Theory of Composite Shells
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Week - 06
Lecture - 02
Development of Levy type finite shell under static and free vibration case

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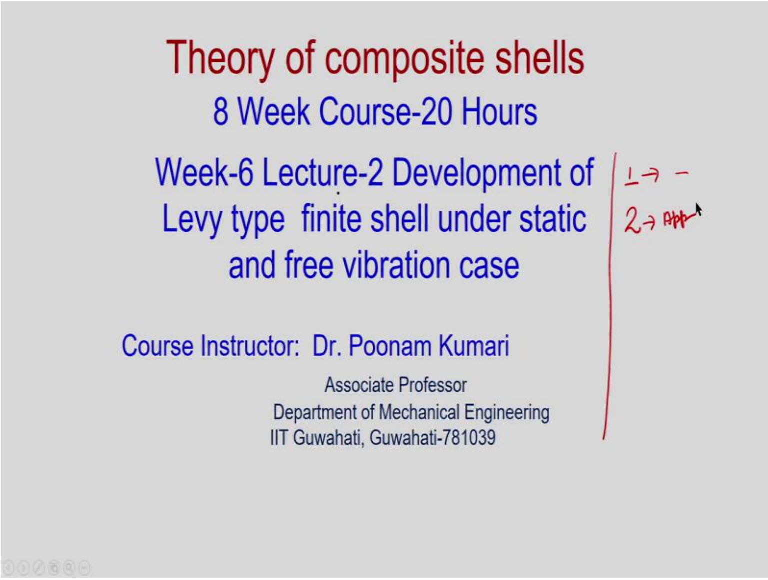

Dear learners welcome to week-06, lecture-02. In this lecture, I will explain the first free vibration solution for all-around simply supported finite cylindrical shell and state of art for developing levy type boundary conditions for a finite shell. Previously, I developed a Navier solution for static bending and an approximate solution for arbitrary supported shell panels.

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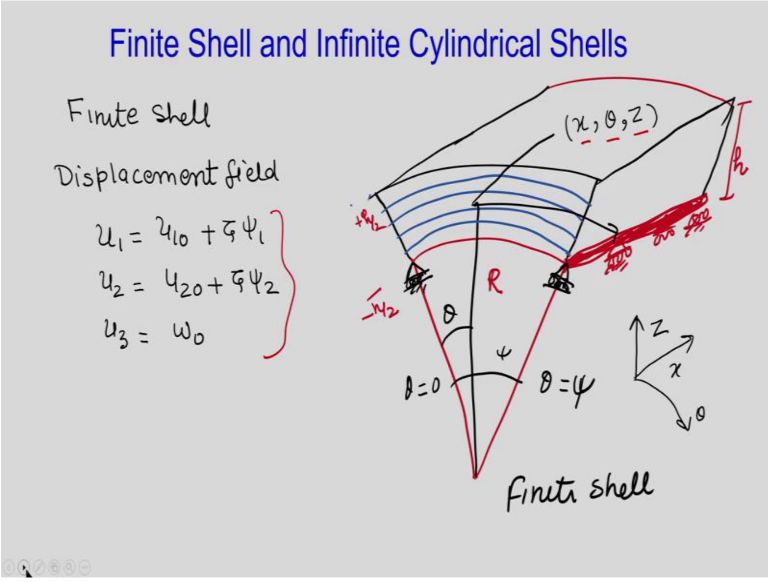

The geometry remains the same, a finite shell panel is considered, coordinate in the first direction is x, in the second direction is θ, and z is in radial or thickness direction, the displacement field is:

$$u_1 = u_{10} + \varsigma\psi_1, \quad u_2 = u_{20} + \varsigma\psi_2, and \quad u_3 = w_0 \ .$$

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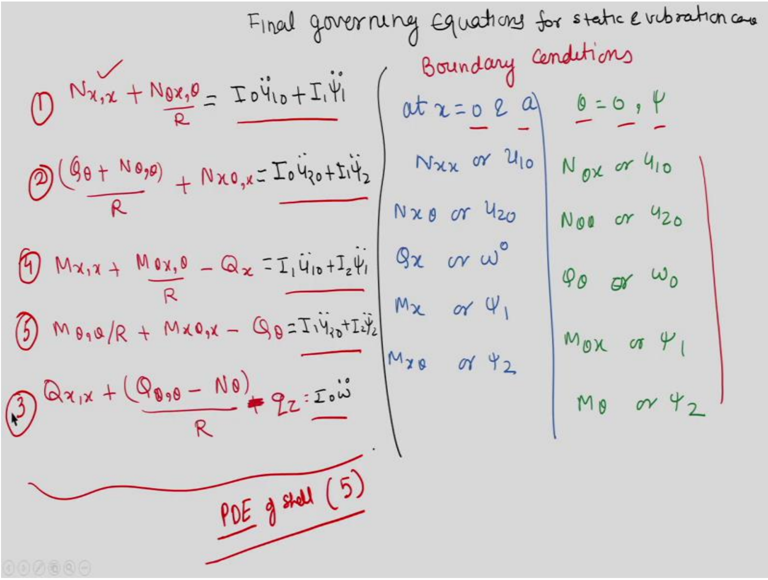

Now, we are directly coming to the governing equations. Following governing equations we derived in week-05; lecture-02 and lecture-03:

$$N_{x,x} + \frac{N_{\theta x,\theta}}{R} = I_0\ddot{u}_{10} + I_1\ddot{\psi}_1 \quad equation(1)$$

$$\frac{\left(Q_\theta + N_{\theta,\theta}\right)}{R} + N_{x\theta,x} = I_0\ddot{u}_{20} + I_1\ddot{\psi}_2 \quad equation(2)$$

$$M_{x,x} + \frac{M_{\theta x,\theta}}{R} - Q_x = I_1\ddot{u}_{10} + I_2\ddot{\psi}_1 \quad equation(4)$$

$$\frac{M_{\theta,\theta}}{R} + M_{x\theta,x} - Q_\theta = I_1\ddot{u}_{20} + I_2\ddot{\psi}_2 \quad equation(5)$$

$$Q_{x,x} + \frac{Q_{\theta,\theta} - N_\theta}{R} - q_z = I_1\ddot{w}_0 \quad equation(3)$$

This time, I included the dynamic terms also. Following are the boundary conditions:

At x = 0 and a: N_{xx} or u_{10} $N_{x\theta}$ or u_{20} Q_x or w_0 M_{xx} or ψ_1 $M_{x\theta}$ or ψ_2, these variables are needed to be specified.

At θ is equal to 0 and ψ: $N_{x\theta}$ or u_{10} $N_{\theta\theta}$ or u_{20} Q_θ or w_0 $M_{x\theta}$ or ψ_1 $M_{\theta\theta}$ or ψ_2, these variables are needed to be specified.

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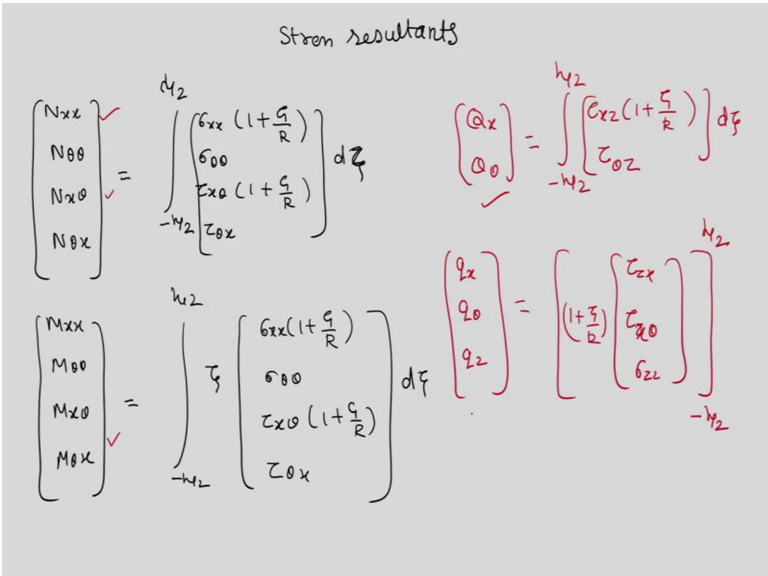

Just for the sake of completeness, I again have defined stress resultants, moment results, shear resultants, and the loading:

$$\begin{bmatrix} N_{xx} \\ N_{\theta\theta} \\ N_{x\theta} \\ N_{\theta x} \end{bmatrix} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \begin{bmatrix} \sigma_{xx}\left(1+\frac{\varsigma}{R}\right) \\ \sigma_{\theta\theta} \\ \tau_{x\theta}\left(1+\frac{\varsigma}{R}\right) \\ \tau_{\theta x} \end{bmatrix} d\varsigma \;; \quad \begin{bmatrix} M_{xx} \\ M_{\theta\theta} \\ M_{x\theta} \\ M_{\theta x} \end{bmatrix} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \varsigma \begin{bmatrix} \sigma_{xx}\left(1+\frac{\varsigma}{R}\right) \\ \sigma_{\theta\theta} \\ \tau_{x\theta}\left(1+\frac{\varsigma}{R}\right) \\ \tau_{\theta x} \end{bmatrix} d\varsigma$$

$$\begin{bmatrix} Q_x \\ Q_\theta \end{bmatrix} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \begin{bmatrix} \tau_{xz}\left(1+\frac{\varsigma}{R}\right) \\ \tau_{\theta z} \end{bmatrix} d\varsigma \text{ and } \begin{bmatrix} q_x \\ q_\theta \\ q_z \end{bmatrix} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \left[\left(1+\frac{\varsigma}{R}\right)\begin{bmatrix} \tau_{xz} \\ \tau_{x\theta} \\ \sigma_{zz} \end{bmatrix}\right] d\varsigma$$

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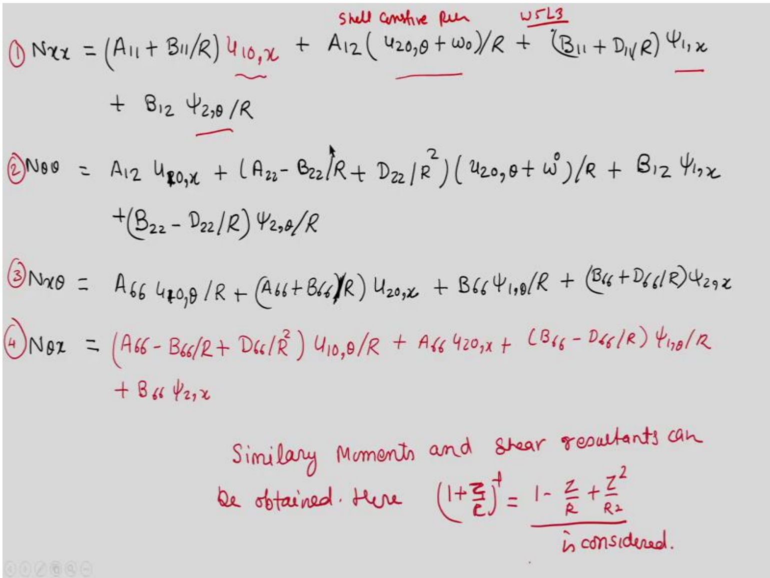

Following are the shell constitutive relations, which we have derived in week 5, lecture 03:

$$N_{xx} = \left(A_{11}+\frac{B_{11}}{R}\right)u_{10,x} + A_{12}\frac{\left(u_{20,\theta}+w_0\right)}{R} + \left(B_{11}+\frac{D_{11}}{R}\right)\psi_{1,x} + B_{12}\frac{\psi_{2,\theta}}{R}$$

$$N_{\theta\theta} = A_{12}u_{10,x} + \left(A_{22}-\frac{B_{22}}{R}+\frac{D_{22}}{R^2}\right)\frac{\left(u_{20,\theta}+w_0\right)}{R} + B_{12}\psi_{1,x} + \left(B_{22}-\frac{D_{22}}{R}\right)\frac{\psi_{2,\theta}}{R}$$

$$N_{x\theta} = A_{66}\frac{u_{10,\theta}}{R} + \left(A_{66}+\frac{B_{66}}{R}\right)u_{20,x} + \frac{B_{66}\psi_{1,\theta}}{R} + \left(B_{66}+\frac{D_{66}}{R}\right)\psi_{2,x}$$

$$N_{\theta x} = \left(A_{66}-\frac{B_{66}}{R}+\frac{D_{66}}{R^2}\right)\frac{u_{10,\theta}}{R} + A_{66}u_{20,x} + \left(B_{66}-\frac{D_{66}}{R}\right)\frac{\psi_{1,\theta}}{R} + B_{66}\psi_{2,x}$$

Here, you can see, they are expressed in terms of primary variables; $u_{10,x}$, $u_{20,x}$, $\psi_{1,x}$,

$\psi_{2,x}$, and so on.

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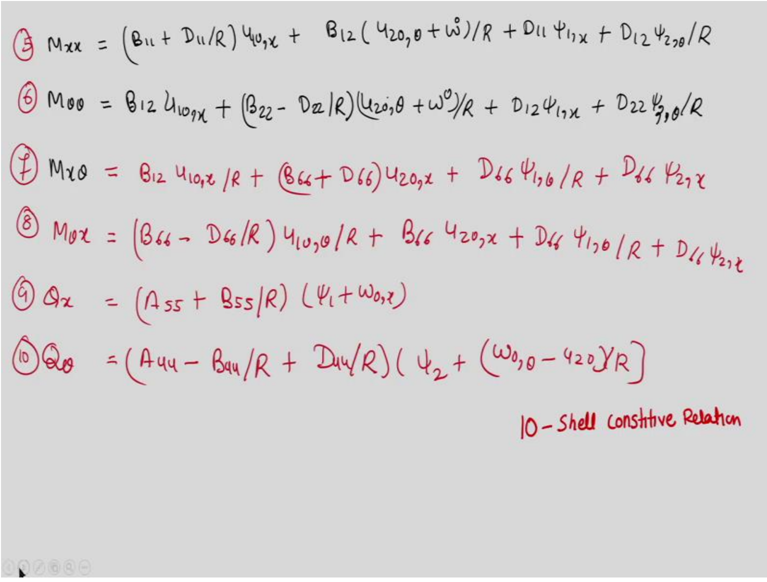

Similarly, the moments and shear resultants will be:

$$M_{xx} = \left(B_{11} + \frac{D_{11}}{R}\right)u_{10,x} + B_{12}\frac{\left(u_{20,\theta} + w_0\right)}{R} + D_{11}\psi_{1,x} + D_{12}\frac{\psi_{2,\theta}}{R}$$

$$M_{\theta\theta} = B_{12}u_{10,x} + \left(B_{22} - \frac{D_{22}}{R}\right)\frac{\left(u_{20,\theta} + w_0\right)}{R} + D_{12}\psi_{1,x} + D_{22}\frac{\psi_{2,\theta}}{R}$$

$$M_{x\theta} = B_{12}\frac{u_{10,x}}{R} + \left(B_{66} + D_{66}\right)u_{20,x} + \frac{D_{66}\psi_{1,\theta}}{R} + D_{66}\psi_{2,x}$$

$$M_{\theta x} = \left(B_{66} - \frac{D_{66}}{R}\right)\frac{u_{10,\theta}}{R} + B_{66}u_{20,x} + \frac{D_{66}\psi_{1,\theta}}{R} + D_{66}\psi_{2,x}$$

$$Q_x = \left(A_{55} + \frac{B_{55}}{R}\right)\left(\psi_1 + w_{0,x}\right)$$

$$Q_\theta = \left(A_{44} - \frac{B_{44}}{R} + \frac{D_{44}}{R}\right)\left(\psi_2 + \frac{\left(w_{0,\theta} - u_{20}\right)}{R}\right)$$

We have 10 shell constitutive relations.

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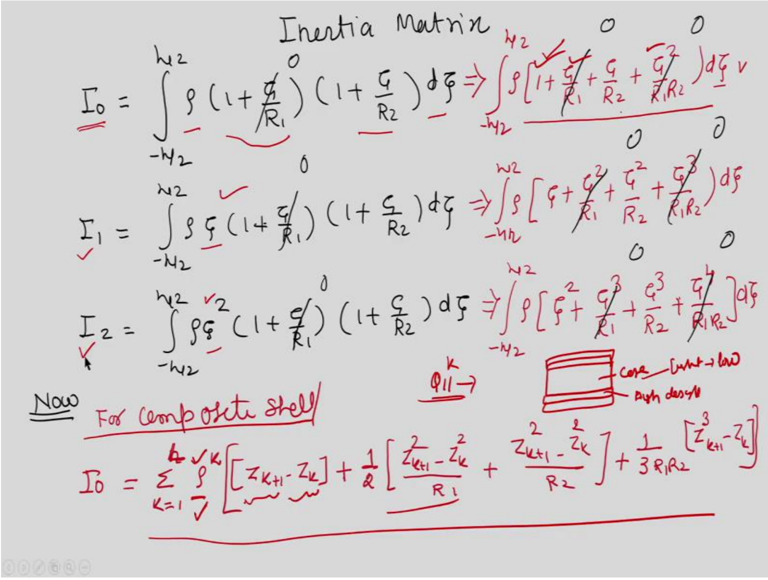

Now, the inertia matrix: the definition of $I_0 = \int_{-h/2}^{h/2} \rho\left(1+\frac{\varsigma}{R_1}\right)\left(1+\frac{\varsigma}{R_2}\right)d\varsigma$.

For the present case, in the first direction radius $R_1 = \infty$, $\left(1+\frac{\varsigma}{R_1}\right)$ term will not contribute.

Let us say, for a doubly curved, this term may exist. If you further open it, it will be:

$$\int_{-h/2}^{h/2} \rho\left(1+\frac{\varsigma}{R_1}+\frac{\varsigma}{R_2}+\frac{\varsigma^2}{R_1R_2}\right)d\varsigma.$$

But, for the present case, $\frac{\varsigma}{R_1}$ and $\frac{\varsigma^2}{R_1R_2}$ will not exist, it will contain $\int_{-h/2}^{h/2} \rho \left(1+\frac{\varsigma}{R_2}\right) d\varsigma$.

And similarly, $I_1 = I_1 = \int_{-h/2}^{h/2} \rho\varsigma\left(1+\frac{\varsigma}{R_1}\right)\left(1+\frac{\varsigma}{R_2}\right)d\varsigma \Rightarrow \int_{-h/2}^{h/2} \rho\left(\varsigma+\frac{\varsigma^2}{R_1}+\frac{\varsigma^2}{R_2}+\frac{\varsigma^3}{R_1R_2}\right)d\varsigma$

And $I_2 = \int_{-h/2}^{h/2} \rho\varsigma^2\left(1+\frac{\varsigma}{R_1}\right)\left(1+\frac{\varsigma}{R_2}\right)d\varsigma \Rightarrow \int_{-h/2}^{h/2} \rho\left(\varsigma^2+\frac{\varsigma^3}{R_1}+\frac{\varsigma^3}{R_2}+\frac{\varsigma^4}{R_1R_2}\right)d\varsigma$.

If it is an isotropic shell, then we can directly integrate and we can write ρ h or $\rho\,\frac{S^2}{2}$ and so on. But, for a composite shell or a laminated shell, each layer may have a different density and different reduced stiffness coefficients k times.

If we talk about a sandwich shell, the core is very light and the face is having stronger material and may have more density. And, this core is a light material, we have low density. In that case, ρ is a function of the kth layer which means $\left(z_{k+1}-z_k\right)$, this is the first term contribution.

Similarly, the second term contribution, third term contribution can be found for the case of a composite shell. In this way I_0 will be:

$$\sum_{K=1}^{L}\rho^k\left[\left(z_{k+1}-z_k\right)+\frac{1}{2}\left(\frac{z_{k+1}^2-z_k^2}{R_1}+\frac{z_{k+1}^2-z_k^2}{R_2}\right)+\frac{1}{3R_1R_2}\left(z_{k+1}^3-z_k^3\right)\right].$$

Similarly, I_1 and I_2 can be found.

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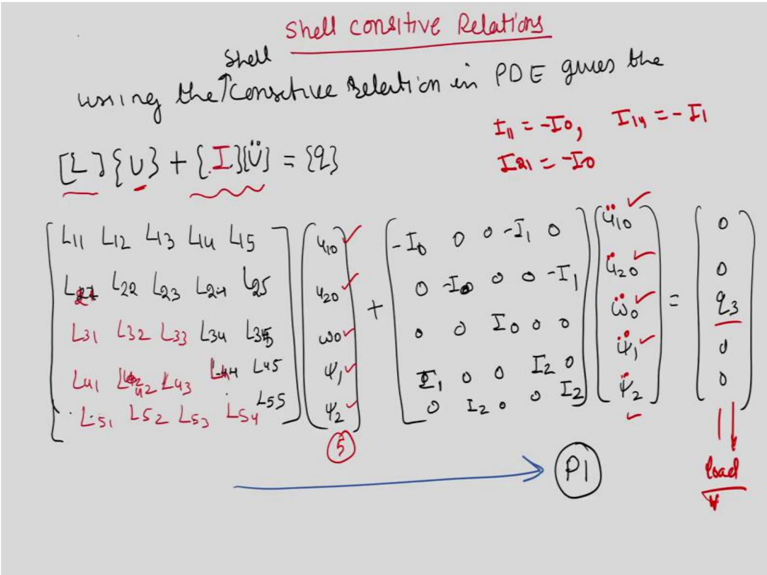

Now, using the shell constitutive relations, if we substitute into the 5 partial differential equations that lead to an equation (p1):

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} & L_{15} \\ L_{21} & L_{22} & L_{23} & L_{24} & L_{25} \\ L_{31} & L_{32} & L_{33} & L_{34} & L_{35} \\ L_{41} & L_{42} & L_{43} & L_{44} & L_{45} \\ L_{51} & L_{52} & L_{53} & L_{54} & L_{55} \end{bmatrix}\begin{bmatrix} u_{10} \\ u_{20} \\ w_0 \\ \psi_1 \\ \psi_2 \end{bmatrix}+\begin{bmatrix} -I_0 & 0 & 0 & -I_1 & 0 \\ 0 & -I_0 & 0 & 0 & -I_1 \\ 0 & 0 & I_0 & 0 & 0 \\ I_1 & 0 & 0 & I_2 & 0 \\ 0 & I_2 & 0 & 0 & I_2 \end{bmatrix}\begin{bmatrix} \ddot{u}_{10} \\ \ddot{u}_{20} \\ \ddot{w}_0 \\ \ddot{\psi}_1 \\ \ddot{\psi}_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ q_3 \\ 0 \\ 0 \end{bmatrix} equation-p1$$

$$[L]\{U\}+[I][\ddot{U}]=\{q\}$$

Where, $[L]$ is a linear operator, $[\ddot{U}]$ contains the primary displacement variables u_{10}, u_{20}, w_0, ψ_1, and ψ_2.

This time you see a dynamic matrix comes into the picture. Let us say a matrix I.

$I_{11} = -I_0$, $I_{14} = -I_1$ and $I_{21} = -I_0$.

These are the components of inertia matrix and time derivative of primary variables, $\ddot{u}_{10}$, $\ddot{u}_{20}$, $\ddot{w}_0$, $\ddot{\psi}_1$, and $\ddot{\psi}_2$ and load matrix q_3. It is a column matrix; the array is written like this. First, I shall explain the free vibration because I already explained the static response of a finite cylindrical shell subjected to all-round simply supported case.

In this lecture, I shall explain the free vibration of a finite circular cylindrical shell subjected to all-round simply supported boundary conditions.

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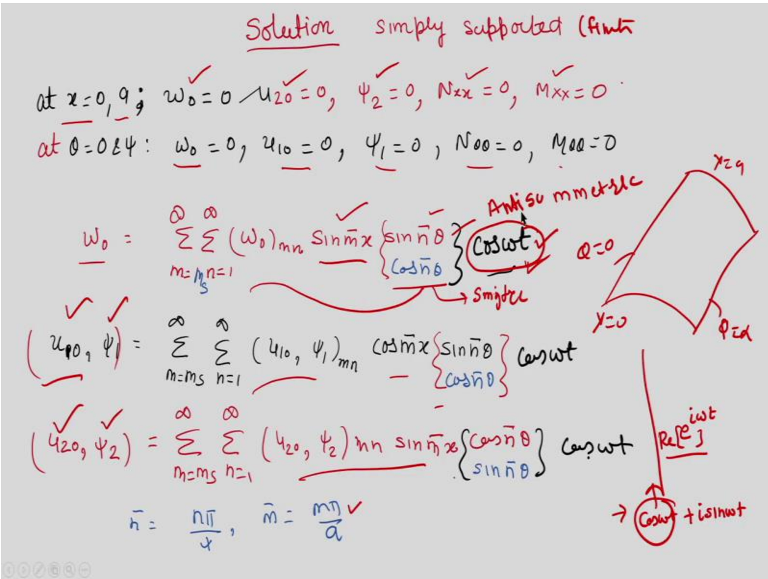

For the case of simply supported, a finite shell,

$$at\ x=0,a \quad w_0=0;\ u_{20}=0;\ \psi_2=0;\ N_{xx}=0;\ \ M_{xx}=0$$

$$at\ \theta=0,\alpha \quad w_0=0;\ u_{10}=0;\ \psi_1=0;\ N_{\theta\theta}=0;\ \ M_{\theta\theta}=0.$$

If these variables are specified, then we can assume the solution that along x axis:

$$w_0 = \sum_{m=ms}^{\infty}\sum_{n=1}^{\infty}(w_0)_{mn} \sin \bar{m}x \begin{Bmatrix} \sin \bar{n}\theta \\ \cos \bar{n}\theta \end{Bmatrix} \cos \omega t \text{, where } \bar{m} = \frac{m\pi}{a}.$$

The blue term $\cos \bar{n}\theta$ is for the symmetric case when loading is symmetric and $\sin \bar{n}\theta$ is for an anti-symmetric case.

Similarly, $(u_{10}, \psi_1) = \sum_{m=ms}^{\infty}\sum_{n=1}^{\infty}(u_{10}, \psi_1)_{mn} \cos \bar{m}x \begin{Bmatrix} \sin \bar{n}\theta \\ \cos \bar{n}\theta \end{Bmatrix} \cos \omega t$ and $(u_{20}, \psi_2) = \sum_{m=ms}^{\infty}\sum_{n=1}^{\infty}(u_{20}, \psi_2)_{mn} \sin \bar{m}x \begin{Bmatrix} \cos \bar{n}\theta \\ \sin \bar{n}\theta \end{Bmatrix} \cos \omega t$. Already, we are aware of this type of combination, the new thing here is $\cos \omega t$, it is a time derivative. Sometimes people used to take $\mathrm{Re}[\mathrm{e}]^{i\omega t}$.

If you open, it becomes $\cos \omega t + i \sin \omega t$. We are taking only this portion, instead of writing in this form, we can assume $\cos \omega t$. So, if we take that $\cos \omega t$ is the variation of variable w_0 in the time domain and the along the space domain, which satisfies the simply supported boundary conditions.

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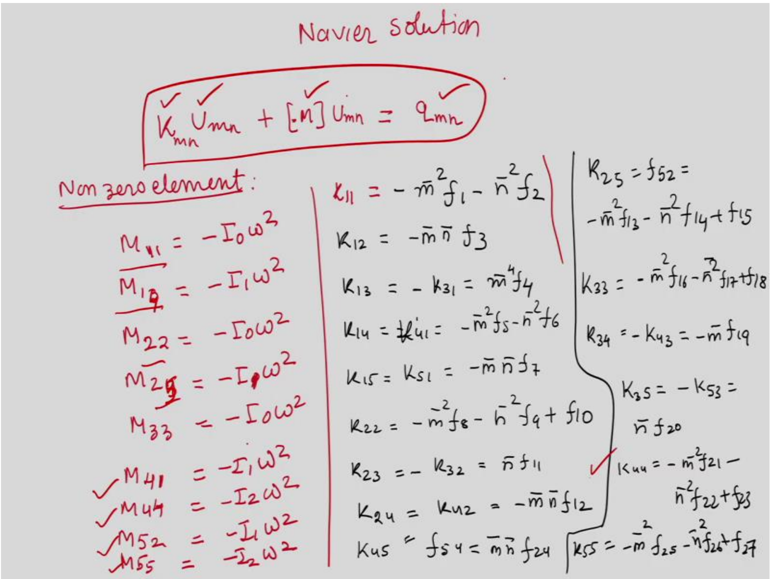

If we substitute this Fourier expansion and time expansion into the p1 equation, then it reduces to:

$$K_{mn}U_{mn} + [M]U_{mn} = q_{mn}.$$

The non-zero component of a matrix M will be:

$M_{11} = -I_0\omega^2$; $M_{14} = -I_1\omega^2$; $M_{22} = -I_0\omega^2$; $M_{25} = -I_1\omega^2$; $M_{33} = -I_0\omega^2$;
$M_{41} = -I_1\omega^2$; $M_{44} = -I_2\omega^2$; $M_{52} = -I_1\omega^2$; $M_{55} = -I_2\omega^2$

And, the stiffness matrix non-zero components will be:

$$K_{11} = -\bar{m}^2 f_1 - \bar{n}^2 f_2; \;\; K_{12} = -\bar{m}\bar{n} f_3; \;\; K_{13} = -K_{31} = -\bar{m}^2 f_5 - \bar{n}^2 f_6; \;\; K_{15} = K_{51} = -\bar{m}\bar{n} f_7;$$
$$K_{22} = -\bar{m}^2 f_8 - \bar{n}^2 f_9 + f_{10}; \;\; K_{23} = K_{32} = -\bar{n} f_{11}; \;\; K_{24} = K_{42} = -\bar{m}\bar{n} f_{12};$$
$$K_{25} = K_{52} = -\bar{m}^2 f_{13} - \bar{n}^2 f_{14} + f_{15}; \; K_{33} = -\bar{m}^2 f_{16} - \bar{n}^2 f_{17} + f_{18}; \;\; K_{34} = K_{43} = -\bar{m}\bar{n} f_{19};$$
$$K_{35} = K_{53} = -\bar{n} f_{20}; \;\; K_{44} = -\bar{m}^2 f_{21} - \bar{n}^2 f_{22} + f_{23} \;\; K_{45} = K_{54} = \bar{m}\bar{n} f_{24}; \;\; K_{55} = -\bar{m}^2 f_{25} - \bar{n}^2 f_{26} + f_{27}$$

For the case of free vibration, there will be no loading, therefore,

$$K_{mn} U_{mn} + M U_{mn} = 0 \;.$$

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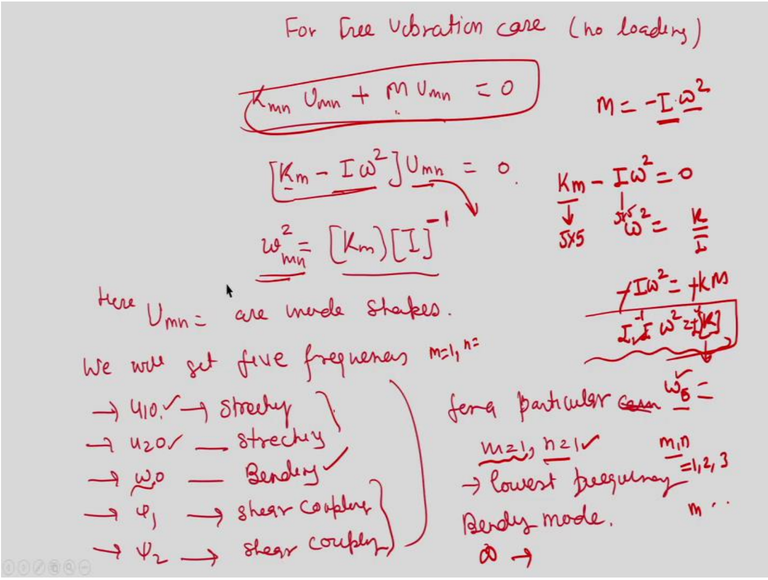

Ultimately, M = $-I\omega^2$.

K_m - $I\omega^2$ = 0, this is an eigenvalue problem where U_{mn} is the mode shape.

From here, w_{mn}^2 = $[K_m] - [I]^{-1}$, it is just a single equation like in the case of a beam. If it is just a single equation, then we can say that ω^2 = $\dfrac{K}{I}$, for the case of a beam or a spring. But here, K_m is a 5 by 5 matrix, and I is also a 5 by 5 matrix, therefore, we can say that $I\omega^2$ = - K_m , minus and minus get canceled, if we multiply with I^{-1}, then

$$I^{-1} I\omega^2 = I^{-1}[K].$$

It becomes an identity and, in this way, 5 different values of ω can be obtained at a time. For a particular combination of m and n, these are Fourier numbers that may vary from 1, 2, 3, and so on.

Let us say, for m is equal to 1, and n is equal to 1, we will get 5 frequencies at a time.

$$u_{10} \Rightarrow \textit{stretching}$$
$$u_{20} \Rightarrow \textit{stretching}$$
$$w_0 \Rightarrow \textit{bending}$$
$$\psi_1 \Rightarrow \textit{shear couples}$$
$$\psi_2 \Rightarrow \textit{shear couples}$$

For a particular combination of m is equal to 1, we will get frequencies in bending stretching and shear modes. If we increase that, we will get an infinite set of frequencies. When a cylindrical shell is subjected to all-round simply supported, the calculation of frequency is very easy like a case of a plate, but if we talk about a levy type support condition, then it is slightly difficult.

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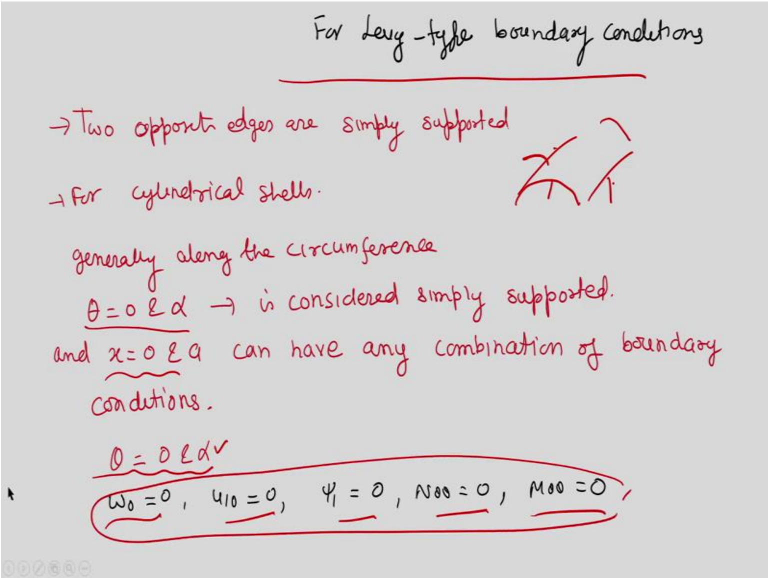

For the levy type boundary condition, the very first condition is that two opposite edges are simply supported. When we say that any structure is subjected to levy type support condition, in that case, we are saying that any two opposite edges either x = 0 to a or θ = 0 to α is to be simply supported. If that is the case, then we can develop a levy solution.

In the case of a cylindrical shell generally along θ direction, θ = 0 and α, is considered simply supported. Another axis; which is x = 0 and x = a, can have any combination of the boundary conditions. For the present case, if we assume that θ = 0 and α, then the following variables are needed to be 0:

$$w_0 = u_{10} = u_{20} = \psi_1 = \psi_2 = 0$$

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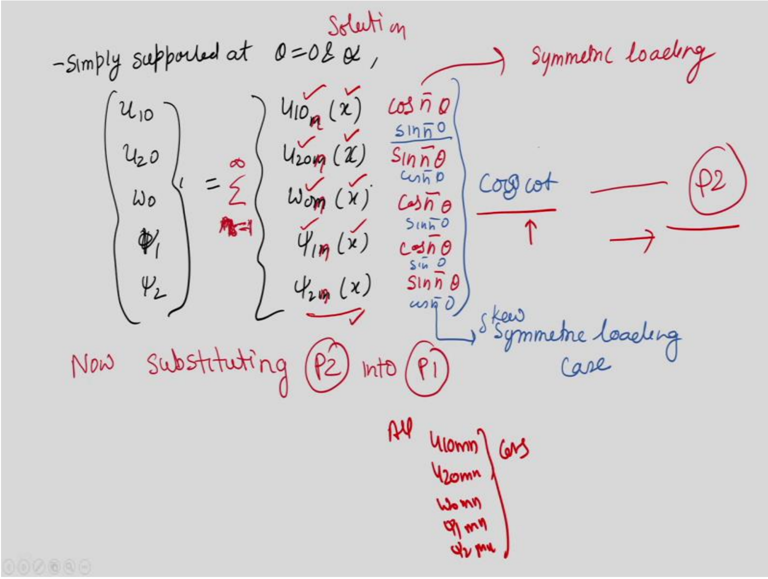

If this is the case, we can assume the primary variables in terms of a single series or sin series. The first case is due to the symmetric loading. In the case of free vibration, loading is not, but boundary conditions may be symmetric and if it is a static case, then loading may be symmetric or antisymmetric.

So, we can assume the solution like this:

$$\begin{bmatrix} u_{10} \\ u_{20} \\ w_0 \\ \psi_1 \\ \psi_2 \end{bmatrix} = \sum_{ms=1}^{\infty} \begin{bmatrix} u_{10n}(x) & \begin{Bmatrix} \cos\bar{n}\theta \\ \sin\bar{n}\theta \end{Bmatrix} \\ u_{20n}(x) & \begin{Bmatrix} \sin\bar{n}\theta \\ \cos\bar{n}\theta \end{Bmatrix} \\ w_{0n}(x) & \begin{Bmatrix} \cos\bar{n}\theta \\ \sin\bar{n}\theta \end{Bmatrix} \\ \psi_{1n}(x) & \begin{Bmatrix} \cos\bar{n}\theta \\ \sin\bar{n}\theta \end{Bmatrix} \\ \psi_{2n}(x) & \begin{Bmatrix} \sin\bar{n}\theta \\ \cos\bar{n}\theta \end{Bmatrix} \end{bmatrix} \cos\omega t \quad equation - p2$$

$\cos\omega t$ is the variation of time. Now, substituting this expression into again p1 equation will give this big expression.

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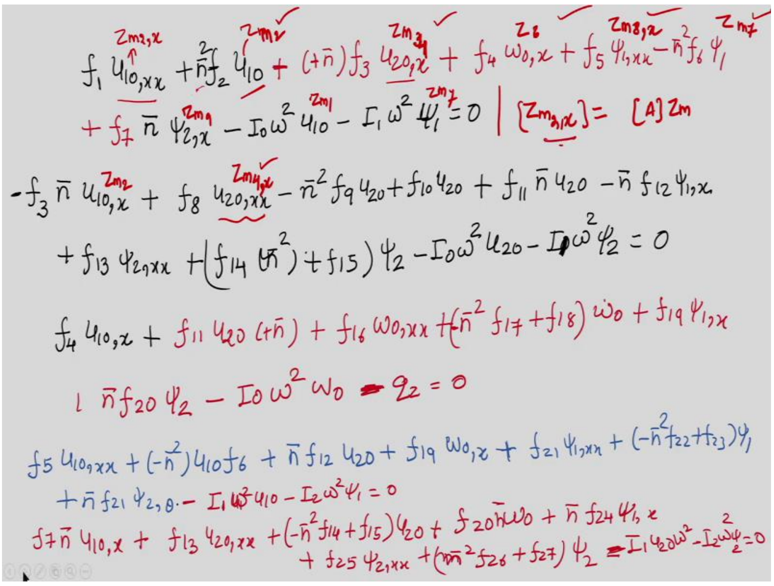

$$f_1 u_{10,xx} + \bar{n}^2 f_2 u_{10} + \bar{n} f_3 u_{20,x} + f_4 w_{0,x} + f_5 \psi_{1,xx} - \bar{n}^2 f_6 \psi_1 +$$
$$f_7 \bar{n} \psi_{2,x} - I_0 \omega^2 \psi_1 = 0 \quad equation-1$$
$$-f_3 \bar{n} u_{10,x} + f_8 u_{20,xx} + \bar{n}^2 f_9 u_{20} + f_{10} u_{20} + f_{11} \bar{n} u_{20} - \bar{n} f_{12} \psi_{1,x} + f_{13} \psi_{2,xx} +$$
$$\left(f_{14}\left(\bar{n}^2\right) + f_{15} \right) \psi_2 - I_0 \omega^2 u_{20} - I_1 \omega^2 \psi_2 = 0 \quad equation-2$$
$$f_4 u_{10,x} + f_{11} u_{20}\left(+\bar{n}\right) + f_{16} w_{0,xx} + \left(-\bar{n}^2 f_{17} + f_{18}\right) w_0 + f_{19} \psi_{1,x} +$$
$$\bar{n} f_{20} \psi_2 - I_0 \omega^2 w_0 - q_z = 0 \quad equation-3$$
$$f_5 u_{10,xx} + \left(-\bar{n}^2\right) u_{10} f_6 + \bar{n} f_{12} u_{20} + f_{19} w_{0,x} + f_{21} \psi_{1,xx} + \left(-\bar{n}^2 f_{22} + f_{23}\right) \psi_1 +$$
$$\bar{n} f_{21} \psi_{2,\theta} - I_1 \omega^2 u_{10} - I_1 \omega^2 \psi_1 = 0 \quad equation-4$$
$$f_7 \bar{n} u_{10,x} + f_{13} u_{20,xx} + \left(-\bar{n}^2 f_{14} + f_{15}\right) u_{20} + f_{20} \bar{n} w_0 + \bar{n} f_{24} \psi_{1,x} + f_{25} \psi_{2,xx} +$$
$$\left(\bar{n}^2 f_{26} + f_{27}\right) \psi_2 - I_1 \omega^2 u_{20} - I_1 \omega^2 \psi_2 = 0 \quad equation-5$$

Here one can see that the variation along θ direction vanishes and $\bar{n}^2$ comes into the picture.

And, in u_{10}; there is no derivative, but derivative along x-axis remains. If, I go to the previous slide: Here, u_{10}, u_{20}, w_0, ψ_1 all are a function of x. In the previous case: when all-round simply supported condition, u_{10mn}, u_{20mn}, w_{0mn}, ψ_{1mn}, ψ_{2mn}, all were constant.

If only two opposite edges are simply supported, then these are the function of the remaining variable along the x-axis. If we substitute in that partial differential equation that gives us an ordinary differential equation in the x coordinate.

It is a second-order highest degree in all the equations, you will find that the second derivative of x and the first derivative of x of all the variables exist in equations 1 to 5. The solution of this equation can be done in different ways.

One way is that we can directly solve, using some techniques, the second-order ordinary differential equations, or we can convert it into the most suitable form. This solution technique is presented by Khadir et al, in that paper, these equations were converted into a first-order form.

For the first-order differential form ordinary differential equation, the solution is straightforward.

We are trying to convert these 5 equations into a first-order differential equation.

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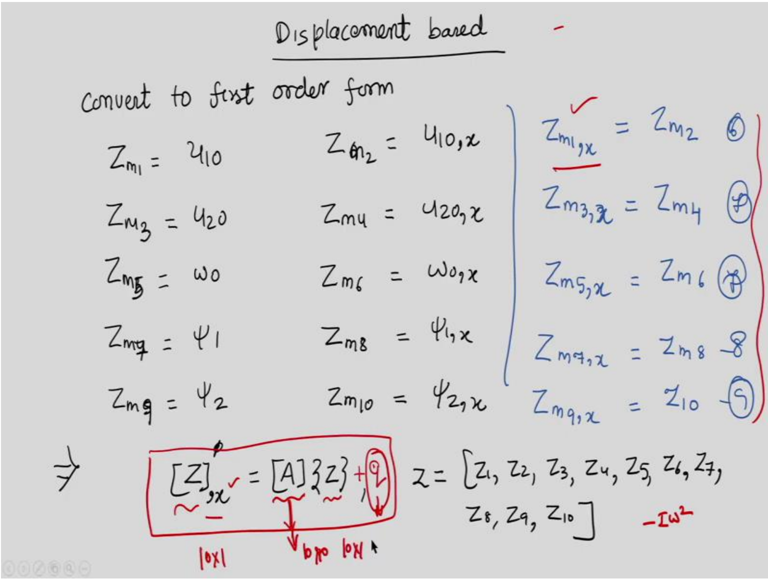

We are defining another set of variables, let us add a variable $Z_{m1} = u_{10}$, $Z_{m2} = u_{10x}$, first derivative is assumed as a variable Z_{m2}. Then,

$$Z_{m3} = u_{20} \quad Z_{m4} = u_{20,x}; \; Z_{m5} = w_0 \quad Z_{m6} = w_{0,x}$$

$$Z_{m7} = \psi_1 \;\; Z_{m8} = \psi_{1,x}; \;\; Z_{m9} = \psi_2 \quad Z_{m10} = \psi_{2,x}.$$

In this way we have defined 10 variables. We have 5 equations from this:

$$Z_{m1,x} = Z_{m2}; \;\; Z_{m3,x} = Z_{m4}; \;\; Z_{m5,x} = Z_{m6}; \;\; Z_{m7,x} = Z_{m8}; \;\; Z_{m9,x} = Z_{m10}$$

In equation (1):

$$f_1 u_{10,xx} + \bar{n}^2 f_2 u_{10} + \bar{n} f_3 u_{20,x} + f_4 w_{0,x} + f_5 \psi_{1,xx} - \bar{n}^2 f_6 \psi_1 + f_7 \bar{n} \psi_{2,x} - I_0 \omega^2 \psi_1 = 0$$

$u_{10,xx} = Z_{m2,x}$, $u_{10} = Z_{m2}$, $u_{20,x} = Z_{m4}$, $w_{0,x} = Z_{m6}$, $\psi_{1,xx} = Z_{m8,x}$, $\psi_1 = Z_{m7}$, $\psi_{2,x} = Z_{m10}$, $\psi_1 = Z_{m7}$.

We can see that the first equation using the concept of another variable Z, is converted to a first-order differential equation, where the highest derivative is along x-direction.

$$f_1 Z_{m2,x} + \bar{n}^2 f_2 Z_{m1} + \bar{n} f_3 Z_{m4} + f_4 Z_{m6} + f_5 Z_{m8,x} - \bar{n}^2 f_6 Z_{m7} + f_7 \bar{n} Z_{m10} - I_0 \omega^2 Z_{m7} = 0$$

From this, the first equation can be rewritten like this: $\left[Z_{m2,x}\right] = [A] Zm$, x is kept inside and the rest of the variables are kept outside because these are having no derivative. Similarly, the second equation will be expressed.

After mathematically simplification or rearranging, we will get:

$$[Z]_{,x} = [A]\{Z\} + q.$$

TZ contains 10 variables plus the load vector. Again, here the concept of $-I_0\omega^2$ is taken inside the matrix A.

This is the first-order ordinary differential equation with non-homogeneous and where A is the constant matrix, the solution of this equation is easy. This is a 10 by 10 matrix.

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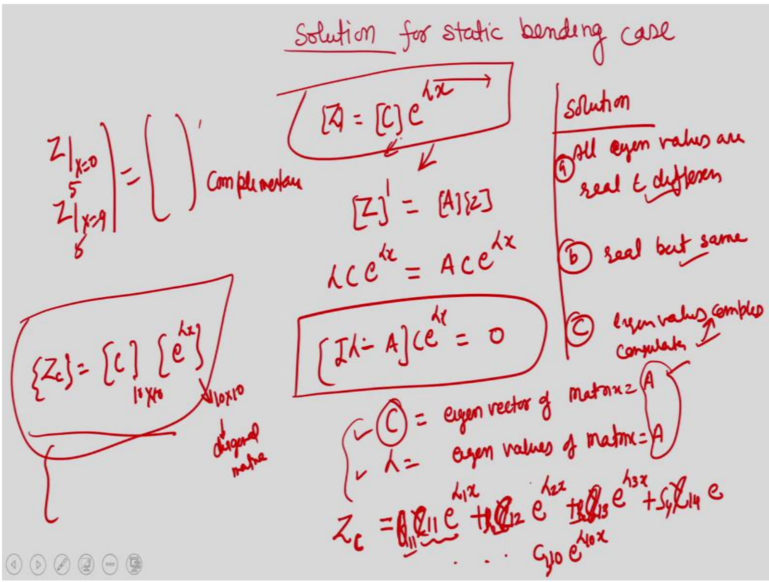

The standard solution can be written as $[Z] = [C]e^{\lambda x}$, where λ, I will explain later on.

If you substitute this complementary solution for the equation $[Z]^1 = [A][Z]$, if you substitute it here it becomes $\lambda C e^{\lambda x} = A C e^{\lambda x}$.

Ultimately, $[I\lambda - A]Ce^{\lambda x}$ = 0, this is an eigenvalue problem, here, C is eigenvector of matrix A, and λ is the eigenvalue of matrix A.

For the solution, we must know only the matrix A and we can get the eigenvector and eigenvalue of that matrix. Then, we can write the solution. Let us say, all eigenvalues are real and different, there will be 3 cases.

1st case: All eigenvalues are real and different.

2nd case: eigenvalues are real but identical

3rd case: eigenvalues are conjugate complex

For each case, the complementary solution will be different and this has been already explained in the book of mathematics or in the paper on 3-dimensional solutions where the first-order differential equation is solved. For the present case, I will write, eigenvalues are real and different.

In that case, we can write the solution:

$$Z_c = S_1C_{11}e^{\lambda_1 x} + S_2C_{12}e^{\lambda_2 x} + S_3C_{13}e^{\lambda_3 x} + S_4C_{14}e^{\lambda_4 x} + ... + S_{10}C_{10}e^{\lambda_{10} x}.$$

From λ_1 to λ_{10}, we have 10 eigenvalues and corresponding eigenvectors.

Ultimately, the complementary solution can be written as $ACe^{\lambda x}$, where C is having a 10 by 10 matrix and $e^{\lambda x}$ is a diagonal matrix of 10 by 10. In this way, a complementary solution can be found.

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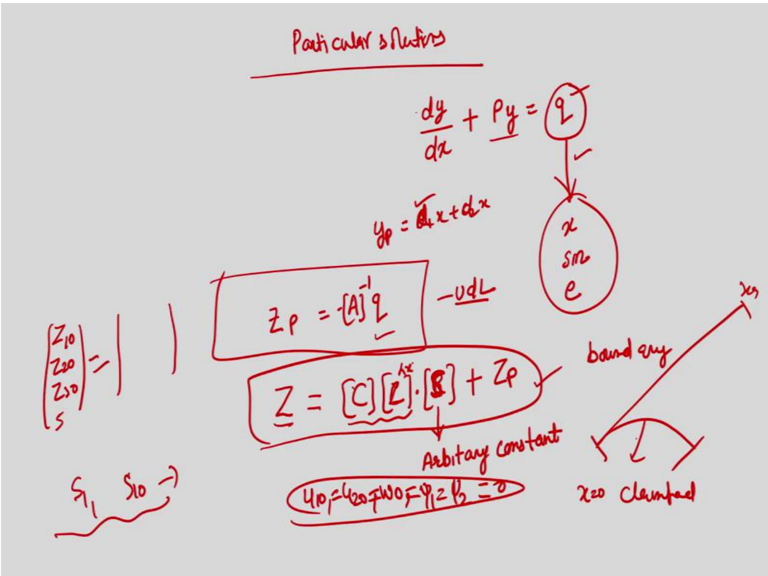

Now, the concept of a particular solution depends upon the type of loading.

I would like to explain with the help of a simple example, $\frac{dy}{dx} + py = q$, how do we assume a particular solution, if it is a constant or it is a function of x or it is a function of sin or exponential, the same way we assume a particular solution of that thing, let us say, $y_p = d_1 x + d_2 x$.

In the same way, if the loading is uniform, for that case: $Z_p = -[A]^{-1} q$, but if the load is something else, then, we have to evaluate the particular solution.

Now, we can say that the total solution will be:

$Z = [C][e]^{\lambda x}.[S] + Z_p$, where, Z_p is the arbitrary constant.

These are the constant that comes into the picture when we write a solution. These constants S can be found by implementing the boundary conditions. If I say that along x = 0 and x = a, x = 0: for the clamped case; the variables $u_{10} = u_{20} = w_0 = \psi_1 = \psi_2 = 0$

We can make a solution matrix, let us say, it will come up Z_{10}, Z_{20}, Z_{30} all are 0 and putting x = 0, we will get these 5 variables. And, when you put x is equal to a, maybe at some other boundary condition or clamped case there will be 5 variables again.

This will be a 5 by 5 matrix; a new combination comes up there. From here, this S_{11}, S_{10} can be solved. Arbitrary constants can be found by subjecting the boundary conditions in this equation.

Once we know the boundary conditions, we can sort them out and the final solution can be written like this. In this way, the problem is solved for the case of levy, when it is having a transverse load. For the case of a free vibration, there will be no load.

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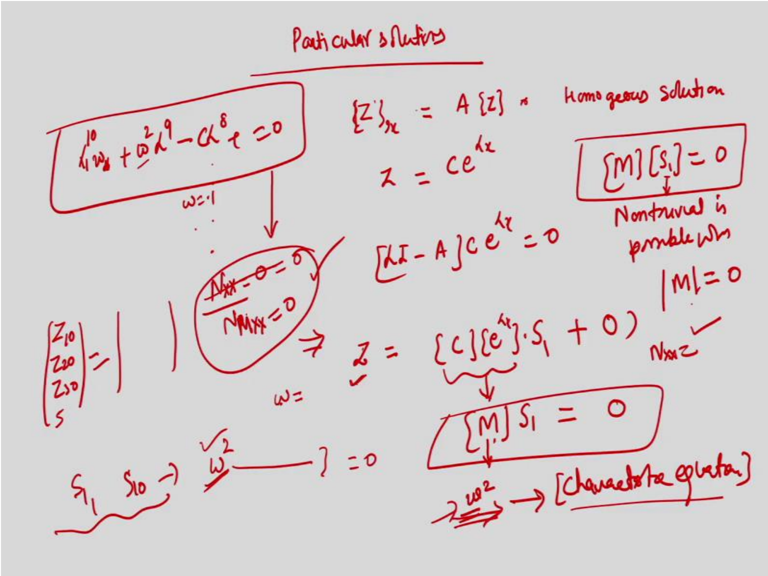

In that case, $\{Z\}_{,x} = A\{Z\}$,there will be no loading, it will be only a homogeneous solution.

Again, we can assume a solution, $Z = Ce^{\lambda x}$, and it becomes $[\lambda I - A]Ce^{\lambda x} = 0$. Finally, we can write a solution of Z = $[C][e^{\lambda x}]S_1 + 0$.

If we subject all these boundary conditions; let us say $[M] = 0$, in this M we have ω term, ω^2 is there. We do not know the natural frequency of the system, we have a system like this $[M][S_1] = 0$, a homogeneous equation. The non-trivial solution is that $[S_1] = 0$.

If all constants are 0, then what is the fun of doing these things? $[S_1]$ cannot be 0, we are trying to say that, the non-trivial solution is possible only when the determinant of M = 0. In this matrix, there is ω, if you put a determinant of M = 0, it leads to an equation that is known as the characteristic equation.

We solve that equation through an iterative process or you can directly solve because these days some algorithms are available in MATLAB. If you give this characteristic equation, they will give you the roots and you can write the code. If it is ω^2 inside the equation, and we do not know the ω, we will try to solve it by substituting the initial guess. $\lambda_1^{10}\omega + \omega^2\lambda^9 - C\lambda_e^8 = 0$.

If ω is the solution of this equation, then it will satisfy the equation. Otherwise, it will not satisfy this equation. Let us say, w = 0, 1, 2, and 3, then through iterative processor, we find it. There are some algorithms in which we can find the solutions by processing i.e., moving from initial guess to final.

Generally, for these cases, we try to guess the simply supported condition. A simply supported shell will have a higher natural frequency compared to a clamped case. We have to reduce the guess to get the frequencies. This is the technique where the boundary conditions are to be satisfied in the displacement form.

If we say that the cylindrical shell is free, in that case: N_{xx}, M_{xx} need to be specified.

For the present case: we do not have variables of N_{xx}, M_{xx}. We need to satisfy in an average sense, which means using the concept of constitutive relations we satisfy.

In lecture-1, week 6, I said that in an extended Kantorovich method a mixed form is used. Similarly, for the case of a levy solution, the mixed formulation can be used. In that case, we assume that N_{xx}, M_{xx} are our variables.

And then using the 5 equations from the shell constitutive relations, 5 equations from the partial differential equation, or maybe some more variables, these form a first-order differential equation, then we can solve the problem.

The only advantage of using a mixed formulation is that those boundary conditions are satisfied. But, in the present displacement case: N_{xx}, M_{xx} are not our variables, using the shell constitutive relations they are satisfied.

From an accuracy point of view, there is no difference, the only difference comes up near the support at the very edge. If you plot a circumferential variation along that very clamped edge or at a very free edge, then some boundary effect may appear. To accurately predict that behavior the mixed formulation is preferred mostly.

Today, I explained to develop a Levy solution for a circular cylindrical shell for static bending case and free vibration case.

Thank you very much.