\theta}$ will be:

$N_{x\theta} = -x\gamma R \sin\theta + f_1(\theta)$, because it is partial derivative along x.

$f_1$ will be a function in the $\theta$ direction. Now, we can substitute this expression in the 1st equation, here $N_{xx,x}$ will be:

$N_{xx,x} = -\frac{1}{R} N_{x\theta,\theta}$, and if you take the derivative with respect to $\theta$, then it will be:

$$-\frac{1}{R} \left(-x\gamma R \cos\theta + f_1'(\theta)\right).$$

Integrating with respect to x gives you this expression:

$$N_{xx} = \frac{rx^2}{2}\cos\theta \; - \frac{1}{R} f_1'(\theta) x + f_2(\theta),$$

Here, $f_2$ is another constant function of $\theta$, now, we have to find, $f_1$, $f_1'$, and $f_2$.

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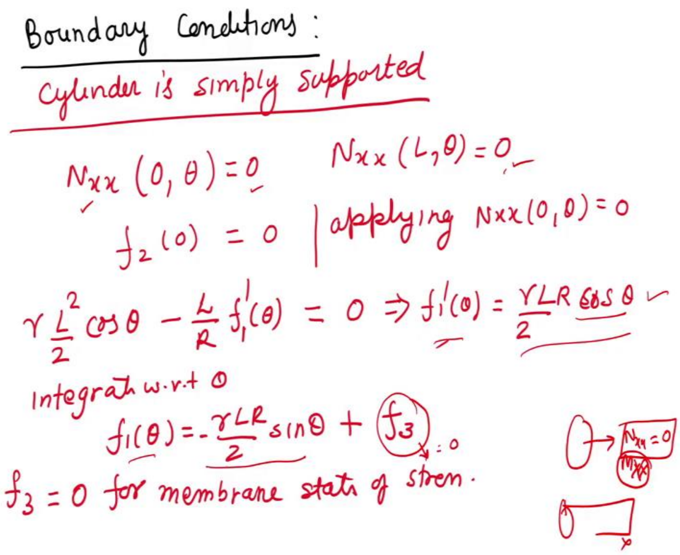


The cylinder is simply supported which means the axial force and axial moment are going to be 0, here, we are talking about the membrane stress the moment is not there. When you say that x = 0 of a shell and x = L, then, $N_{xx}$ = 0.

If we apply this condition,

If we put x = 0, then $\frac{rx^2}{2}\cos\theta$ = 0, $\frac{1}{R} f_1'(\theta)x$ = 0, and $f_2(\theta)$ = 0.

And, if we put that x = L, then $\frac{rL^2}{2}\cos\theta - \frac{L}{R} f_1'(\theta)$ = 0 and $f_1'(\theta) = \frac{rLR}{2}\cos\theta$.

If, we integrate with respect to $\theta$, then,

$f_1(\theta) = -\frac{rLR}{2}\sin\theta + f_3$, integrating constant come, but it is not a membrane state of stress, therefore, this integrating constant $f_3$ = 0.

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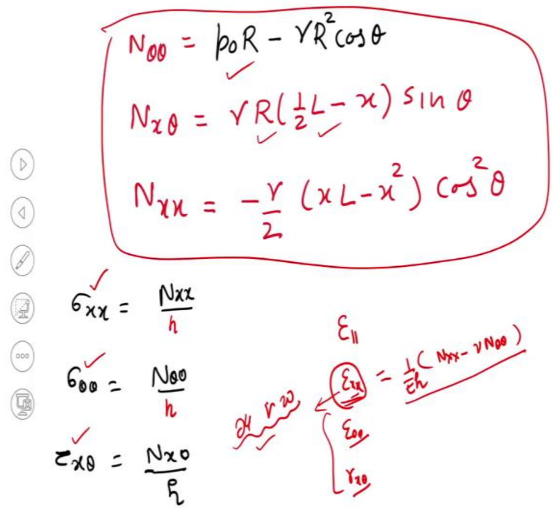

From here we can find all the stress resultant in terms of pressure, radius, and length:

$$N_{\theta\theta} = p_0 R - \gamma R^2 \cos\theta ;$$

$$N_{x\theta} = \gamma R \left( \frac{1}{2} L - x \right) \sin\theta ;$$

$$N_{xx} = -\frac{\gamma}{2} \left( xL - x^2 \right) \cos^2\theta ,$$

They are varying accordingly. Now, we can say that $\sigma_{xx} = \frac{N_{xx}}{h}$, these are the average stresses, $\sigma_{\theta\theta} = \frac{N_{\theta\theta}}{h}$, and $\tau_{x\theta} = \frac{N_{x\theta}}{h}$.

And, after that we have already derived the expression of $\varepsilon_{xx}$, $\varepsilon_{\theta\theta}$, and $\gamma_{x\theta}$, from there we can find the strains in terms of $\varepsilon_{xx}$ :

$$\varepsilon_{xx} = \frac{1}{Eh} \left( N_{xx} - \mu N_{\theta\theta} \right)$$

Then, $\varepsilon_{xx}$, $\varepsilon_{\theta\theta}$, and $\gamma_{x\theta}$ are having terms like, $\delta u$, $\delta v$, or $\delta w$, from these three equations, we can solve displacements by integrating, because they are having some derivatives.

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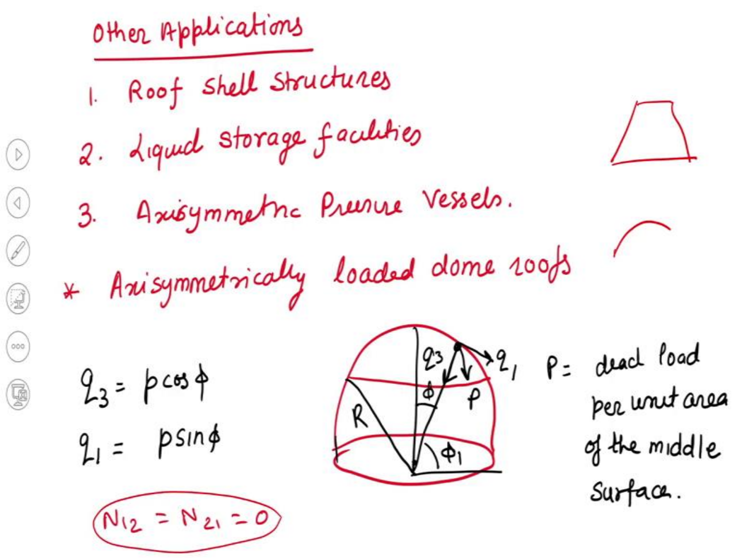


Now, there are some other applications, i.e., roof shell structures, when we design a roof of a building, which is under self-weight, it may be a spherical dome, a cylindrical shell, a frustum kind of thing, a conical shell, or any other kind.

2nd application is liquid storage facilities and the 3rd application is Axisymmetric pressure vessels. When you talk about an axisymmetric loaded dome roof or a spherical dome roof, then the main concept is that first, we have to find the component of loads.

If we correctly define the components of load, then the problem will be easy to solve. If, the loading components are not defined properly then it will be a problem, because, once you know the loading the equations become very simple. And, further for the case of axisymmetric, in-plane component of stress resultant $N_{12} = N_{21} = 0$.

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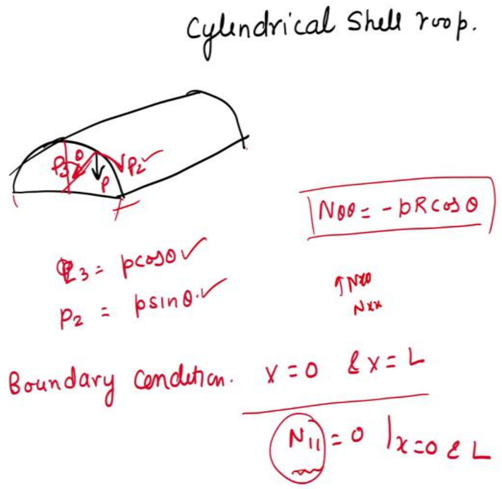


Next in the case of the cylindrical shell, we have to define the components of loading and the boundaries that, maybe simply supported x = 0 and x = L. From here, one can find $N_{\theta\theta}$, $N_{x\theta}$, and $N_{xx}$, the gist of this problem is that in membrane shell theory the loading component should be clearly defined, and then depending upon the case we can solve the problem.

And, the solution of a cylindrical shell is very easy. In that case, first, we solve the 3rd equation and then, 2nd and 1st equations. For the case of cylindrical shell, it is very easy and boundary conditions are satisfied in the terms of stress resultants.

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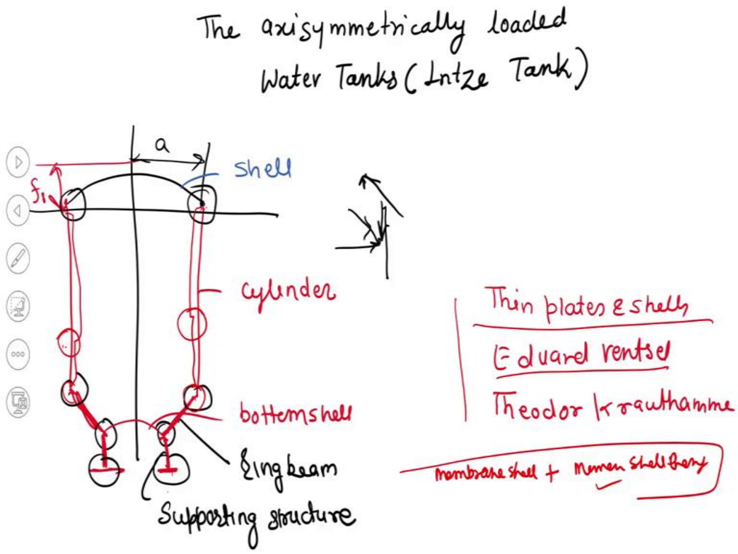


Now, there are cases of some mixed shells, like a cylindrical, spherical, ring, or conical shape.

How do we get the solution in that case? First, we have to find the solution in 3 domains, and then we have to club the solution. But, at the joints, the solutions will not be valid, because there will be a change in curvature.

The stress moments may generate at the curvature change point or the thickness if the thickness of the shell is changing. The membrane shell theory cannot predict the accurate behavior of all these junctions and the supporting structures, where the boundary conditions are expressed in terms of displacement.

These types of designs are given in detail in the book Thin Plates and Shell by Edward Ventsel. If you are interested, can go and see how to solve these kinds of problems. You are aware that the membrane shell theory cannot give the solution here, then the concept of moment shell theory comes, and then, we can get the solution. For a complete structure, we add those which means, there are areas where membrane shell theory is applicable and areas where moment shell theory is applicable.

It is a combination of both. If, you do not consider the membrane shell theory, then, your design is not reliable because, at the joint, it may have a large number of stresses, we need to assess that.

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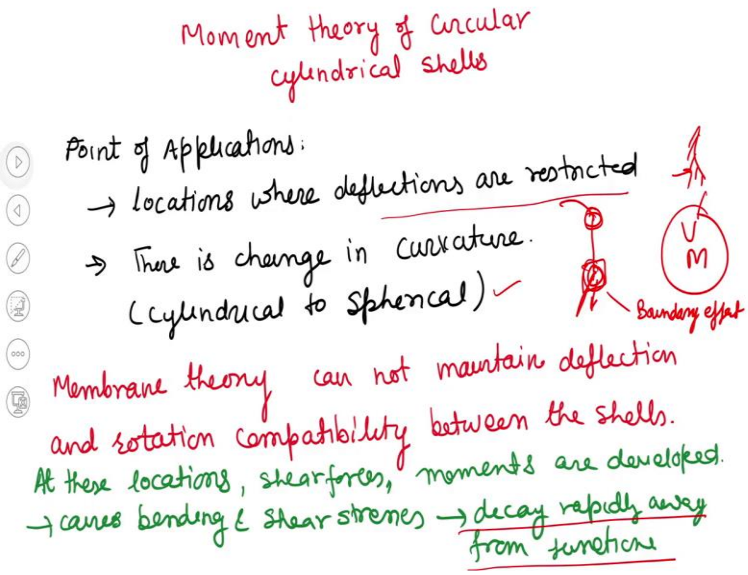


Next, is the moment theory of circular cylindrical shell, we are talking about the only circular cylindrical shell, because the moment theory, in general, is very complex. We cannot get the solution for all the cases; we have an analytical solution for typical cases.

The solution for a circular cylindrical shell is possible in some cases. Most of the time the circular cylindrical shells are used extensively for structural applications and for the locations where deflections are restricted which means the boundary conditions are expressed in terms of deflections.

If, you say that in the case of a simply supported we have satisfied only axial stress we did not satisfy that $w = 0$. If, we have to satisfy the clamped edge, let us say, the cylinder is clamped at some points, then your deflection is 0 and slope is 0.

Because at the clamped edge, there will be shear force and moment both. It will not come into a membrane case; because in the membrane, we do not assume shear force and a moment or the horizontal force.

There is a change in the curvature like, cylindrical to spherical, spherical to conical, or some other kind, membrane theory cannot maintain deflection and rotation compatibility between the shells. At these locations, shear forces and moments are generated and which causes shear stresses and bending.

The very important observation is that at the junctions though the moments, shear forces, bending stresses, or shear stresses are generated, but it decays it vanish very rapidly when we go away from that corner or the joint.

That is why we say that it is a local effect, near the juncture it is high, but as you go away, it vanishes very rapidly sometimes we called it is a boundary effect.

The moment shell theory is used to find the boundary effect at the junctions, because the small thickness or a small bending moment may cause large stresses in the shell.

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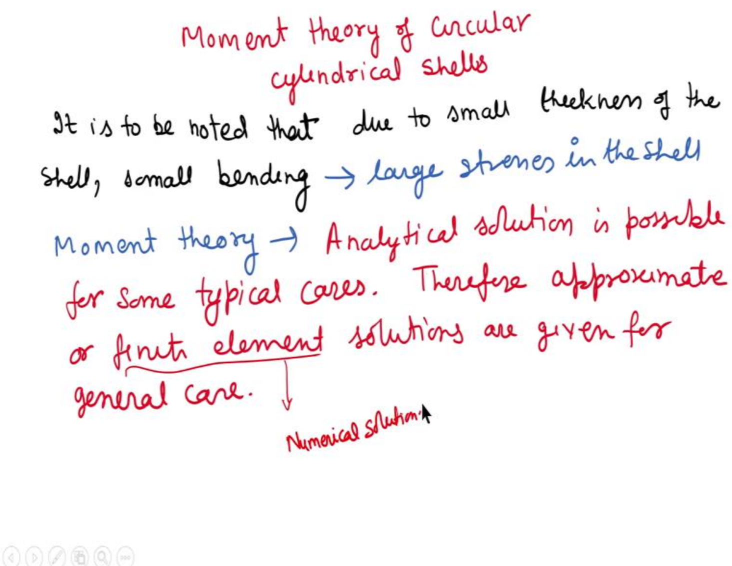


Already, I said that an analytical solution is possible, for some typical cases, therefore, if you are interested to find the solution, then we have to go for an approximate or a numerical solution, these days, we say, finite element solution or numerical solutions.

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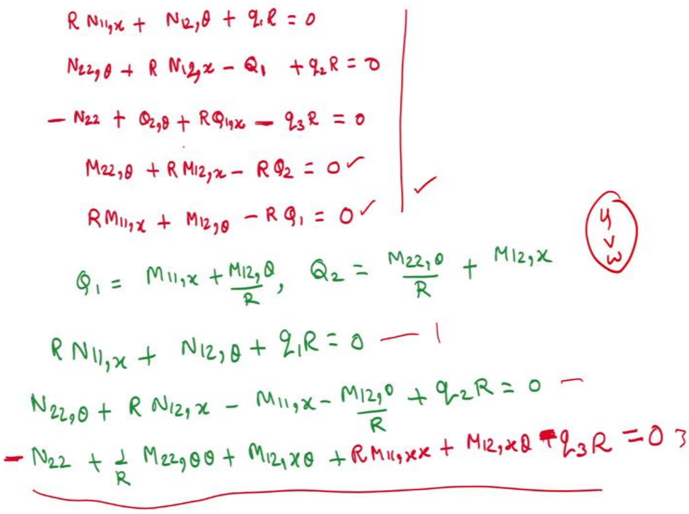

Let us say for the case of moment shell theory, general equations in a case of the circular cylindrical shell can be expressed like this.

$$R.N_{11,x} + N_{12,\theta} + q_1 R = 0 \quad equation\,(1)$$
$$N_{22,\theta} + R.N_{12,x} + Q_1 + q_2 R = 0 \quad equation\,(2)$$
$$M_{22,\theta} + R.M_{12,x} - RQ_2 = 0 \quad equation\,(3)$$
$$R.M_{11,x} + M_{12,\theta} - RQ_1 = 0 \quad equation\,(4)$$
$$-N_{22} + Q_{2,\theta} + RQ_{1,x} - q_3 R = 0 \quad equation\,(5)$$

Ultimately, these 5 equations can be written into 3 equations, by using the concept

$$Q_1 = M_{11,x} + \frac{M_{12,\theta}}{R} \quad \text{and} \quad Q_2 = M_{12,x} + \frac{M_{22,\theta}}{R}$$

And substituting in the 5th equation, this gives you:

$-N_{22} + \frac{1}{R} M_{22,\theta\theta} + M_{12,x\theta} + RM_{11,xx} + M_{12,x\theta} - q_3 R = 0$ , and it becomes the 3rd equation.

Equation (1) and equation (2) remain the same.

We cannot solve in terms of moments; we have to convert these in terms of primary variables. We have to find the value of $N_{11}$, $N_{12}$, $N_{22}$, $M_{11}$, $M_{12}$, $M_{22}$ in terms of u, v, and w, then we can substitute it here. That is why it is difficult, getting all these terms.

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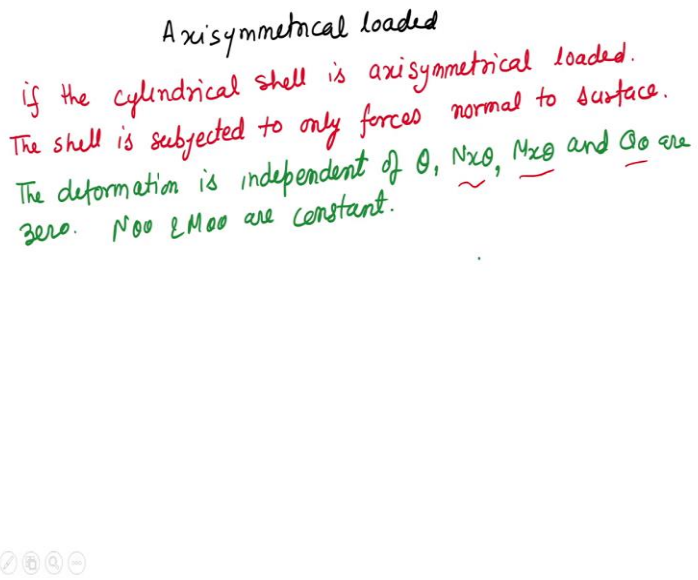


Ultimately, the very simple solution, I am going to solve here is that axis-symmetric bending, for that case, the deformation is independent of $\theta$ , $N_{x\theta}$ , $M_{x\theta}$ , and $Q_\theta$ are considered 0.

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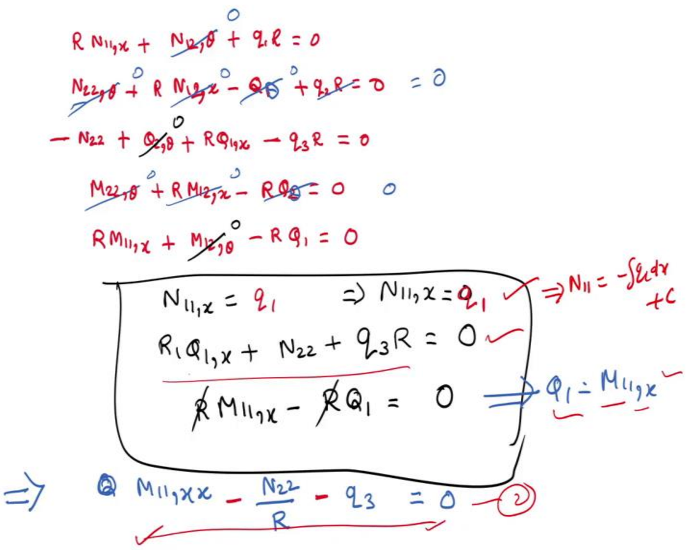

If you take consideration of this, it will give you these three equations.

$$N_{11,x} = q_1 \quad equation\,(1)$$
$$R_1 Q_{1,x} + N_{22} + q_3 R = 0 \quad equation\,(2)$$
$$R.M_{11,x} - RQ_1 = 0 \quad equation\,(3)$$

From the 3rd equation, $Q_1 = M_{11,x}$, and if you substitute it in the equation (2), this gives:

$M_{11,xx} - \dfrac{N_{22}}{R} - q_3 = 0$, 2nd equation modified.

The 1st equation can be directly found that $N_{11} = -\int q_1 dx + C$.

If you substitute all these things, that we can solve from the 2nd equation.

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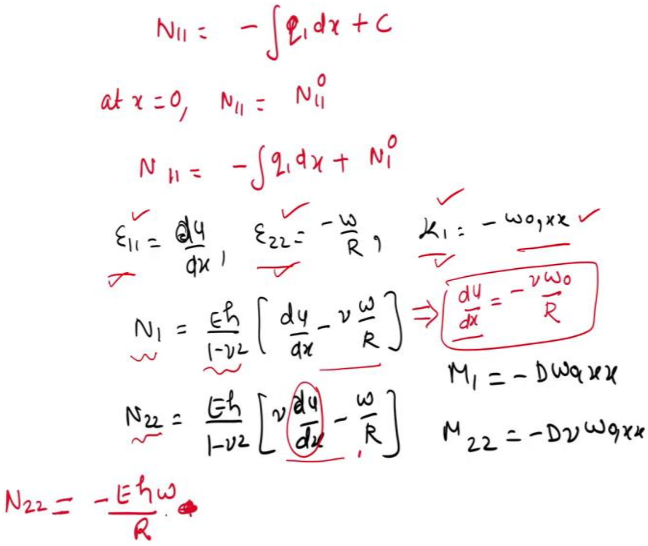


What is the definition of $\varepsilon_{11}$, $\varepsilon_{22}$, and curvature $K_1$ for this case? Derivative with respect to $\theta$ = 0, and v = 0. If we choose all these things, then strain components are expressed like this $\varepsilon_{11} = \dfrac{du}{dx}$; $\varepsilon_{22} = \dfrac{-w}{R}$; $K_1 = -w_{0,xx}$.

Now, $N_{11} = Q_{11}\,\varepsilon_{11} + \mu\, Q_{12}$,

And integrating,

$$N_{11} = \frac{Eh}{1-\mu^2}\left(\frac{du}{dx} - \mu\frac{w}{R}\right).$$

Now, we say that $N_{xx}$ = 0, from here, $\dfrac{du}{dx} = -\mu\dfrac{w_0}{R}$ and

$$N_{22} = \frac{Eh}{1-\mu^2}\left(\mu \frac{du}{dx} - \frac{w}{R}\right),$$

If you substitute this expression here, that gives you the expression of $N_{22} = \frac{-Ehw}{R}$.

Ultimately, we are expressing all the variables in terms of transverse deflection w. The moment $M_{11} = -Dw_{0,xx}$ and $M_{22} = -D\mu w_{0,xx}$.

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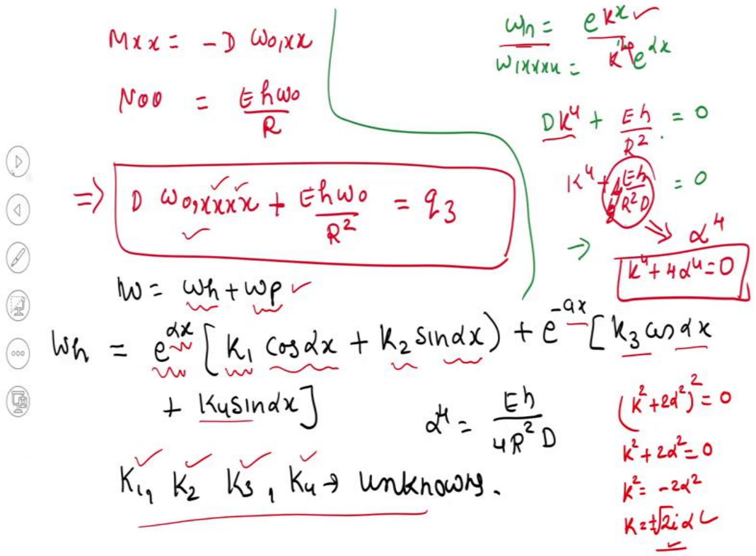


If you substitute in the 2nd equation, it leads to a fourth-order differential equation in x. This equation:

$$Dw_{0,xxxx} + \frac{Ehw_0}{R^2} = q_3 \text{ can be solved}$$

And there will be a homogeneous solution $wh$ and a particular solution $wp$.

We can assume a homogeneous solution $wh = e^{Kx}$. If, you substitute:

$$DK^4 + \frac{Eh}{R^2} = 0.$$

For finding the roots, we multiply with h and divide by h.

We can assume $\frac{Eh}{4R^2D} = \alpha^4$.

Ultimately, the equation becomes $K^4 + 4\alpha^4 = 0$.

We can write the equation like $\left(K^2 + 2\alpha^2\right)^2$ = 0, we can say that, $K^2 + 2\alpha^2$ = 0,

Then $K^2 = 2\alpha^2$, then $K = \pm\sqrt{2i\alpha}$.

In this way, there will be 4 roots, 2 roots will be the same and there are complex roots.

The solution $wh = e^{\alpha x}\left(K_1 \cos\alpha x + K_2 \sin\alpha x\right) + e^{-\alpha x}\left(K_3 \cos\alpha x + K_4 \sin\alpha x\right)$.

$K_1$, $K_2$, $K_3$ and $K_4$ are unknowns, it is a fourth-order equation, so there will be 4 roots. For solving these we need four boundary conditions.

We can solve a maximum of 4 variables and what will be the particular $wp$ solution?

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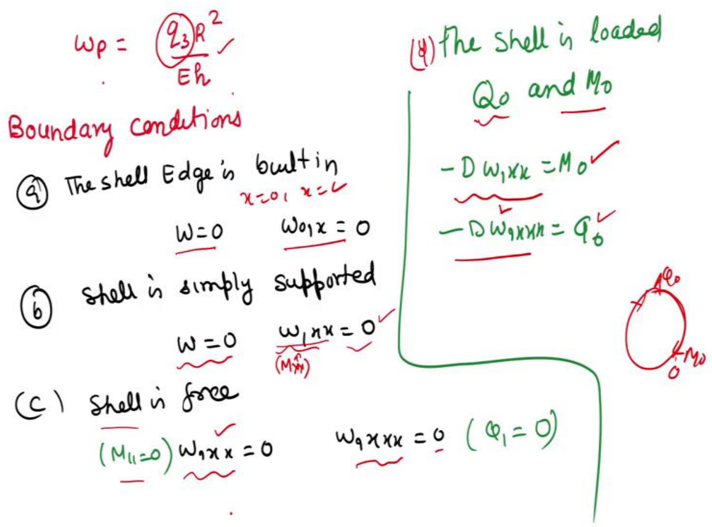


The particular solution can be written as $\frac{q_3R^2}{Eh}$. We have assumed a udl, for that case, $wp = \frac{q_3R^2}{Eh}$. The boundary conditions; if a shell edge is built in means clamped, then we say that w = 0, $w_{0,x}$ = 0, x = 0 and x = L, we will have a fourth boundary condition.

We can solve $K_1$, $K_2$, $K_3$ and $K_4$. Then, for the case of simply supported shell, the deflection is 0 and the moment is 0. So, from there $w_{,xx}$ = 0. Similarly, when the shell is free the moment and shear force are 0.

The expression of the moment $M_{11}$ = 0, $w_{,xx} = 0$, and $w_{,xxx} = 0$. By following this boundary condition, we can solve them. Then, the next is the D case when the shell is loaded with $Q_0$ and $M_0$, because at the edges we can say there is some shear force $Q_0$ applied and the moment $M_0$ is applied. For that case, $M_0 = -Dw_{,xx}$ and shear force $Q_0 = -Dw_{,xxx}$.

In this way, we can solve the problems. I have explained the membrane shell theory, and moment shell theory, the one can do the application part, the details are given in any thin shell theory book. Next, I am going to provide you a general solution to solve a general cylindrical shell for a simply supported shell. In the next lecture, I will explain the basic solution of a shell.

Thank you very much.