Theory of Composite Shells

Dr. Poonam Kumari
Department of Mechanical Engineering
Indian Institute of Technology, Guwahati

Week - 06

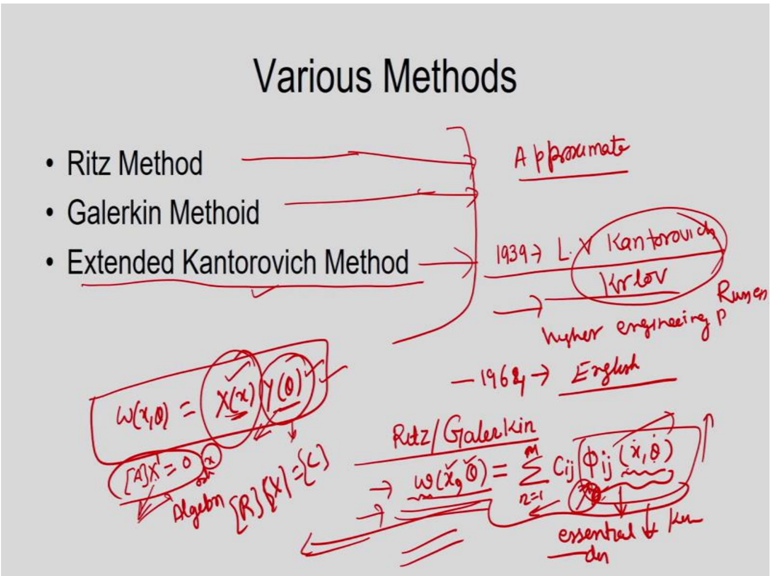
Lecture - 01

Development of Approximate solution of finite shell for arbitrary supported conditions

Dear learners, welcome to week 6 lecture-01, “Development of Approximate solutions for finite shell subjected to arbitrary support conditions”. In the previous lecture, I have presented the solution for an infinite shell and finite shell subjected to the simply supported boundary conditions. But as you know that these are various simplified forms, it is not always possible to have only simply supported boundary conditions.

Can we get the solutions, if a panel is subjected to clamped, free, or any combination of boundary conditions? Yes, we can get the solutions, if it is an analytical solution, comes under the approximate solution and there are purely numerical solutions like finite element solutions.

(Refer Slide Time: 01:38)



In these techniques: the Ritz method and Galerkin method are the approximate techniques and are very popular if we are interested to develop a solution subjected to arbitrary support conditions. And the extended Kantorovich method was first proposed in 1939 by L. V. Kantorovich and Krlov in a book on higher engineering mathematics which was initially published in the Russian language. And in 1968 or 1964 this book was published in English. Professor Kantorovich and Krlov try to further improve this method. Later on, this method is known as Kantorovich method. They try to improve the Ritz method or Galerkin method.

I would like to briefly explain the limitation of the Ritz or Galerkin method. In Ritz or Galerkin method; the function we choose, let us say, if we are going to solve for a plate or a shell, we take a variable in the present case, we say that the variable is a function of x and θ, $w(x,\theta)$.

We can express this variable: $\sum_{n=1}^{m} C_{ij}\phi_{ij}\left(x,\theta\right)$.

Here this function ϕ_{ij} needs to satisfy the essential boundary conditions. Essential means the kinematic boundary condition or displacement boundary condition.

If I say that the shell is simply supported, clamped, or free, then, this ϕ_{ij} should satisfy those boundary conditions, then we can substitute those into the weak form and try to solve them. Ultimately, Ritz and Galerkin equations lead to an algebraic equation like this:

$$[R][X] = [C]$$

The very first limitation is that it is very difficult to find a function that satisfies the other boundary condition easily, i.e., clamped, free, and so on.

If you are not able to find the approximate function, then the convergence issue will be there the result will not converge. This depends upon the initial choice; if the initial choice is very close then we will get the result, If the initial choice is far away then we will not get the accurate result.

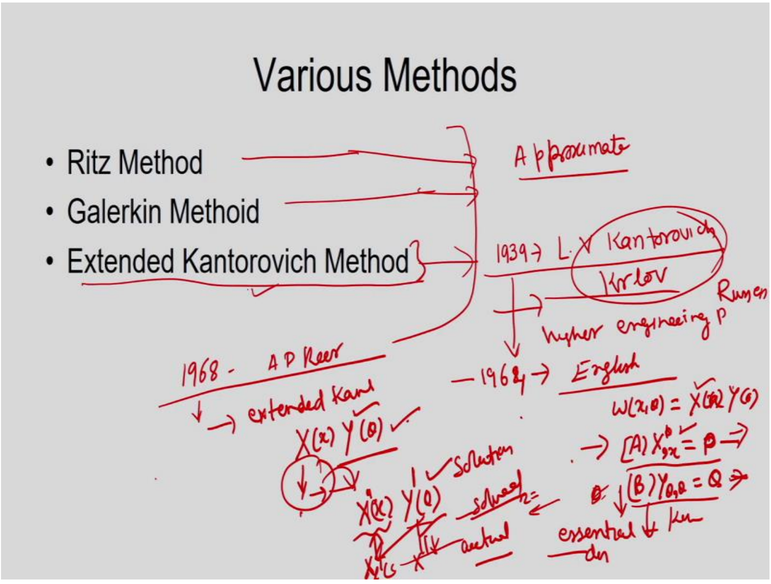
To eliminate this dependence, Kantorovich in 1939, proposed a technique to express this: Let us say, $X^1(x)Y^1(\theta)$;

X is a function of x and Y is a function of θ, in a bivariate system. Initially, it is a single term that X is a function of x and Y is a function of θ, if we substitute it into a weak form, then it reduces to a differential equation.

If we are assuming a solution along θ direction, and interested to solve along x-direction, A will be some constant, $[A]X^1$ = 0. It leads to an ordinary differential equation in x-direction, this differential equation is solved exactly and then subjected to the boundary condition.

In this way, the assumptions are along the x-direction and the solution is not dependent on x-direction, it is only dependent on one direction. In the previous case, the initial guess depends on both x and θ.

(Refer Slide Time: 06:57)



In 1968, A D Keer further extended this Kantorovich method. From 1939 to 1968, it is known as Kantorovich method, and is briefly described in most of the books based on energy methods, one page is devoted to that only.

In 1968, A D Keer extended this method and it is known as extended Kantorovich. What was the approach in that? We assume first, in the Y direction, and solve for X direction, in the next step, we assume for X-direction and solve for Y.

In the first case, $Y^1(\theta)$ is assumed, and $X^1(x)$ is known now as per the actual boundary condition. And now $X^1(x)$ is assumed and $Y^1(\theta)$ is solved. We can apply the different support condition and $Y^1(\theta)$ is subjected to actual boundary condition which is there.

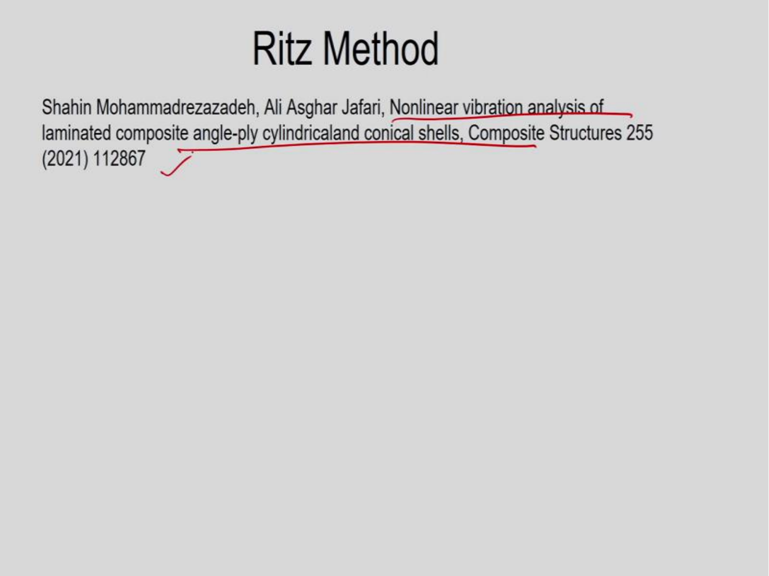
In this way, this process is iterated, once one is known we can solve the other. This x and θ ultimately give the two set of differential equations.

One is $[A]X_{,x} = p$, and the other is $[B]Y_{\theta,\theta} = Q$.

There will be two sets of ordinary differential equations. When $Y^1(\theta)$ is known this equation is used, when $X^1(x)$ is known, $Y^1(\theta)$ is solved. By using this iterative process, we can find an accurate solution subjected to the various boundary conditions. First, we presented for the case of a plate, and then this method has been extended to beam, shell, bending, free vibration, buckling, etc.

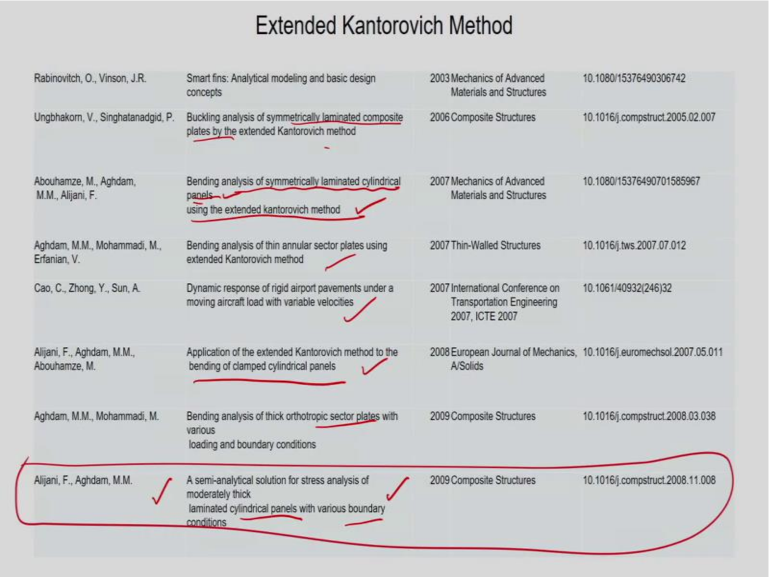
Recently, in 2011 or 2012, Professor Kapuria and I developed this method for a three-dimensional case. And we and our group of students are working on it and recently, we developed a solution for a three-dimensional cylindrical shell subjected to arbitrary support conditions. In this lecture, I will explain, how to develop a solution using the extended Kantorovich method for a cylindrical shell panel.

(Refer Slide Time: 10:41)



Ritz method: it is a very recent article on non-linear vibration analysis of laminated composite angle-ply, cylindrical-conical shells. In this paper, the approach of the Ritz method is presented and the results are also presented. Interested learners can go through this paper, to know how to apply the Ritz method for the present case.

(Refer Slide Time: 11:12)



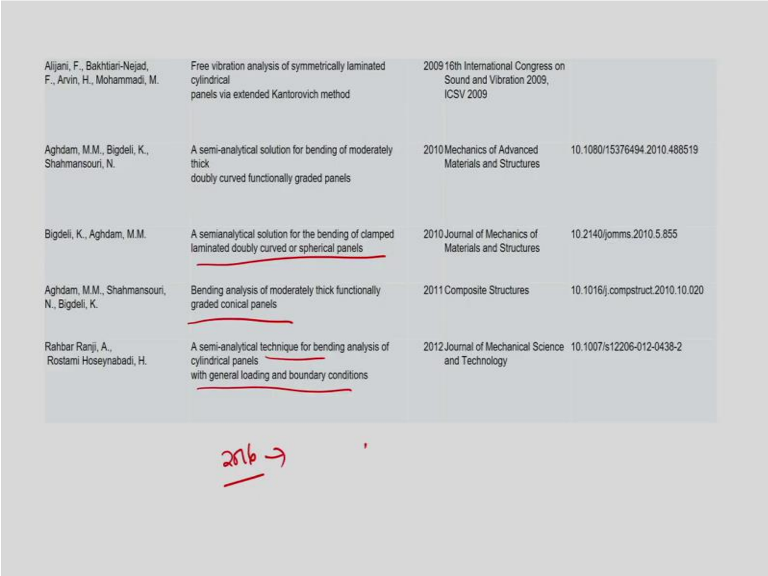
Extended Kantorovich Method

Rabinovitch, O., Vinson, J.R.	Smart fins: Analytical modeling and basic design concepts	2003	Mechanics of Advanced Materials and Structures	10.1080/15376490306742
Ungbhakorn, V., Singhatanadgid, P.	Buckling analysis of symmetrically laminated composite plates by the extended Kantorovich method	2006	Composite Structures	10.1016/j.compstruct.2005.02.007
Abouhamze, M., Aghdam, M.M., Alijani, F.	Bending analysis of symmetrically laminated cylindrical panels using the extended kantorovich method	2007	Mechanics of Advanced Materials and Structures	10.1080/15376490701585967
Aghdam, M.M., Mohammadi, M., Erfanian, V.	Bending analysis of thin annular sector plates using extended Kantorovich method	2007	Thin-Walled Structures	10.1016/j.tws.2007.07.012
Cao, C., Zhong, Y., Sun, A.	Dynamic response of rigid airport pavements under a moving aircraft load with variable velocities	2007	International Conference on Transportation Engineering 2007, ICTE 2007	10.1061/40932(246)32
Alijani, F., Aghdam, M.M., Abouhamze, M.	Application of the extended Kantorovich method to the bending of clamped cylindrical panels	2008	European Journal of Mechanics, A/Solids	10.1016/j.euromechsol.2007.05.011
Aghdam, M.M., Mohammadi, M.	Bending analysis of thick orthotropic sector plates with various loading and boundary conditions	2009	Composite Structures	10.1016/j.compstruct.2008.03.038
Alijani, F., Aghdam, M.M.	A semi-analytical solution for stress analysis of moderately thick laminated cylindrical panels with various boundary conditions	2009	Composite Structures	10.1016/j.compstruct.2008.11.008

In this slide, the papers related to the shell are compiled. The first paper is about the buckling analysis of symmetrically laminated composite plate by extended Kantorovich method, so and then the cylindrical panels using the extended Kantorovich. Another is bending analysis of symmetrical laminated cylindrical panels using extended Kantorovich method, then is bending analysis of thin annular sector plates, different pavements, and bending of clamped cylindrical by extended Kantorovich method.

This is about bending analysis of thick orthotropic sector plates which are having some curvature. It is a semi-analytical solution for a thick laminated cylindrical panel with various boundary conditions. In the present lecture, I shall explain the state of art presented in this paper, so that the learners will have an idea about how to move ahead or how to use this method for developing a solution for a cylindrical shell panel using first-order shear deformation theory.

(Refer Slide Time: 12:20)

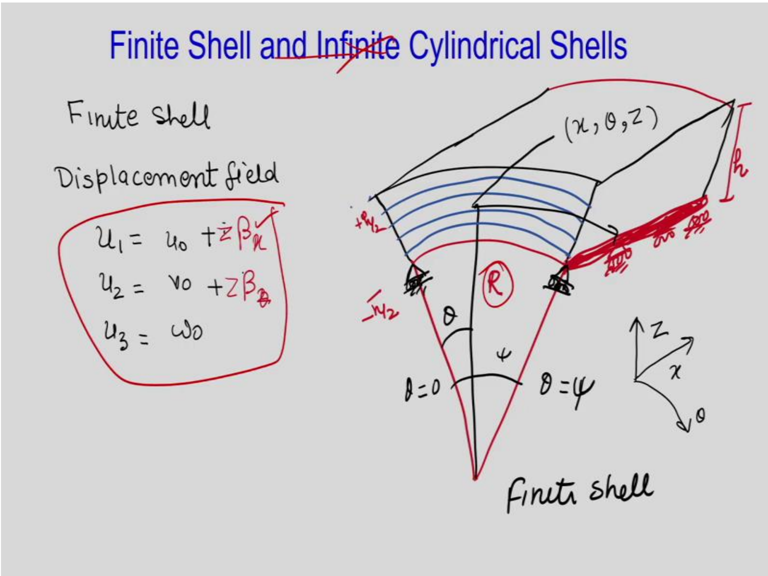


Alijani, F., Bakhtiari-Nejad, F., Arvin, H., Mohammadi, M.	Free vibration analysis of symmetrically laminated cylindrical panels via extended Kantorovich method	2009	16th International Congress on Sound and Vibration 2009, ICSV 2009	
Aghdam, M.M., Bigdeli, K., Shahmansouri, N.	A semi-analytical solution for bending of moderately thick doubly curved functionally graded panels	2010	Mechanics of Advanced Materials and Structures	10.1080/15376494.2010.488519
Bigdeli, K., Aghdam, M.M.	A semianalytical solution for the bending of clamped laminated doubly curved or spherical panels	2010	Journal of Mechanics of Materials and Structures	10.2140/jomms.2010.5.855
Aghdam, M.M., Shahmansouri, N., Bigdeli, K.	Bending analysis of moderately thick functionally graded conical panels	2011	Composite Structures	10.1016/j.compstruct.2010.10.020
Rahbar Ranji, A., Rostami Hoseynabadi, H.	A semi-analytical technique for bending analysis of cylindrical panels with general loading and boundary conditions	2012	Journal of Mechanical Science and Technology	10.1007/s12206-012-0438-2

Then is the free vibration analysis of symmetrically laminated cylindrical panels, solution for bending of moderately thick doubly curved functionally graded shell panels, and then spherical shell panels, and the conical shell panels.

All kinds of different varieties of shell panels have been studied using the extended Kantorovich method. In 2016, a professor from Iran developed a layer-wise theory, cylindrical shell panels, I will discuss that paper in detail.

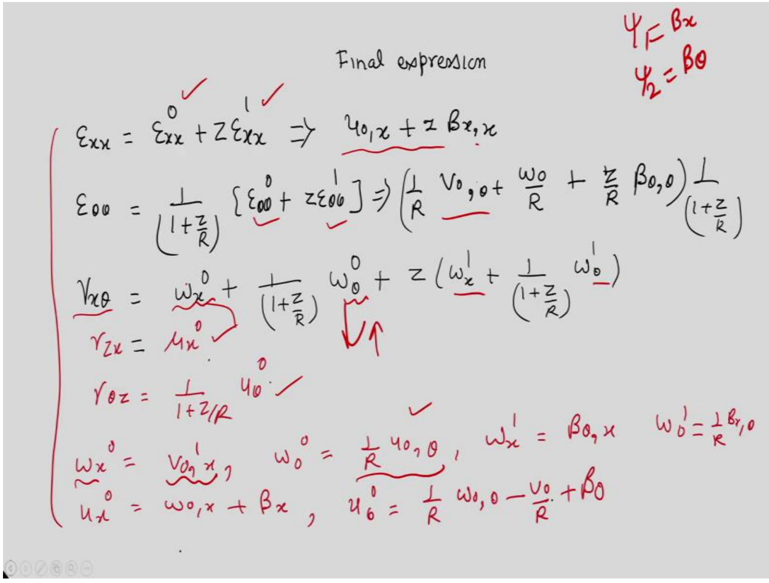
(Refer Slide Time: 13:03)



First of all, here a finite shell is considered to study, the displacement field in this paper is assumed like this: $u_1 = u_0 + z\beta_x;\;\; u_2 = v_0 + z\beta_\theta;\;\; u_3 = w_0$

Instead of ψ_1 and ψ_2 they are using β_x and β_θ, but the concept remains the same x, θ, and z, and the mean radius R, are the strain components.

(Refer Slide Time: 13:35)



The major difference is that the name is slightly different and they have made components like this:

$$\varepsilon_{xx} = \varepsilon_{xx}^0 + z\varepsilon_{xx}^1 \Rightarrow u_{0,x} + z\beta_{x,x}$$

$$\varepsilon_{\theta\theta} = \frac{1}{\left(1+\frac{z}{R}\right)}\left(\varepsilon_{\theta\theta}^0 + z\varepsilon_{\theta\theta}^1\right) \Rightarrow \left(\frac{1}{R}v_{0,\theta} + \frac{w_0}{R} + \frac{z}{R}\beta_{\theta,\theta}\right)\frac{1}{\left(1+\frac{z}{R}\right)}$$

$$\gamma_{x\theta} = w_x^0 + \frac{1}{\left(1+\frac{z}{R}\right)}w_\theta^0 + z\left(w_x^1 + \frac{1}{\left(1+\frac{z}{R}\right)}w_\theta^1\right)$$

$$\gamma_{zx} = \mu_x^0 \qquad \gamma_{\theta z} = \frac{1}{\left(1+\frac{z}{R}\right)}\mu_\theta^0$$

The terms are same, only their notation is slightly different.

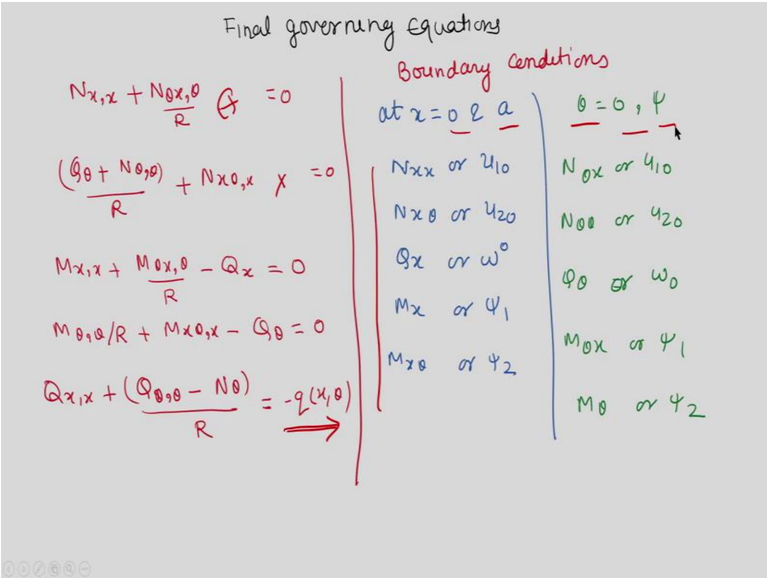
Instead of a ψ_1, they have used β_x , instead of a ψ_2, they used β_θ. And the second difference is the notation; w_θ^1, w_x^1, w_θ^0, w_x^0 and so on.

Here, $w_x^0 = v_{0,x}^1$; $w_\theta^0 = \frac{1}{R} u_{0,x}$; $w_x^1 = \beta_{\theta,x}$; $w_\theta^1 = \frac{1}{R}\beta_{x,\theta}$; $u_x^0 = w_{0,x} + \beta_x$; and

$$u_\theta^0 = \frac{1}{R} w_{0,\theta} - \frac{v_0}{R} + \beta_0 \ .$$

They have written like this in a short form instead of this, these are the strain displacement relations.

(Refer Slide Time: 14:42)



Following are the 5 sets of governing differential equations:

$$N_{x,x} + \frac{N_{\theta x,\theta}}{R} = 0$$

$$\frac{Q_\theta + N_{\theta,\theta}}{R} + N_{x\theta,x} = 0$$

$$M_{x,x} + \frac{M_{\theta x,\theta}}{R} - Q_x = 0$$

$$\frac{M_{\theta,\theta}}{R} + M_{x\theta,x} - Q_\theta = 0$$

$$Q_{x,x} + \frac{Q_{\theta,\theta} - N_\theta}{R} = -q_z(x,\theta)$$

It is the same, but for the present case, there is no in-plane loading along x-direction and

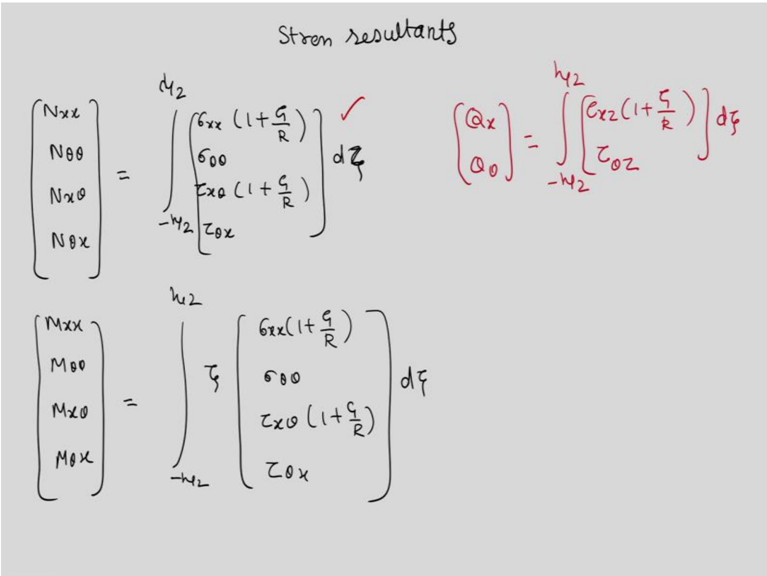
θ direction only transverse loading is explained. Following are the boundary conditions at x = 0 and x = a:

$$N_{xx} \text{ or } u_{10,} \quad N_{x\theta} \text{ or } u_{20,} \quad Q_x \text{ or } w_{0,} \quad M_{xx} \text{ or } \psi_{1,} \quad M_{x\theta} \text{ or } \psi.$$

Boundary condition for θ = 0 and θ = ψ will be:

$$N_{x\theta} \text{ or } u_{10} \quad N_{\theta\theta} \text{ or } u_{20} \quad Q_\theta \text{ or } w_0 \quad M_{x\theta} \text{ or } \psi_1 \quad M_{\theta\theta} \text{ or } \psi_2$$

(Refer Slide Time: 15:14)

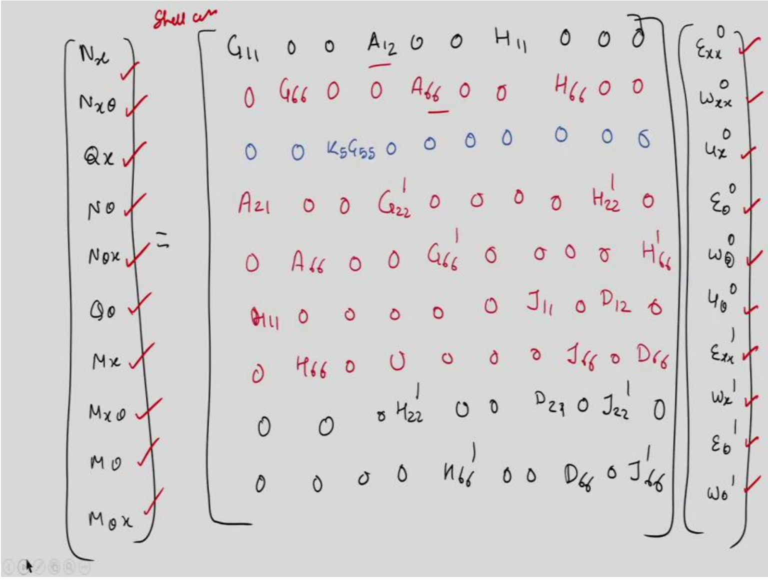


The Stress resultant is defined as we defined in the previous case:

$$\begin{bmatrix} N_{xx} \\ N_{\theta\theta} \\ N_{x\theta} \\ N_{\theta x} \end{bmatrix} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \begin{bmatrix} \sigma_{xx}\left(1+\frac{\varsigma}{R}\right) \\ \sigma_{\theta\theta} \\ \tau_{x\theta}\left(1+\frac{\varsigma}{R}\right) \\ \tau_{\theta x} \end{bmatrix} d\varsigma \ ; \quad \begin{bmatrix} M_{xx} \\ M_{\theta\theta} \\ M_{x\theta} \\ M_{\theta x} \end{bmatrix} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \varsigma \begin{bmatrix} \sigma_{xx}\left(1+\frac{\varsigma}{R}\right) \\ \sigma_{\theta\theta} \\ \tau_{x\theta}\left(1+\frac{\varsigma}{R}\right) \\ \tau_{\theta x} \end{bmatrix} d\varsigma \quad \text{and}$$

$$\begin{bmatrix} Q_x \\ Q_\theta \end{bmatrix} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \begin{bmatrix} \tau_{xz}\left(1+\frac{\varsigma}{R}\right) \\ \tau_{\theta z} \end{bmatrix} d\varsigma$$

(Refer Slide Time: 15:26)



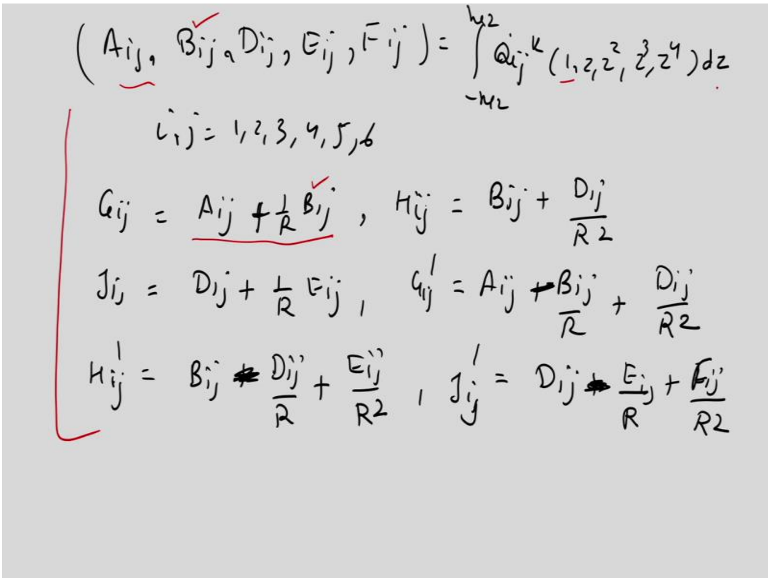
Ultimately, the shell constitutive relation is presented like this:

$$
\begin{bmatrix} N_x \\ N_{x\theta} \\ Q_x \\ N_\theta \\ N_{\theta x} \\ Q_\theta \\ M_x \\ M_{x\theta} \\ M_\theta \\ M_{\theta x} \end{bmatrix}
=
\begin{bmatrix}
G_{11} & 0 & 0 & A_{12} & 0 & 0 & H_{11} & 0 & 0 & 0 \\
0 & G_{66} & 0 & 0 & A_{66} & 0 & 0 & H_{66} & 0 & 0 \\
0 & 0 & G_{55} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
A_{21} & 0 & 0 & G_{22}^1 & 0 & 0 & 0 & 0 & H_{22}^1 & 0 \\
0 & A_{66} & 0 & 0 & G_{66}^1 & 0 & 0 & 0 & 0 & H_{66}^1 \\
H_{11} & 0 & 0 & 0 & 0 & 0 & J_{11} & 0 & D_{12} & 0 \\
0 & H_{66} & 0 & 0 & 0 & 0 & 0 & J_{66} & 0 & D_{66} \\
0 & 0 & 0 & H_{22}^1 & 0 & 0 & D_{21} & 0 & J_{22}^1 & 0 \\
0 & 0 & 0 & 0 & H_{66}^1 & 0 & 0 & D_{66} & 0 & J_{66}^1 \\
 & & & & & & & & &
\end{bmatrix}
\begin{bmatrix} \varepsilon_{xx}^0 \\ w_{xx}^0 \\ u_x^0 \\ \varepsilon_\theta^0 \\ w_\theta^0 \\ u_\theta^0 \\ \varepsilon_{xx}^1 \\ w_x^1 \\ \varepsilon_\theta^1 \\ w_\theta^1 \end{bmatrix}
$$

They are arranged in a matrix form and these are the coefficients of a matrix and these are the strain components. These are the different components they are arranged such that we are getting a matrix like this. The various terms here you see that G_{11}, G_{66}, A_{12}, A_{66}, etc. are same as we defined f_1, f_2, f_3, etc.

Here, $\left[A_{ij},\ B_{ij}\ ,D_{ij},E_{ij},F_{ij}\right]=\int\limits_{\frac{-h}{2}}^{\frac{h}{2}} Q_{ij}^{k}\left(1,\ z,\ z^{2},z^{3},z^{4}\right)dz$.

(Refer Slide Time: 16:13)



These are same; A_{ij} = constant times dz and B_{ij} = z Q_{ij}^{k} and so on. These are the coefficients:

$$G_{ij} = A_{ij} + \frac{1}{R} B_{ij}$$

$$H_{ij} = B_{ij} + \frac{D_{ij}}{R^2}$$

$$J_{ij} = D_{ij} + \frac{1}{R} E_{ij}$$

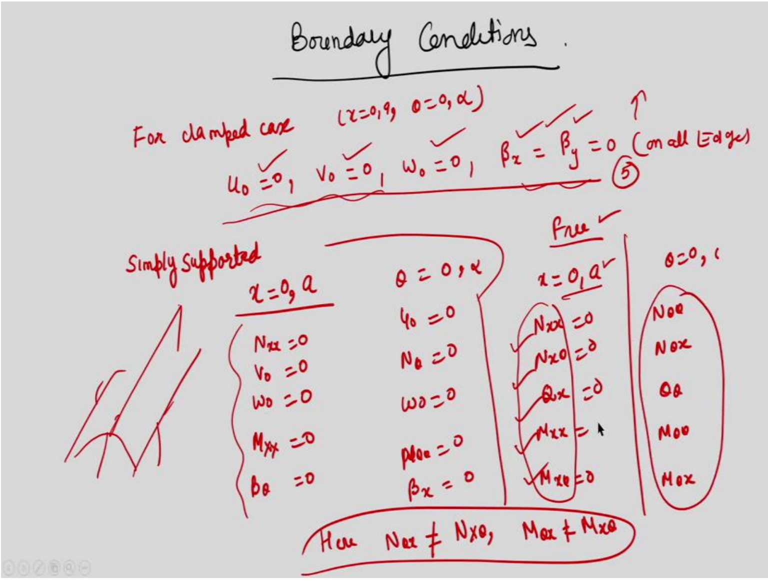
$$G_{ij}^{1} = A_{ij} - \frac{B_{ij}}{R} + \frac{D_{ij}}{R^2}$$

$$H_{ij}^{1} = B_{ij} - \frac{D_{ij}}{R} + \frac{E_{ij}}{R^2}$$

$$J_{ij}^{1} = D_{ij} - \frac{E_{ij}}{R} + \frac{F_{ij}}{R^2} .$$

Now, these are the basic governing equations, for the arbitrary support conditions, first, we will define the variables which need to be specified for a particular boundary condition.

(Refer Slide Time: 16:45)



For the clamped case:

At $x=0,a$ and $\theta=0,\alpha$; $u_0=0$, $v_0=0$, $w_0=0$, $\beta_x=0$, and $\beta_y=0$.

If a panel is subjected to simply supported boundary conditions:

It is very important to understand that there will be a difference;

At the edge $x=0,a:$ $N_{xx}=0;\ v_0=0;\ w_0=0;\ M_{xx}=0;\ and\ \beta_\theta=0$. At $\theta=0,\alpha:$ $N_\theta=0;\ w_0=0;\ N_{\theta\theta}=0; and\ \beta_x=0$

If it is free:

At the edge $x=0,a:$ $N_{xx}=0;\ N_{x\theta}=0;\ Q_x=0;\ M_{xx}=0;\ and\ M_{x\theta}=0$ and

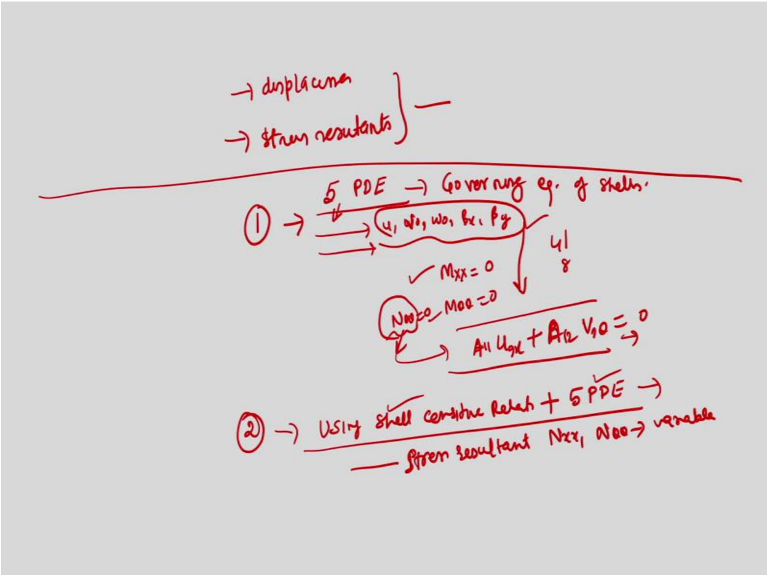
At $\theta=0,\alpha:$ $N_{\theta\theta}=0;\ N_{\theta x}=0;\ Q_\theta=0;\ M_{\theta\theta}=0;\ and\ M_{\theta x}=0$

Here, it is a cylindrical shell, therefore, $N_{x\theta}\neq N_{\theta x}$ and $M_{x\theta}\neq M_{\theta x}$. For the clamped case: all essential variables are 0, which means the kinematic boundary conditions, all displacement, and rotations are 0.

If we are interested to develop a Ritz solution, then the clamped solution can be developed very easily because we can find a function in which all variables are 0. But if a panel is subjected to simply supported conditions, then it will have mixed boundaries.

You have to specify N_{xx}, v_0, w_0, etc. There are different types of formulations, even for the case of free all stress resultant needs to be specified because the boundaries are mixed.

(Refer Slide Time: 20:38)



At some edge we may specify the displacement, at some edge, we may specify the stress resultants or moment resultants. For such cases, the formulation can be done in two different ways.

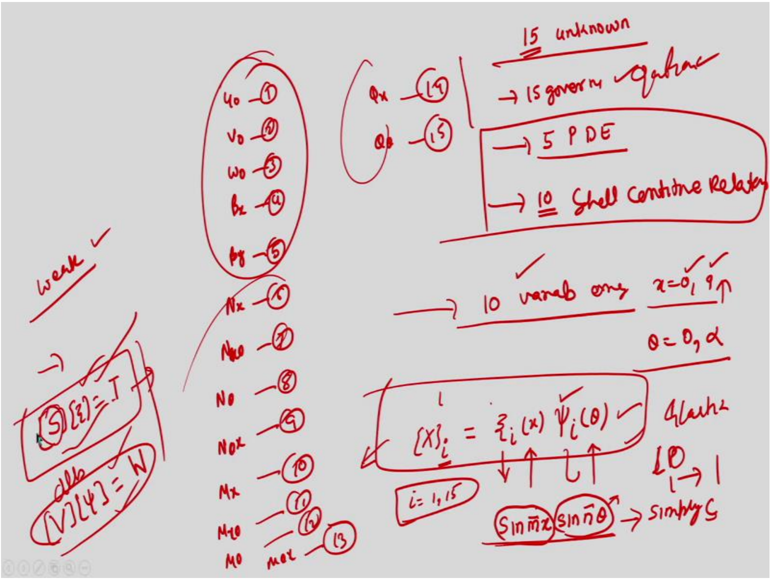
First: we have all 5 partial differential equations or governing equations for a shell, if we substituting the use of shell constitutive relations, we can express in terms of displacements variables and their derivatives in the case of simply supported.

These will be expressed only in these 5 variables. If you say that M_{xx} = 0, or $M_{\theta\theta}$ = 0, or $N_{\theta\theta}$ = 0, in that case, the for getting the boundary condition, we have to first convert it into that form by using the shell constitutive relations.

Let us say, $A_{11}u_{,x} + A_{12}v_{,\theta}$ = 0, we are going to satisfy the boundary condition because $N_{\theta\theta}$ is not our variable at that time. And these governing equations may be 4th order, higher-order, 8 order, and so on.

Second: using the shell constitutive relations and 5 partial differential equations, we make setup in such a way that we are assuming stress resultant as a variable also, like N_{xx}, $N_{\theta\theta}$ are our variables. Then, we can directly specify the boundary because that is our variable, but we have to use the shell constitutive equations in the governing part.

(Refer Slide Time: 23:21)



In this paper, the second approach is used, where, u_0, v_0, w_0, β_x, β_y, N_{xx}, $N_{x\theta}$, $N_{\theta\theta}$, $N_{\theta x}$ M_{xx}, $M_{\theta\theta}$, $M_{x\theta}$, $M_{\theta x}$, Q_x, and Q_θ, these 15 unknown variables are used.

Previously, if it goes for displacement, we had only 5 unknown variables, now we assumed that stress resultants are also unknowns. Therefore, we need 15 governing equations. We have 5 partial differential equations, which we are using every time, and 10 of your shell constitutive relations.

In this way, we will get 15 governing equations. Now, you check the boundary conditions: at an edge, we can provide 5 variables, if we solve in one direction and in the second direction at another edge, we can have 10 variables. In one direction, the maximum we can solve is 10 variables, whether you talk about along x-axis or along θ axis.

Out of these 15 variables, we have to select in such a way that the variables for a particular set, along those boundary conditions will be the primary variables. And rest of the 5 variables will be expressed in terms of these 10 primary variables. Similarly, if we solve along θ direction, then we assume the reverse way, i.e., the variables which come along θ direction are considered as primary variables and other variables can be expressed in terms of primary variables.

In this way, these 5 variables are fixed, and out of these 10 variables, we have to take another 5 variables. For example, if somebody asked me to choose, then I will choose these 5 variables along x = 0 and a, I am going to solve these, I will take these as primary variables. Similarly, along θ direction, I will take these 5 variables.

How do we proceed with the case of Kantorovich method? Let us say, I have given one variable $\left[X_i\right]$, this X is a state variable and I means 1, 2, 3, 4, 5, up to 15. It can be expressed as:

$\left[X_i\right]=\xi_i(x)\psi_i(\theta)$.

ξ_i variable is only a function of x-direction and ψ_i is another variable that is only a function of θ direction.

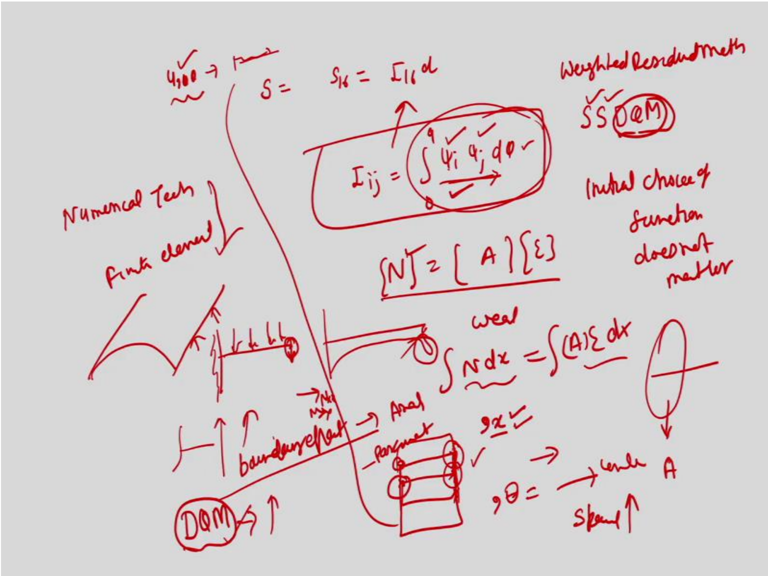
If you remember that in a double Fourier series, we said that $\sin\bar{m}x\sin\bar{n}\theta$, where $\sin\bar{m}$ is solely a function of x and $\sin\bar{n}$ is solely a function of y, but this type of combination is possible only for the simply supported case. Now, we have taken a general function, it may be sin sin or it may be something else.

In this paper, we have chosen the variable like this x_{15} and ψ_{15}. Now, substituting these variables into 15 governing equations and applying the Galerkin technique, because all these are in the form of PDE or ODE, so we have to find in terms of a weak form.

Applying the concept of Galerkin, ultimately, it reduces to let us say, $[S][\xi]=T$ loading vector, algebraic differential equation along ψ direction.

Similarly, we will get another set of the equation: $[V][\psi]=W$, for loading vector it is W, this is also an algebraic differential equation and that can be solved. This is one of the ways that we can convert into a weak form and then using the concept of Kantorovich, we will get two sets of differential equations, one which is solely a function of x and another is the function of θ .

(Refer Slide Time: 30:02)



Here the S is the most important part, it is nothing but it contains all the components. Let us say, S_{16}, there it shows $I_{16}d$, where $I_{ij} = \int_{\theta}^{\alpha} \psi_i \psi_j d\theta$, if we convert it into a weak form.

Initially, we knew ψ_j along θ direction, so we have chosen any function and converted using the concept of weighted residual method.

Through that our concept of let us say, $[N]=[A][\varepsilon]$, that can be converted into a weak form, integrated form $\int Ndx = \int (A)\varepsilon dx$.

From there along the θ direction ψ is known to us, so we can take it as a constant and the variables which are having derivatives along θ will be considered in that governing equation.

In the same way, when the variable along the x-direction is known, then we can find along the θ direction derivatives. This method has been applied and is found that the initial choice of a function does not matter. This means if you say that the initial choice may or may not satisfy the boundary condition but that solution will converge fast.

It is not like Ritz that is mandatory satisfy, because through the iterative processor we can satisfy the boundary conditions, and ultimately, we get the final result. These techniques are developed and recently, have applied for buckling of a shell or a non-linear vibration of a shell or the carbon nanotubes composites, and so on. This is a very efficient method; one can try and can go for the solutions.

It needs to be developed further for special cases and this method has recently even been tried for skew plates and plates having some cut-outs and so on. Later on, this method has been developed. Now, the concept of pure numerical techniques is there like the finite element solutions.

These days, you will find that in literature, if you go for a cylindrical shell, then you will find a lot of research articles, around 22000 articles are shown in the Scopus.

And there you can see that number of works have been published using the finite element techniques, Ritz technique, and other techniques. Even for very general cases or general loading conditions pure numerical techniques can be applied, methods have been developed.

But if you are interested in approximate or analytical solutions, the advantage of analytical solutions over the numerical technique is that the boundary effects can be accurately predicted by the analytical solutions. And the parametric studies can be done easily by the analytical techniques. And these are somewhat closed-form formulas, so one can easily do parametric studies, optimization, and so on.

But if you go for pure numerical techniques the boundary effects cannot be captured.

The boundary effects mean, when it is simply supported then there is no effect of the boundaries, but when it is clamped or free, I would like to explain with the help of a cantilever beam; if a cantilever beam is subjected to such kind of loading; at the free edge, over this edge, N_{xx} needs to be 0, because this is free and moment needs to be 0.

If you want to see the variation, then it may be having some moment near the boundary it is just dipping making some effect, it is not smoothly going there, in the case of a composite plate because it is more prone.

In an isotropic, it is not, but when you are talking about the layered composite plate or a composite shell, because at each layer the material property is different and they cause a sharp variation of stresses, shooting variations either they are increasing or decreasing very sharply, due to the delamination occurs. And in the composites, this is the most prominent problem that we are trying to overcome or we have to understand that how much and what level of stresses are shooting in this area. If this edge is free, then at every edge stress needs to be 0, but just near the edge, it is very high. We are interested to find which causes the delamination in the composite shells.

For that kind of accuracy, we need analytical solutions. These days a special purpose finite element is developed or the concept of numerical techniques plus analytical techniques are combined that is known as a very famous technique SS DQM, which means state space and differential quadrature matter.

These days another method DQM (differential quadrature method) is most popular. If you go through a recent research article, you can find that in the case of shells. I think a lot of papers are presented using the DQM approach, and two books specifically from Italy have been published using this differential quadrature method.

Differentiation means the derivative is presented in a form of a series, like we want to solve $u_{,\theta}$, here, u is our variable. Can we express this variable in terms of a linear series, in some functions?

Solving a differential equation in the first order and fourth-order is typical. But if you can convert it into an algebraic form, then it becomes easy. With the help of DQM, these derivatives are converted in terms of some algebraic form, so that we can get the solution. With this, I end this lecture.

Thank you very much.