Theory of Composite Shells

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Week - 05

Lecture - 01

Development of Navier solution of finite shell

Dear learners, welcome to week-05 lecture-01. In this lecture, I shall explain first, the special case. Already we have developed a generalized governing equation that is valid for all regular shells. In this particular lecture, I shall present the solution for a cylindrical shell under cylindrical bending or sometimes we call it a cylindrical shell panel.

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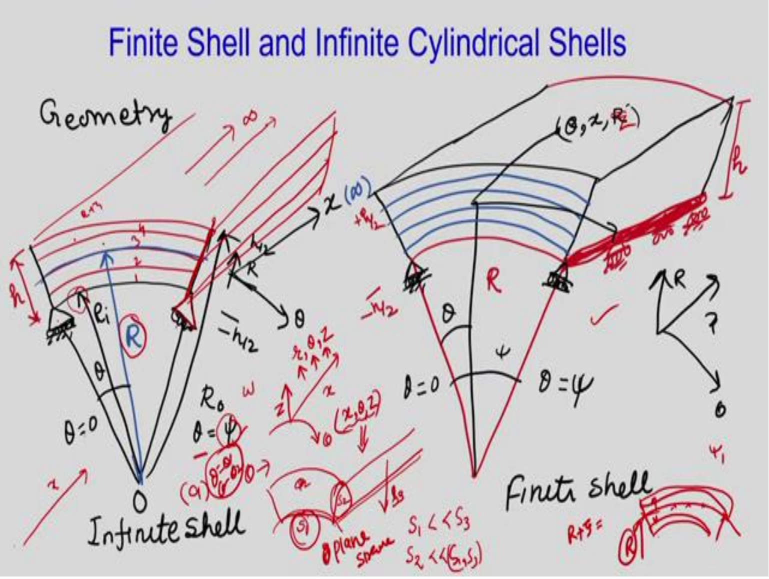

Let us first understand, what is the meaning of a cylindrical shell panel, finite shell panel, and infinite shell panels. When we talk about those shells, if it is a closed shell the solution is different, because we cannot get the boundaries. We have to satisfy in terms of periodic boundary conditions.

But if we have some panel form means all edges where we can provide a boundary condition shell panel.

Let us say, panel ‘a’ has an infinite length along the longitudinal axis, then it is treated as infinite cylindrical shells. The examples may be in some roofs, or the long cylindrical tunnels, and the roof of a cylindrical bridge.

In that way, we can analyze those structures. In aerospace also the wings can be treated as an infinite shell that is very very long.

The theoretical concept is that; if this arc length, let us say S_1 or S_2 is very very small as compared to this length or S_3 . $S_1 \ll S_3$ and $S_2 \ll \left(S_1, S_3\right)$.

Along S_2 there is a thickness, S_1 is along the circumference, and S_3 is along the longitudinal direction. We can treat it as infinite panels, and we call it infinite cylindrical shell panels. They are generalized under plane strain assumption which means the solution is obtained using the plane strain assumptions.

One length is very very large, we can assume that all the entities; displacement, and stresses will be independent of this coordinate. In most of the books which are based on thin cylindrical shells or thick cylindrical shells, the coordinate system is assumed like that along the length it is assumed x, along circumference it is θ, and along the thickness direction, the coordinate is taken as z. It is taken as x, θ, z .

If you want to compare with a standard cylindrical system, which you have seen at the undergraduate level or the post-graduate level the cylindrical system that is r, θ, z; where r is the thickness along the radius, θ is along the circumference, and z is along the longitudinal. Here the system is slightly changed, we have taken x along the longitudinal, and θ along the circumference, and z along the thickness direction.

And the cylindrical shell panel is considered simply supported. Here, the concept of support is that radial deflection means along that direction deflection w is taken as 0. Depending upon this edge or in the two-dimensional theorem at the reference surface.

Ultimately, we develop the governing equation in terms of a reference surface. The reference surface radius is R, we call it is a mean radius.

If we talk about a composite panel or a panel made of sandwich material, they may have a face, core, and face concept. Then, the CG or the geometric center will be taken as the reference plane having radius R.

And above and top can be modeled at R $+\varsigma$. At the center or at any point above can be written as R + ς or R -ς.

The inner radius is titled as R_i of a panel and the outer radius is R_o. θ is the circumference. θ may vary from 0 to ψ at any angle or you can because we have assumed that ψ_1 and ψ_2.

You can take another also that $\theta = \theta_1$ and θ_2 also there is no issue. We are assuming that a cylindrical infinite shell panel having some layers, 1, 2, 3, 4, and so on.

And, its reference surface is having radius R, the inner radius = R_i and the outer radius $= R_o$. And the panel is subjected to simply supported boundary condition at the edges; $\theta = 0$ and $\theta = \psi$. The panel is known as the infinite shell panel.

In the finite shell panel, the length along the x-axis is finite. The entity is the displacement and stresses are taken care of; these all are not the function of x, θ, z. In the case of infinite shell panel all entities are independent of x-direction.

Whenever, a derivative with respect to x come up we are going to put 0 there. Our first step is to express the displacement field, strain field and the governing equation in this nomenclature whatever the coordinate system we assumed in that direction.

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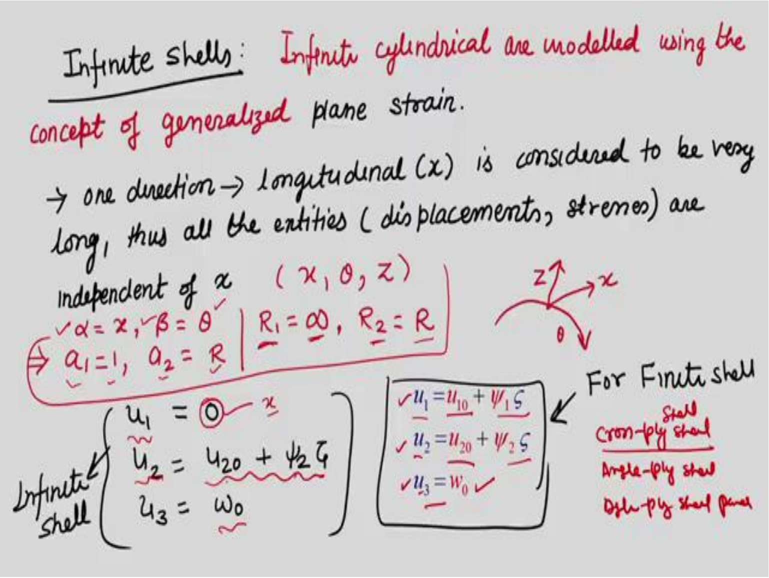

For the case of infinite shell panel, the displacement along x-direction is considered 0. For a cross-ply shell panel u_1 = 0, but for angle ply then u_1 = constant u_0.

In the present lecture, I am going to explain how to get a solution or generalized equation for cross-ply panels. Later on, I will discuss the angle ply shell panels or the finite shell panels.

Let us assume: $u_1 = u_0$, $u_2 = u_{20} + \psi_2\varsigma$, and $u_3 = w_0$.

If we talk about a finite shell panel, then the references will be the same that we have taken in the starting of the development of shell theory:

$u_1 = u_{10} + \psi_1\varsigma$, $u_2 = u_{20} + \psi_2\varsigma$, and $u_3 = w_0$.

For this case it is a cylindrical shell panel, the curvilinear parameters $\alpha = x$ and $\beta = \theta$, lame's parameters $a_1 = 1$ and $a_2 = R$, the radius of curvature $R_1 = \infty$ and $R_2 = R$. These parameters are the same for infinite as well as finite shell panel.

The only difference comes up whenever we are going to have a derivative along x, then we are going to put it 0, then it reduces to an infinite shell panel. If we do not put it 0, then it will give us a solution or the equation for a finite shell panel.

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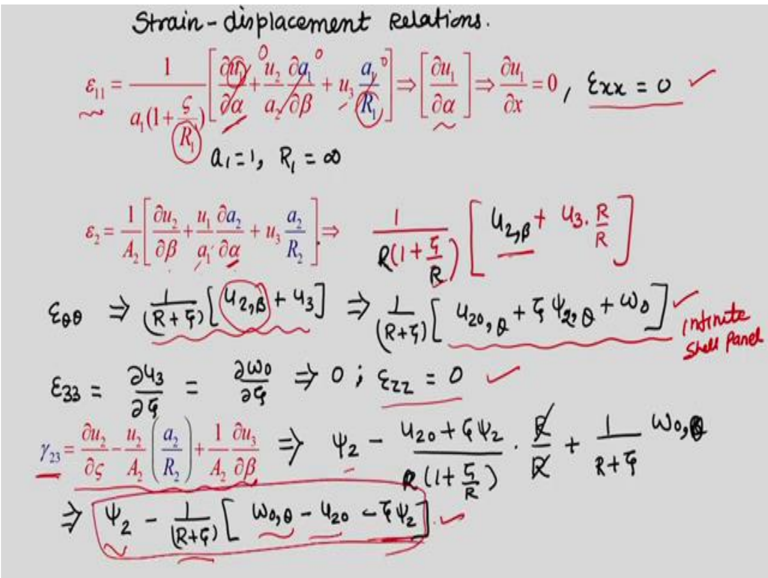

Let us see how the strain displacement relations reduce. In the strain displacement relations $R_1 = \infty$.

$$\varepsilon_{11} = \frac{1}{a_1\left(1+\dfrac{\varsigma}{R_1}\right)}\left(\frac{\partial u_1}{\partial \alpha} + \frac{u_2}{a_2}\frac{\partial a_1}{\partial \beta} + u_3\frac{a_1}{R_1}\right)$$

$$\frac{1}{a_1\left(1+\dfrac{\varsigma}{R_1}\right)} = \frac{1}{a_1}.$$

Here, $\dfrac{\partial u_1}{\partial \alpha}$ with respect to α means with respect to x and u_1 = 0. $\dfrac{\partial u_1}{\partial \alpha}$ will not contribute, then in the term $\dfrac{\partial a_1}{\partial \beta}$, a_1 is constant if you want to take derivative with respect to β, this term $\dfrac{u_2}{a_2}\dfrac{\partial a_1}{\partial \beta}$ will not exist. And in the term $u_3\dfrac{a_1}{R_1}$, $R_1 = \infty$ and $\dfrac{\partial u_1}{\partial x}$ = 0.

In this way, ε_{11} or ε_{xx} = 0.

Now, ε_{22} or $\varepsilon_{\theta\theta}$.

Here, the term $\dfrac{1}{a_2\left(1+\dfrac{\varsigma}{R_2}\right)}$ = $\dfrac{1}{R\left(1+\dfrac{\varsigma}{R}\right)}$ because, a_2 = R and R_2 = R and then $\dfrac{\partial u_2}{\partial \beta}$ =

$u_{2,\beta}$, with respect to x-axis.

$\dfrac{u_1}{a_1}\dfrac{\partial a_2}{\partial \alpha}$ = 0 and $u_3\dfrac{a_2}{R_2}$ = $u_3.\dfrac{R}{R}$

Here, $\dfrac{R}{R}$ gets canceled and it becomes like this: $\varepsilon_{\theta\theta} = \dfrac{1}{(R+\varsigma)}\left(u_{2,\beta} + u_3\right)$.

Ultimately, $\varepsilon_{\theta\theta} = \dfrac{1}{(R+\varsigma)}\left(u_{20,\theta} + \varsigma\psi_{2,\theta} + w_0\right)$

This is strain along θ direction, for infinite shell panel. We are interested to find ε_{33}.

$$\varepsilon_{33} = \frac{\partial u_3}{\partial \varsigma} = \frac{\partial w_0}{\partial \varsigma}$$

∂u_3 = ∂w_0, which is a function of α and β , or you can say the function of x and θ, not a function of thickness direction.

Therefore , $\varepsilon_{zz} = 0$.

Now, $\gamma_{23} = \dfrac{\partial u_2}{\partial \varsigma} - \dfrac{u_2}{A_2}\left(\dfrac{a_2}{R_2}\right) + \dfrac{1}{A_2}\dfrac{\partial u_3}{\partial \beta}$, this is the initial expression.

If you substitute the values, it reduces to:

$$\psi_2 - \frac{u_{20} + \varsigma\psi_2}{R\left(1+\dfrac{\varsigma}{R}\right)}.\frac{R}{R} + \frac{1}{R+\varsigma}w_{0,\theta}.$$

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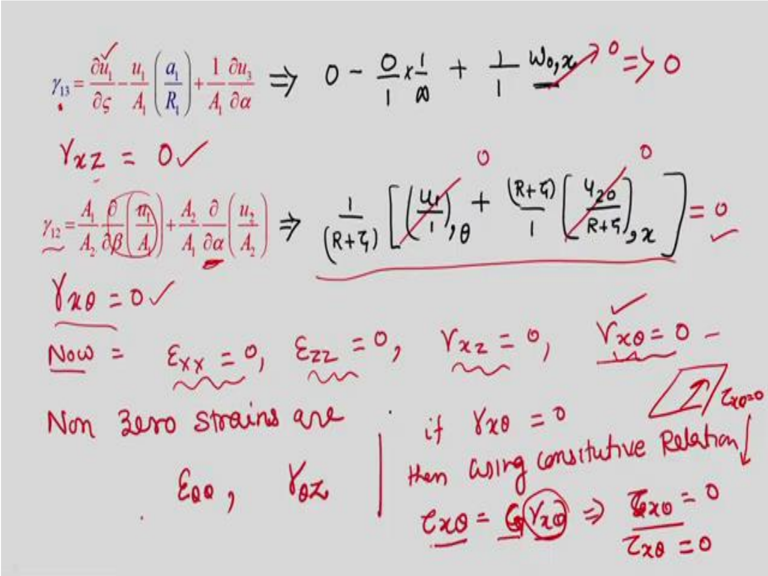

$$\gamma_{13} = \frac{\partial u_1}{\partial \varsigma} - \frac{u_1}{A_1}\left(\frac{a_1}{R_1}\right) + \frac{1}{A_1}\frac{\partial u_3}{\partial \beta}.$$

Here, u_1 = 0, therefore, $\frac{\partial u_1}{\partial \varsigma}$ will not contribute;

$R_1 = \infty$, $\frac{u_1}{A_1}\left(\frac{a_1}{R_1}\right)$ will not contribute.

$\frac{\partial u_3}{\partial \beta} = w_{0,x}$, because it is an infinite panel not a function of x, therefore, it is going to be 0.

Ultimately, γ_{13} or $\gamma_{xz} = 0$.

γ_{12} or $\gamma_{x\theta}$ = 0, because $\gamma_{12} = \frac{A_1}{A_2}\frac{\partial}{\partial \beta}\left(\frac{u_1}{A_1}\right) + \frac{A_2}{A_1}\frac{\partial}{\partial \alpha}\left(\frac{u_2}{A_2}\right)$

If you substitute the values, $\gamma_{12} = \frac{1}{(R+\varsigma)}\left[\left(\frac{u_1}{1}\right)_{,\theta} + \frac{(R+\varsigma)}{1}\left(\frac{u_{20}}{(R+\varsigma)}\right)_{,x}\right] = 0$

From here, we can see that $\varepsilon_{xx} = \varepsilon_{zz} = \gamma_{xz} = \gamma_{x\theta} = 0$.

If $\gamma_{x\theta}$ = 0, and we use the constitutive relations that $\tau_{x\theta} = G\gamma_{\theta x}$, then, $\tau_{x\theta}$ = 0.

In this way, that transfer strain, in-plane shear $\tau_{x\theta}$ is not going to contribute. This is a very important concept, because the strain is 0 and it is a cross-ply, for that case, $\tau_{x\theta}$ = 0, but if it is an angle ply panel, then it may have some value.

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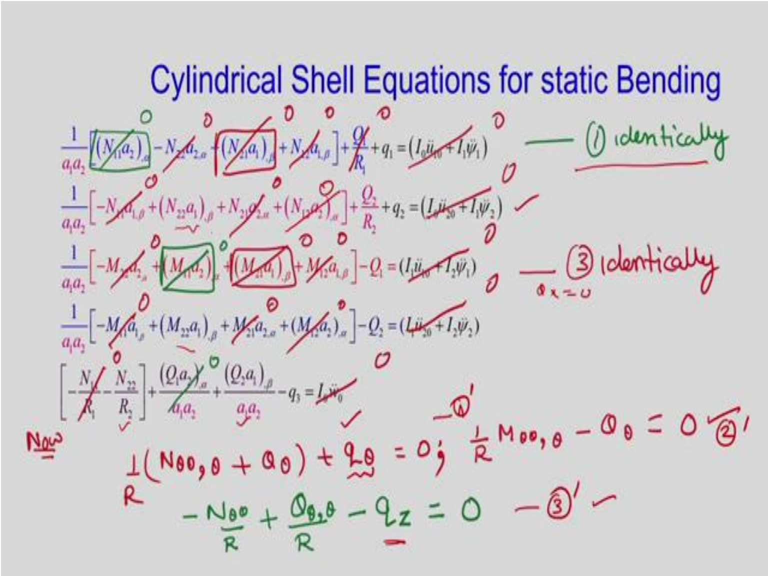

Now, we are going to put it into the main governing equations. I am going to explain with the help of linear governing differential equations:

$$\frac{1}{a_1a_2}\left[\left(N_{11}a_2\right)_{,\alpha}-N_{22}a_{2,\alpha}+\left(N_{21}a_1\right)_{,\beta}+N_{12}a_{1,\beta}\right]+\frac{Q_1}{R_1}+q_1=\left(I_0\ddot{u}_{10}+I_1\ddot{\psi}_1\right) \quad equation(1)$$

$$\frac{1}{a_1a_2}\left[-N_{11}a_{1,\beta}+\left(N_{22}a_1\right)_{,\beta}+N_{21}a_{2,\alpha}+\left(N_{12}a_2\right)_{,\alpha}\right]+\frac{Q_2}{R_2}+q_2=\left(I_0\ddot{u}_{20}+I_1\ddot{\psi}_2\right) \quad equation(2)$$

$$\frac{1}{a_1a_2}\left[-M_{22}a_{2,\alpha}+\left(M_{11}a_2\right)_{,\alpha}+\left(M_{21}a_1\right)_{,\beta}+M_{12}a_{1,\beta}\right]-Q_1=\left(I_1\ddot{u}_{10}+I_2\ddot{\psi}_1\right) \quad equation(3)$$

$$\frac{1}{a_1a_2}\left[-M_{11}a_{1,\beta}+\left(M_{22}a_1\right)_{,\beta}+M_{21}a_{2,\alpha}+\left(M_{12}a_2\right)_{,\alpha}\right]-Q_2=\left(I_1\ddot{u}_{20}+I_2\ddot{\psi}_2\right) \quad equation(4)$$

$$\left(-\frac{N_{11}}{R_1}-\frac{N_{22}}{R_2}\right)+\frac{\left(Q_1a_2\right)_{,\alpha}}{a_1a_2}+\frac{\left(Q_2a_1\right)_{,\beta}}{a_1a_2}-q_3=I_0\ddot{w}_0 \quad equation(5)$$

You can see that $\left(N_{11}a_2\right)_{,\alpha}$, it is derivative with respect to the first coordinates which is why $\left(N_{11}a_2\right)_{,\alpha}=0$

In term $N_{22}a_{2,\alpha}$, a_2 = R, but it is with respect to α, $N_{22}a_{2,\alpha}$ = 0.

$\left(N_{21}a_1\right)_{,\beta}$ = 0 and $N_{12}a_{1,\beta}$ = 0.

$\frac{Q_1}{R_1}$ = 0 because $R_1 = \infty$

The first equation is identically satisfied, it will not contribute. Then come to the second equation; in the second equation, $N_{11}a_{1,\beta}$ and $N_{21}a_{2,\alpha}$ = 0. $\left(N_{12}a_2\right)_{,\alpha}$ = 0.

Initially, I explained the static bending of an infinite shell panel. Dynamic terms will also be going to be 0.

Then, the third equation is identically 0, it will not contribute. In the fourth equation, $\left(M_{22}a_1\right)_{,\beta}$ - Q_2 is going to contribute.

And in the fifth equation, $\frac{N_{22}}{R_2}$ and $\frac{\left(Q_2a_1\right)_{,\beta}}{a_1a_2}$ are going to contribute.

Equation (1') = $\frac{1}{R}\left(N_{\theta\theta,\theta}+Q_\theta\right)$ + q_θ = 0,

Where, q_θ is in-plane loading along θ the direction.

Equation (2') = $\frac{1}{R}M_{\theta\theta,\theta}$ - Q_θ = 0.

And equation (3') = $-\frac{N_{\theta\theta}}{R}$ + $\frac{Q_{\theta,\theta}}{R}$ - q_z = 0.

Ultimately, these 5 equations are reduced to these 3 equations for an infinite shell panel.

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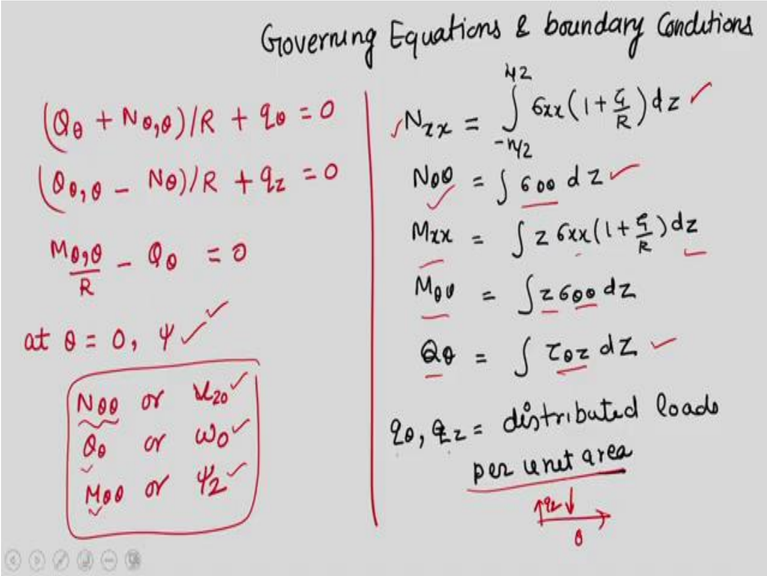

Now, the boundary conditions associated with that. It is an infinite shell panel; therefore, boundary conditions will be along θ = 0 and ψ.

You can prescribe either $N_{\theta\theta}$ or u_{20}, Q_{θ} or w_0, and $M_{\theta\theta}$ or ψ_2. These three variables you can prescribe at one edge and 3 variables at another edge, a maximum of 6 variables you can prescribe. Now, let us see the definition of stress resultant for the present case.

For the present case:

$$N_{xx} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{xx}\left(1+\frac{\varsigma}{R}\right)d\varsigma\,;\;\; N_{\theta\theta} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{\theta\theta} d\varsigma\,;$$

$$M_{xx} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \varsigma\sigma_{xx}\left(1+\frac{\varsigma}{R}\right)d\varsigma\,;\;\; M_{\theta\theta} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \varsigma\sigma_{\theta\theta} d\varsigma\,;$$

And $Q_{\theta} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \tau_{\theta z} d\varsigma$.

Where, q_θ and q_z are distributed load per unit area which we have explained in the previous lectures. q_θ is the in-plane load along θ direction and q_z is the out of the plane load acting in the opposite direction of a surface normal.

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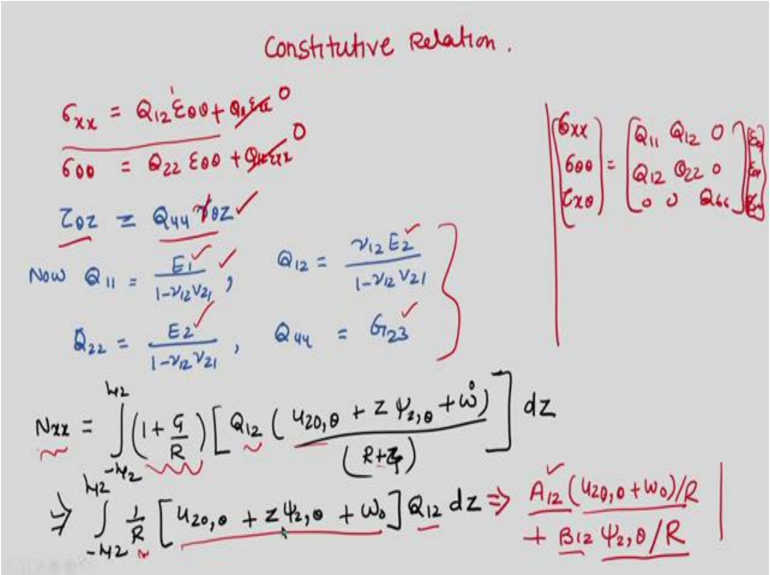

Now, using the constitutive relations, before going to there, we cannot directly get the solution just using the following three equations for the present case:

$$\frac{\left(Q_\theta + N_{\theta,\theta}\right)}{R} + q_\theta = 0 \quad \text{equation (1)}$$

$$\frac{\left(Q_{\theta,\theta} + N_\theta\right)}{R} + q_z = 0 \quad \text{equation (2)}$$

$$\frac{M_{\theta,\theta}}{R} - Q_\theta = 0 \quad \text{equation (3)}$$

We have to convert it into the displacement form.

Already in week -04, I explained that by using the cylindrical shell under axis-symmetric loading, where loading is independent of θ or that cylindrical shell under internal pressure where we have used these 3 equations and we can find the solution.

However, it is not always possible. When we talk about shell panels then we have to work out properly. First, we have to convert this equation into a primary displacement form, then we proceed further.

$\sigma_{xx} = Q_{12}\varepsilon_{\theta\theta} + Q_{11}\varepsilon_{xx}$, because $\varepsilon_{xx} = 0$, therefore, $Q_{11}\varepsilon_{xx}$ will not contribute and ultimately, $\sigma_{xx} = Q_{12}\varepsilon_{\theta\theta}$.

$\sigma_{\theta\theta} = Q_{22}\varepsilon_{\theta\theta}$, and

$\tau_{\theta z} = Q_{44}\gamma_{\theta z}$.

Where, $Q_{11} = \dfrac{E_1}{1-\mu_{21}\mu_{12}}$; $Q_{12} = \dfrac{\mu_{12}E_2}{1-\mu_{12}\mu_{21}}$; $Q_{22} = \dfrac{E_2}{1-\mu_{12}\mu_{21}}$; and $Q_{44} = G_{23}$.

I have given this for a ready reference, because I have explained this in the very first lecture in the first week.

For a composite, we must have the value of E_1, E_2, and all these things. Now, we can evaluate the N_{xx} and $N_{\theta\theta}$. If we substitute it here:

$$N_{xx} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \left(1+\frac{\varsigma}{R}\right)\left[Q_{12}\frac{\left(u_{20,\theta}+\varsigma\psi_{2,\theta}+w_0\right)}{\left(R+\varsigma\right)}\right]d\varsigma \ ,$$

Substituted the value of $\varepsilon_{\theta\theta}$ and Q_{12}.

From here $\left(R+\varsigma\right)$ is going to be cancelled and

$$N_{xx} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \frac{1}{R}\left(u_{20,\theta}+\varsigma\psi_{2,\theta}+w_0\right)Q_{12}d\varsigma \ .$$

Ultimately, $N_{xx} = A_{12}\dfrac{\left(u_{20,\theta}+w_0\right)}{R} + B_{12}\dfrac{\psi_{2,\theta}}{R}$.

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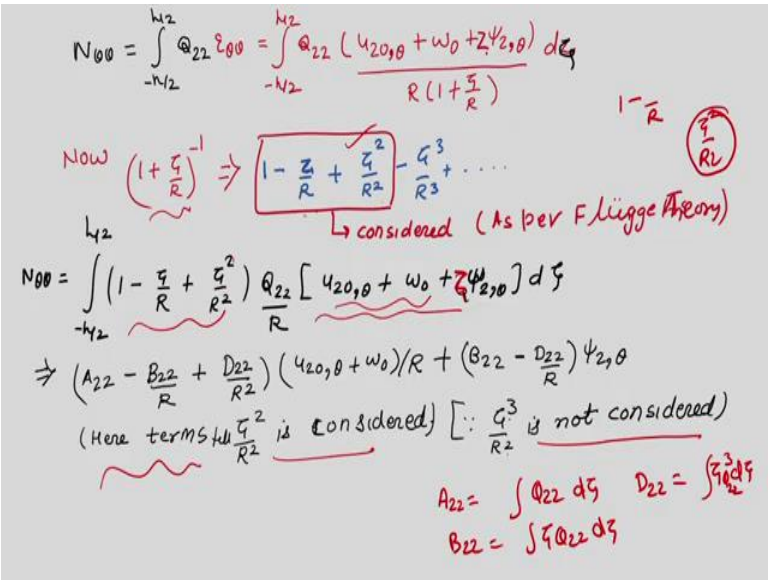

Now, the definition of $N_{\theta\theta} = \int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{22}\varepsilon_{\theta\theta}$.

If we substitute, $N_{\theta\theta} = \int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{22} \frac{\left(u_{20,\theta} + \varsigma\psi_{2,\theta} + w_0\right)}{R\left(1+\frac{\varsigma}{R}\right)} d\varsigma$. $\left(1+\frac{\varsigma}{R}\right)^{-1}$ comes into the picture.

Here, the very important concept is to express this, it leads to a binomial infinite expansion.

As per the flugge assumptions, $\left(1+\frac{\varsigma}{R}\right)^{-1} = 1 - \frac{\varsigma}{R} + \frac{\varsigma^2}{R^2}$.

If we take up to the quadratic $\frac{\varsigma^2}{R^2}$, then it will be a flugge shell theory. In some theories it is considered, $1 - \frac{\varsigma}{R}$, if we say that the shell is very very thin then we consider it 1. But, here, we are saying that the shell is not very thin, it is having some thickness, it can resist the transverse shear.

Therefore, we are taking terms upto $\frac{\varsigma^2}{R^2}$. Even during the integration whenever a term like this comes, we do integration up to $\frac{\varsigma^2}{R^2}$.

Therefore, $N_{\theta\theta} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \left(1 - \frac{\varsigma}{R} + \frac{\varsigma^2}{R^2}\right) \frac{Q_{22}}{R} \left(u_{20,\theta} + w_0 + \varsigma\psi_{2,\theta}\right) d\varsigma$. Because of the term $\varsigma\psi_{2,\theta}$, ς^3 will not be considered.

$N_{\theta\theta} = A_{22} - \frac{B_{22}}{R} + \frac{D_{22}}{R^2} \frac{\left(u_{20,\theta} + w_0\right)}{R} + \left(B_{22} - \frac{D_{22}}{R}\right)\psi_{2,\theta}$.

Ultimately, $A_{22} = \int Q_{22} d\varsigma$; $B_{22} = \int \varsigma Q_{22} d\varsigma$, and $D_{22} = \int \varsigma^3 Q_{22} d\varsigma$.

Already in this slide, I have written that $\frac{\varsigma^2}{R^2}$ is considered, if you multiply with this, a third term comes which is $\frac{\varsigma^3}{R^2}$. Ultimately, we have to restrict somewhere we cannot take all the infinite terms.

The shell is thin radius is very high, these terms are very very small, if we even take up to the second-order, it gives accurate results. If somebody wants to consider further, they can proceed, there is no issue, it is depending upon our requirement.

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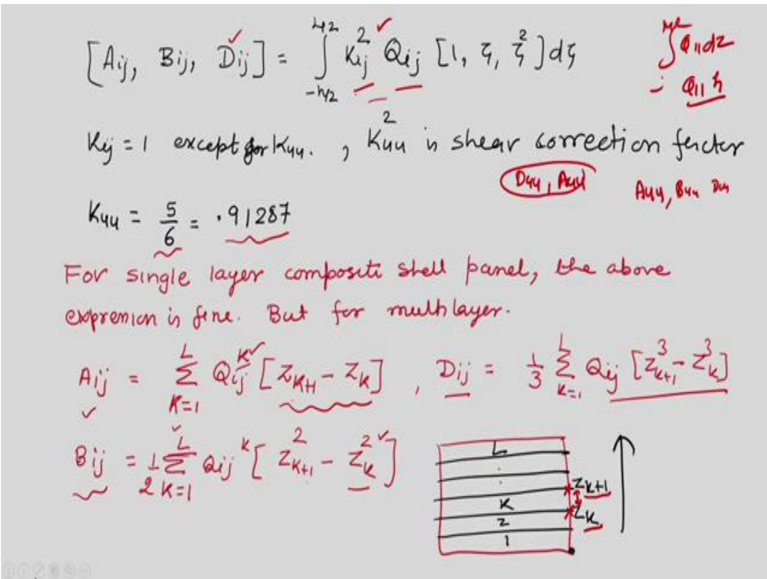

Now, the definition of $\left[A_{ij},\, B_{ij}\,, D_{ij}\right] = \int_{\frac{-h}{2}}^{\frac{h}{2}} K_{ij}^{2} Q_{ij}\left(1,\, \varsigma,\, \varsigma^{2}\right) d\varsigma$.

Already, I have told you that Q_{ij} gives you the definition of A_{ij}, if you multiply with ς, then it gives the definition of B_{ij}, and if you multiply it with ς^{2}, then it will give you the definition of D_{ij} .

When we have a concept of D_{44} or A_{44}, then A in the case of shear we use a shear correction factor, because it is a first-order shear deformation theory.

Transverse shear stresses are not accurate. For getting an accurate result, we multiply with $\frac{5}{6}$ or some factor 0.9128, in the definition itself. Whenever a term A_{44}, B_{44}, and D_{44} is calculated in that Q_{ij} is multiplied with the 0.9128, otherwise, there will be no effect it is taken as 1.

If it is a single layer composite shell, sometimes we analyze a single layer composite an orthotropic material shell, then, this integration works fine. You will take $\int Q_{11} d\varsigma$, but if it is a composite shell then we cannot take, because it is having a number of layers.

Then instead of integration, we have summation: $A_{ij} = \sum_{K=1}^{L} Q_{ij}^{K}\left(Z_{K+1} - Z_{K}\right)$.

Let us say, I have L layers starting from 1, 2, 3, and so on and this is the Kth layer.

The bottom coordinate is denoted as Z_{K} and the top coordinate of this Kth layer is denoted as Z_{K+1}. This is the difference in that it tells you the thickness of that layer.

When we are going for a programming or a composite shell, then similarly definition of $B_{ij} = \frac{1}{2}\sum_{K=1}^{L} Q_{ij}^{K}\left(Z_{K+1}^{2} - Z_{K}^{2}\right)$. And $D_{ij} = \frac{1}{3}\sum_{K=1}^{L} Q_{ij}^{K}\left(Z_{K+1}^{3} - Z_{K}^{3}\right)$.

In this way, the concept of the composite shell is integrated in this direction that you have to integrate in this sense. It is already explained within the plate analysis also though it is the same.

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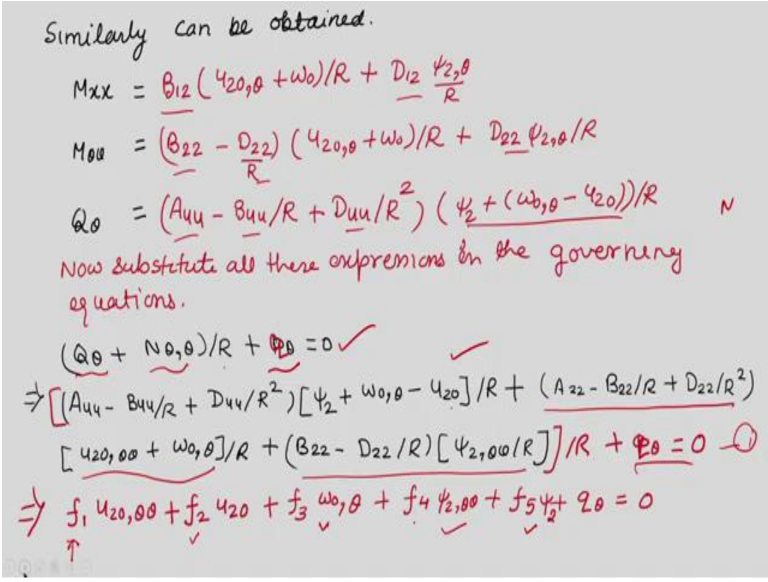

Similarly, we can express:

$$M_{xx} = B_{12}\frac{\left(u_{20,\theta} + w_0\right)}{R} + D_{12}\frac{\psi_{2,\theta}}{R};$$

$$M_{\theta\theta} = \left(B_{22} - \frac{D_{22}}{R}\right)\frac{\left(u_{20,\theta} + w_0\right)}{R} + D_{22}\frac{\psi_{2,\theta}}{R} \text{ and}$$

$$Q_\theta = \left(A_{44} - \frac{B_{44}}{R} + \frac{D_{44}}{R^2}\right)\frac{\left(\psi_2 + w_{0,\theta} - u_{20}\right)}{R}.$$

Now, we have to substitute the expression of $N_{\theta\theta}$, $M_{\theta\theta}$, and Q_θ and their derivatives.

In the very first equation, the definition of $\frac{\left(Q_\theta + N_{\theta,\theta}\right)}{R}$ + q_θ = 0.

If we substitute this will become:

$$\frac{\left[\left(A_{44} - \frac{B_{44}}{R} + \frac{D_{44}}{R^2}\right)\frac{\left(\psi_2 + w_{0,\theta} - u_{20}\right)}{R} + \left(A_{22} - \frac{B_{22}}{R} + \frac{D_{22}}{R^2}\right)\frac{\left(u_{20,\theta\theta} + w_{0,\theta}\right)}{R} + \left(B_{22} - \frac{D_{22}}{R}\right)\left(\frac{\psi_{2,\theta\theta}}{R}\right)\right]}{R} + q_\theta = 0$$

In this equation, we are going to arrange it in such a way that the coefficient of $u_{20,\theta\theta}$ = f_1, coefficient of then u_{20} = f_2, the coefficient of $w_{0,\theta}$ = f_3, the coefficient of $\psi_{2,\theta\theta}$

= f_4, and the coefficient of ψ_2 = f_5. It is written in a very nice form:

$$f_1 u_{20,\theta\theta} + f_2 u_{20} + f_3 w_{0,\theta} + f_4 \psi_{2,\theta\theta} + f_5 \psi_2 + q_\theta = 0 \text{ equation (1)}$$

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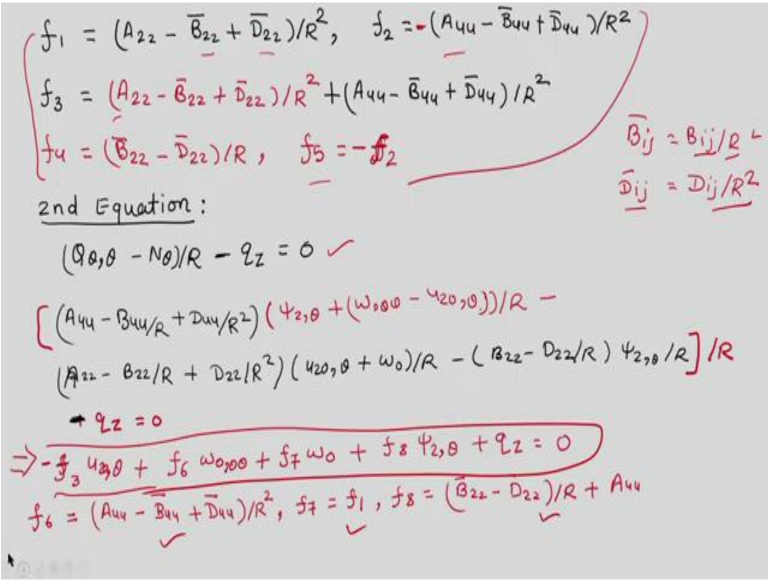

Here $f_1 = \dfrac{\left(A_{22} - \bar{B}_{22} + \bar{D}_{22}\right)}{R^2}$, where $\bar{B}_{22}$ = $\dfrac{B_{ij}}{R^2}$ and $\bar{D}_{22}$ = $\dfrac{D_{ij}}{R^2}$.

Instead of a writing R^2 again and again, to reduce that we have assumed another coefficient $\bar{B}_{22}$ and $\bar{D}_{22}$.

$$f_2 = \frac{\left(A_{44} - \bar{B}_{44} + \bar{D}_{44}\right)}{R^2};$$

$$f_3 = \frac{\left(A_{22} - \bar{B}_{22} + \bar{D}_{22}\right)}{R^2} + \frac{\left(A_{44} - \bar{B}_{44} + \bar{D}_{44}\right)}{R^2};$$

$$f_4 = \frac{\left(\bar{B}_{22} + \bar{D}_{22}\right)}{R^2}; \text{ and } f_5 = -\ f_2$$

In the second equation, $\dfrac{\left(Q_{\theta,\theta} + N_\theta\right)}{R}$ + q_z = 0. If we substitute all these things, the equation becomes like this:

$$\frac{\left[\left(A_{44}-\frac{B_{44}}{R}+\frac{D_{44}}{R^2}\right)\frac{\left(\psi_{2,\theta}+w_{0,\theta\theta}-u_{20,\theta}\right)}{R}-\left(A_{22}-\frac{B_{22}}{R}+\frac{D_{22}}{R^2}\right)\frac{\left(u_{20,\theta}+w_0\right)}{R}+\left(B_{22}-\frac{D_{22}}{R}\right)\left(\frac{\psi_{2,\theta}}{R}\right)\right]}{R}-q_z=0$$

where coefficient f_3, f_6, f_7, and f_8 are written.

$-f_3 u_{20,\theta}+f_6 w_{0,\theta\theta}+f_7 w_0+f_8 \psi_{2,\theta}+q_z=0$ equation (2)

Where, $f_6 = \frac{\left(A_{44}-\bar{B}_{44}+\bar{D}_{44}\right)}{R^2}$; $f_7 = f_1$; and $f_8 = \frac{\left(\bar{B}_{22}-\bar{D}_{22}\right)}{R}$.

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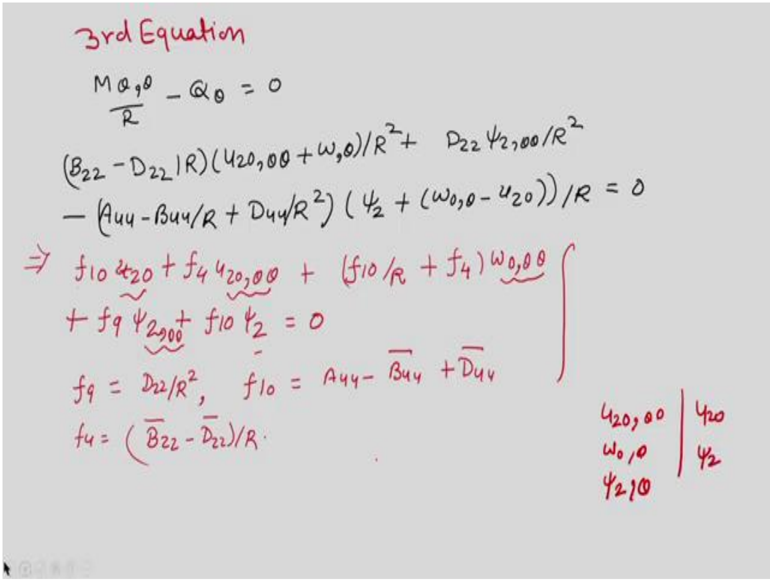

The third equation, $\frac{M_{\theta,\theta}}{R}$ - Q_θ = 0. This gives you:

$$\frac{\frac{\left(B_{22}-D_{22}\right)}{R}\frac{\left(u_{20,\theta\theta}+w_{0,\theta}\right)}{R^2}+\frac{D_{22}\psi_{2,\theta\theta}}{R^2}-\left(A_{44}-\frac{B_{44}}{R}+\frac{D_{44}}{R^2}\right)\left(\psi_2+w_{0,\theta}-u_{20,}\right)}{R}=0$$

$f_{10}u_{20}+f_4 u_{20,\theta\theta}+\left(\frac{f_{10}}{R}+f_4\right)w_{0,\theta\theta}+f_9\psi_{2,\theta\theta}+f_{10}\psi_2=0$ equation (3)

$f_9=\frac{D_{22}}{R^2}$; $f_{10}=A_{44}-\bar{B}_{44}+\bar{D}_{44}$; $f_4=\frac{\left(B_{22}-D_{22}\right)}{R}$

We have represented all 3 equations in terms of primary displacement.

What are the primary displacements? $u_{20,}$ w_0, and ψ_2 are the primary displacements. If you see all are expressed in terms of that, they may have some derivatives, double derivatives, or no derivatives.

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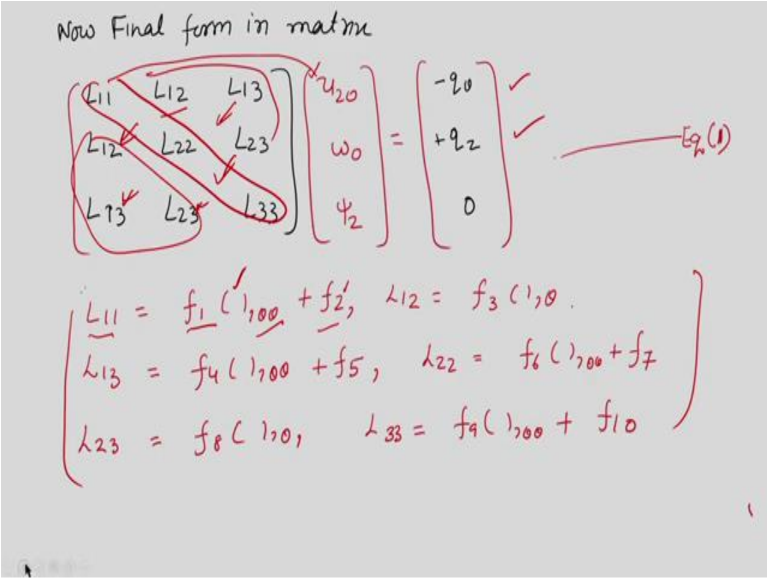

Can we express it in a more suitable form? let us say, I can write in the matrix form:

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{12} & L_{22} & L_{23} \\ L_{13} & L_{23} & L_{33} \end{bmatrix} \begin{bmatrix} u_{20} \\ w_0 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} -q_\theta \\ +q_z \\ 0 \end{bmatrix} \text{ equation (1)}$$

where is the $L_{11} = f_1(\ ),_{\theta\theta} + f_2$. L_{11} is multiplied with u_{20} .

It tells you that $L_{11} = f_1 u_{20,\theta\theta} + f_2$;

$L_{12} = f_3(\ ),_\theta$ or $f_3 w_{0,\theta}$;

$L_{13} = f_4(\ ),_{\theta\theta} + f_5$ *or* $f_4 \psi_{2,\theta\theta} + f_5$;

L_{22} is $f_6(\ ),_{\theta\theta} + f_7$; L_{23} is $f_8(\ ),_\theta$;

L_{33} is $f_9(\ ),_{\theta\theta} + f_{10}$;

And you can see that $L_{12} = L_{13}$.

In a structure, whether you analyze a plate problem or a shell problem, whatever the matrix, you will be able to find the symmetry, it is a symmetric matrix. This is diagonal and the upper half is symmetric to the lower half so, most of the structural problems lead to a symmetric matrix. And then we can think of the solution.

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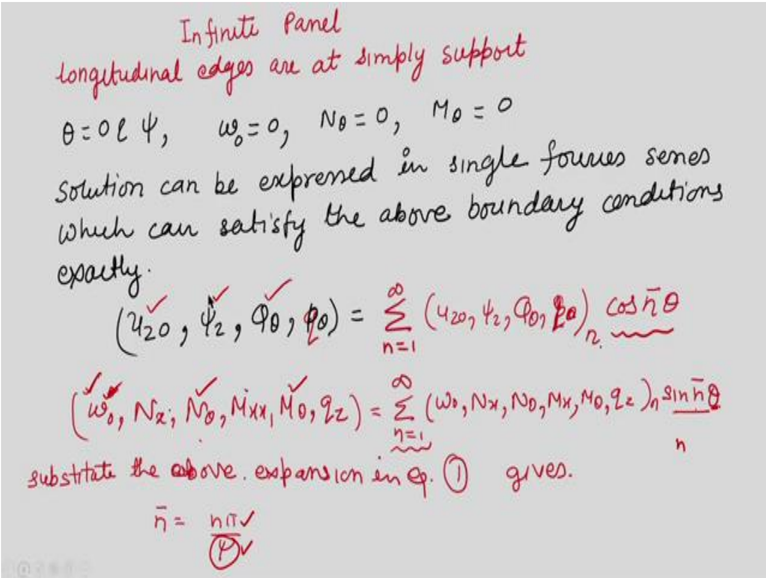

Since it is an infinite panel, longitudinal edges are considered as simply supported, if it is simply supported then we can say that w_0 = 0, normal stress resultant N_θ = 0 and moment M_θ = 0. If the condition is this:

$$\left(u_{20},\psi_2,Q_\theta,q_\theta\right) = \sum_{n=1}^{\infty}\left(u_{20},\psi_2,Q_\theta,q_\theta\right)_n \cos\bar{n}\theta$$

$$\left(w_0,N_\theta,N_x,M_{xx},M_\theta,q_z\right) = \sum_{n=1}^{\infty}\left(w_0,N_\theta,N_x,M_{xx},M_\theta,q_z\right)_n \sin\bar{n}\theta$$

It means we can assume w_0 in the form of a single sin series, where n is going from n to ∞ , where $\bar{n} = \dfrac{n\pi}{\psi}$

If this ψ is the variable, when it is 0, sin θ = 0; when $\psi = \pi$, then $\sin\bar{n}\theta$ = 0, $\theta = \psi$.

We can say that $\dfrac{n\pi}{\psi}$ = 0. Therefore, w_0 is exactly satisfying the boundary condition.

Similarly, the N_θ and M_θ are need to be satisfied. These three variables are to be found. How do you find the value of the expression u_{20} and ψ_2. So, using the concept of strain from there one can find what will be the expression, whether it will come cos or sin so, using those expressions, we can find other variables Q_θ and ψ_2.

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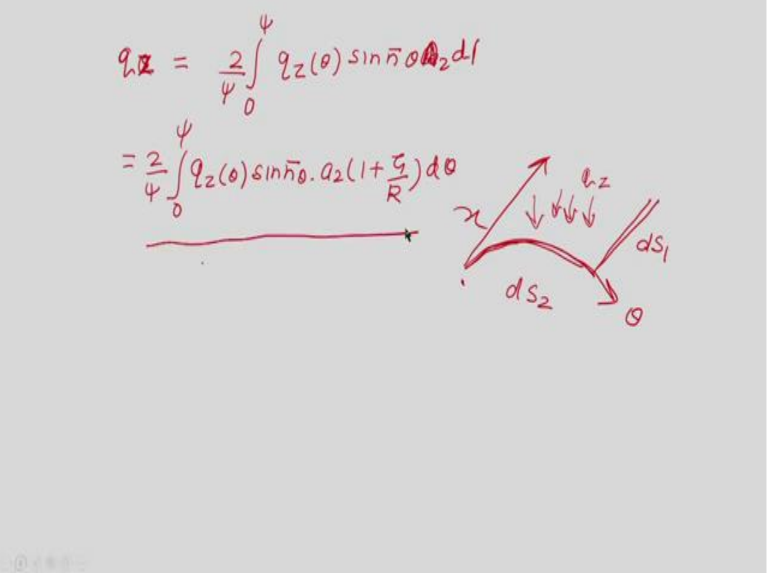

The loading can be expressed in terms of sin or a UDL loading:

$$q_z = \frac{2}{\psi}\int_0^{\psi} q_z(\theta)\sin\bar{n}\theta A_2 d\theta$$

The general Fourier coefficient is expressed like this:

$$\frac{2}{\psi}\int_0^{\psi} q_z(\theta)\sin\bar{n}\theta a_2\left(1+\frac{\varsigma}{R}\right)d\theta.$$

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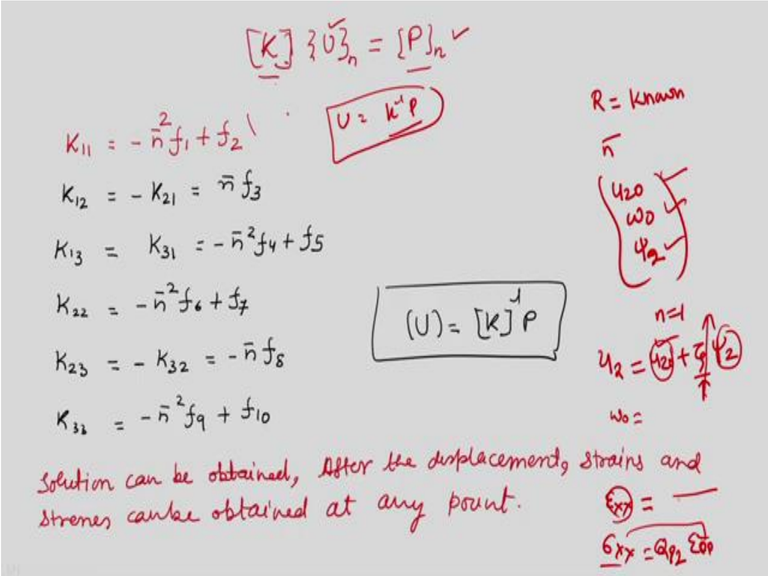

Now, substituting this expression of $\sin n \cos\theta$ into this equation (1), gives you another equation which is: $[K]\{U\}_n = [P]_n$, where K is now having no derivatives.

$K_{11} = -\bar{n}^2 f_1 + f_2$;

$K_{12} = -K_{12} = \bar{n} f_3$;

$K_{13} = K_{31} = -\bar{n}^2 f_4 + f_5$;

$K_{22} = -\bar{n}^2 f_6 + f_7$;

$K_{23} = -K_{32} = \bar{n} f_8$; and

$K_{33} = -\bar{n}^2 f_9 + f_{10}$.

For our case, the mean radius is known to you, $\bar{n}$ for each coefficient give you some value u_{20}, w_0, and ψ_2. For each n value, it is valid we can find n = 1. It depends upon the loading also, if loading is sin loading then the first term is sufficient enough.

Because, the other terms do not contribute if loading is UDL then odd terms you need to consider 1, 3, 5, 7, and so on, depending upon the type of loading. Mostly in literature, for explaining point of view, sinusoidal loading is considered; for the case of sinusoidal loading, the only first term is required. If you put n = 1, then, you can evaluate this matrix K.

It is a standard that you can do in MATLAB also that $U = [K]^{-1} P$. Once you know the variable u_{20}, w_0, and ψ_2, then you can find $u_2 = u_{20} + \varsigma\psi_2$. We are going to vary the z coordinates, then you can get the variation, because u_{20} and ψ_2 is a reference value and with the help of this, you can find the variation of u_2 along the thickness.

Similarly, the variation of w_0 is constant and then you can find the strains σ_{xx}, and ultimately you can find the stresses. You can find the strains using these things and then multiply with the constitutive relations that $\sigma_{xx} = Q_{12}\varepsilon_{\theta\theta}$, stresses will be calculated at each point along the thickness.

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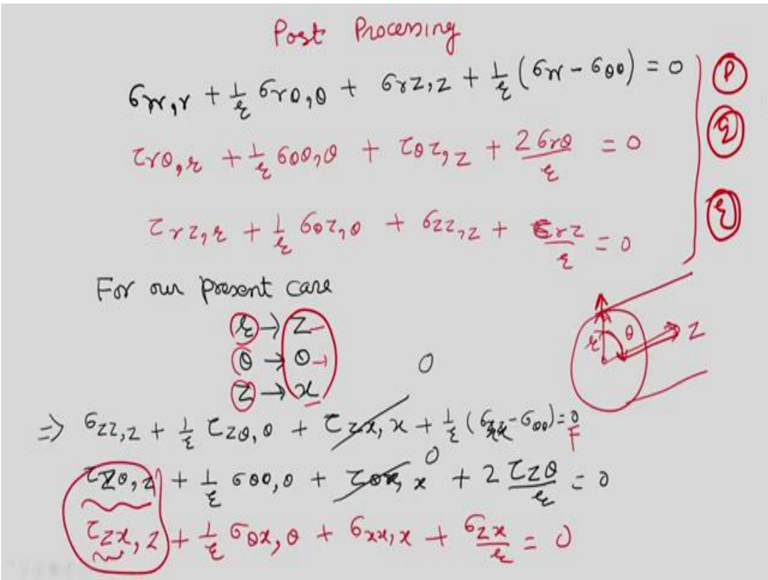

There is another concept, the concept of post-processing, whatever the stresses $\tau_{z\theta}$ or τ_{zx}, these are the transfer shear stresses. They may not be accurate; we usually do the post-processing. We use a 3-dimensional equation of motion in cylindrical coordinate.

First, I will explain the following 3 equations:

$$\sigma_{rr,r} + \frac{1}{r}\sigma_{r\theta,\theta} + \sigma_{rz,z} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0 \text{ equation (p)}$$

$$\tau_{r\theta,r} + \frac{1}{r}\sigma_{\theta\theta,\theta} + \tau_{\theta z,z} + \frac{2\sigma_{r\theta}}{r} = 0 \text{ equation (q)}$$

$$\tau_{rz,r} + \frac{1}{r}\sigma_{\theta z,\theta} + \sigma_{zz,z} + \frac{\sigma_{rz}}{r} = 0 \text{ equation (r)}$$

We are going to have these 3 equations and these are the standard equations given in any undergraduate or a post-graduate book in the cylindrical coordinate system. In this system r is considered along the radial thickness direction, θ is along the circumferential direction, and z is along the longitudinal direction.

But, for our present case, we say that r becomes z and z becomes x but θ remains the same. Therefore, we have to convert the equations p, q, and r into the coordinate system, and then apply the concept of infinite shell panel i.e., derivative along x = 0.

$$\textit{For the present case:}$$
$$\sigma_{zz,z} + \frac{1}{r}\sigma_{z\theta,\theta} + \sigma_{zx,x} + \frac{1}{r}\left(\sigma_{zz} - \sigma_{\theta\theta}\right) = 0$$
$$\tau_{z\theta,z} + \frac{1}{r}\sigma_{\theta\theta,\theta} + \tau_{\theta x,x} + \frac{2\sigma_{z\theta}}{r} = 0$$
$$\tau_{zx,z} + \frac{1}{r}\sigma_{\theta x,\theta} + \sigma_{xx,x} + \frac{\sigma_{zx}}{r} = 0$$

And then we can find $\tau_{z\theta,z}$ and $\tau_{zx,z}$ along the thickness.

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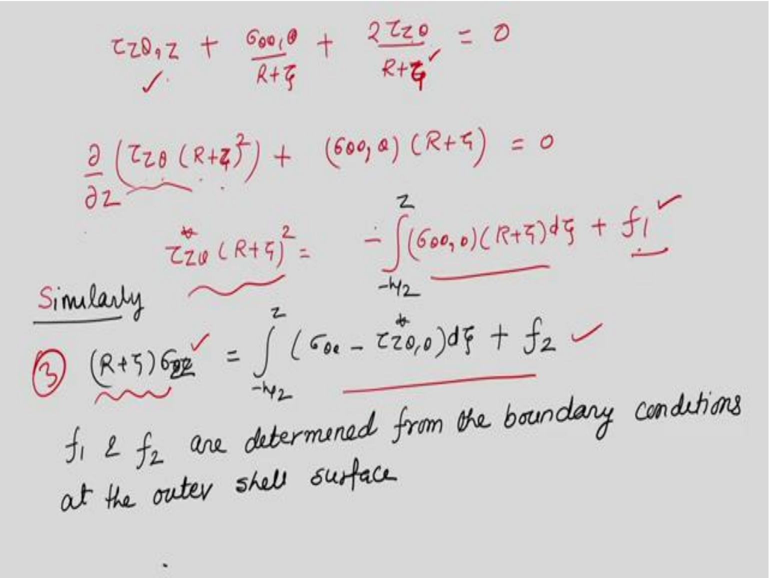

If you take the second equation will lead to:

$$\tau_{z\theta,z} + \frac{\sigma_{\theta\theta,\theta}}{R+\varsigma} + \frac{2\tau_{z\theta}}{R+\varsigma} = 0\text{, because r = } R+\varsigma\text{ .}$$

You can take ς or z, but be consistent. These two terms are there $\tau_{z\theta,z}$ and $\frac{2\tau_{z\theta}}{R+\varsigma}$, we can write like this:

$$\frac{\partial}{\partial z}\left(\tau_{z\theta}\left(R+\varsigma\right)^2\right)+\left(\sigma_{\theta\theta,\theta}\right)\left(R+\varsigma\right)=0\,.$$

If you open it will be:

$$\tau_{z\theta}\left(R+\varsigma\right)^2 = -\int_{-h/2}^{z}\left(\sigma_{\theta\theta,\theta}\right)\left(R+\varsigma\right)d\varsigma + f_1\text{ .}$$

You take the right-hand side and integrate with respect to z. Ultimately, some constants will come up, and these constants can be found out by setting up the boundary conditions at the top of the shell.

$$\left(R+\varsigma\right)\sigma_{zz} = \int_{-h/2}^{z}\left(\sigma_{\theta\theta} - \tau_{z\theta,\theta}\right)d\varsigma + f_2\text{ .}$$

Using the post-processing technique, we can get the accurate behavior or the estimation of transfer stresses along the thickness direction.

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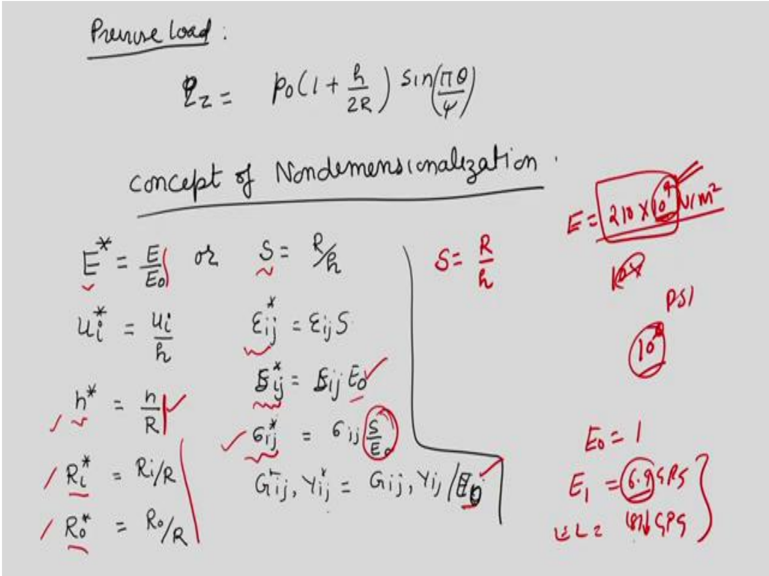

Now, there is a concept of non-dimensionalization, which I have not explained till now, I am explaining that how to develop a theory, how to develop a solution. But, whenever you are going to develop a solution, you first have to find a matrix K. And if you invert it then you can find the variables u_{20}, w_0, ψ_1, and ψ_2.

During the programming if you take all the variables; let us say, you have to give Young's modulus, shear modulus and ultimately find q_1, q_2 and then you have to find A_{11}, A_{12}, and so on, and then finally, compute this. If you do not do the non-dimensionalization, then some numerical instability may come up. Because the actual unit of E whatever you take either in a GPA or in MPA in any form, let us say for a case of steel it is $E = 210\, X\, 10^9\, N / m^2$

This 10^9 is creating the problem, or you can take 10^3 or sometimes in if it is in PSI system, then it will be 10^6. These powers make the problem difficult, if you consider it as it is, you will not get the right solution.

Initially, we have to non dimensionalize it, so that the terms will be in the order of 100 not more than that. For that case, whatever elastic modulus you have if you multiply with some E_0, $E^* = \dfrac{E}{E_0}$.

Let us say, E_1 = 6.9 GPA and E_2 = 181 GPA for the case of a glass fiber composite or

$\frac{172}{6.9}$, we get 23 or something in ratio form.

If we do so then the problem becomes slightly easy and we can get the solution. The very first assumption is that as we find an exact ratio that is known as the radius to thickness ratio $S = \frac{R}{h}$. And, then the thickness needs to be with respect to R both the dimensions are the same h in meter or mm whatever you take R is also meter or mm, you have to find h^*. Then similarly inner radius and outer radius if you divide with R mean radius, then you will get some factor.

The Young's modulus or the compliance is $S_{ij}^* = S_{ij} E_0$.

If you want to work in the form of γ_{ij} or G_{ij} shear modulus and Young modulus, then $G_{ij}^*, \gamma_{ij}^* = \frac{G_{ij}, \gamma_{ij}}{E_0}$.

Similarly, the stresses and strains, $\sigma_{ij}^* = \sigma_{ij} \frac{S}{E}$, these things help in getting less numerical instability.

And the most important part is that if you do all, I have not given the complete non-dimensionalization. But, if you do in such a way, the whole equations, whatever we are going to have, will be converted into a final form which means there will be nothing after that.

Every equation will be non dimensionalized and ultimately reporting the resulting point of view. Also, that we report that non dimensionalized result, because the length and span angle radius may change if so, we report in terms of a non dimensionalized form. Ultimately, because it is a linear solution, whatever we are going to present will be valid for different lengths, different radius, because it will be directly incremented.

If you say that I have presented a solution for θ = 60° and the radius is 2 or 3, then the thickness = 5 mm or 10 mm, then, the solution is only this. Non dimensionalized solutions can be obtained for different thicknesses, different radius, and different span angles. Results are non dimensionalized in such a way that anybody can get results for a different configuration.

In the next lecture, I will explain the brief to develop a MATLAB coding for such type of cylindrical shell, specifically whatever I have presented in this lecture that how to develop the solution for a cylindrical shell. I will explain the basic steps to write a code in MATLAB. What is the purpose of doing that? In the future, if somebody is interested to develop their code, at least at the starting point of you can think of doing it like this.

Later on, you can develop very complex codes, or very big codes for different shell types or very advanced or different means advanced theories, or taking more terms and so on. But, during the theoretical courses like the theory of plates and theory of shells students can understand, but when some projects come up that you have to write an ultimate code and you have to present the results then they find a problem.

I am just going to give you exposure to writing a code for such simple problems, later on, you can make bigger codes using these approaches.

Thank you very much.