Theory of Composite Shells
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Week - 07
Lecture - 01
Shell governing equation

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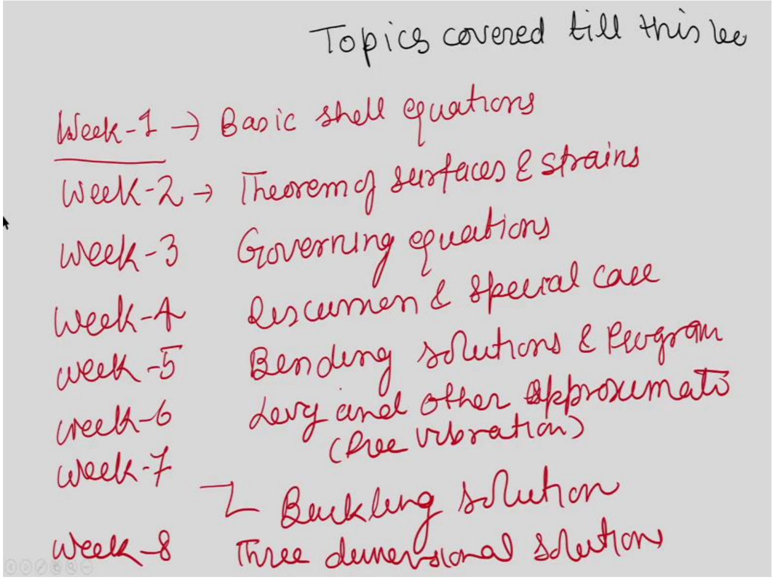

Dear learners once welcome to week-07, lecture-01. Before going to this, I briefly discuss whatever we have covered to date. In week-01, I discussed the composites, material, and the basic Shell equations. Week-02 was related to the development of the theorem of surfaces and strains.

Under week-03, governing equations for a doubly curved shell were developed. Under week-04, discussions and various special cases have been discussed. In week-05, the bending solution and MATLAB program was elaborated.

In week-06, the extended Kantorovich method and Levy solutions were presented. And in this week-07, I am going to discuss the buckling of the shells. And in week-08 I shall develop the three-dimensional solutions.

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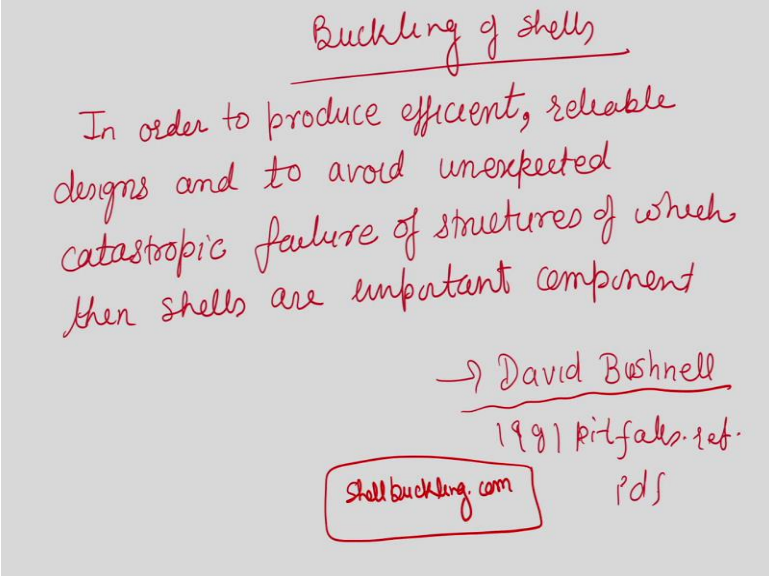

First of all, why do we want to study the buckling of shells? This is the statement given by David Bushnell, he is a pioneer done a lot of work in shell buckling, both experimentally as well as theoretical.

And there is a website called shellbuckling.com is devoted to the analysis of buckling of various types of shells, papers and other experimental works are discussed in detail on shellbuckling.com, it was prepared by David Bushnell.

To produce efficient or reliable design and to avoid unexpected catastrophic failure of structure shell are important components. It means that the shells are the important component, we are interested to produce efficient reliable designs. It is very important to study buckling in the shell as compared to plates or beams because shells are very thin and buckling may occur due to a variety of reasons.

In the case of a plate or case of a beam, it is due to the axial compressive load. But in the case of a shell, it is beside the axial compressive load the buckling may be due to the external pressure or the combined effect.

There may be a local buckling due to the concentrated load or may be due to some geometrical imperfections, earthquake, temperature, initial stresses, or residual stresses. There are a variety of reasons due to which buckling may happen.

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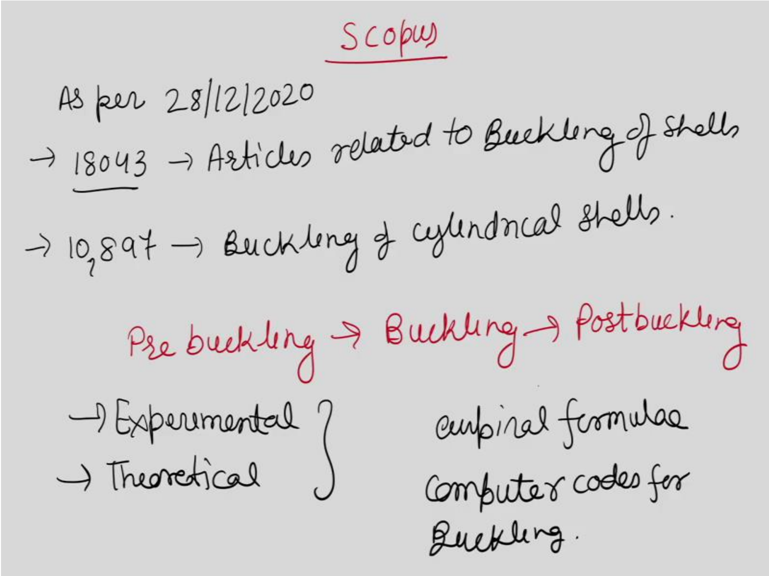

Before going into the subject of buckling of shell, let us review, today is 28 December, I am recording this video, it is showing there are 18000 articles in which the buckling of shells is discussed, and they may be theoretical, experimental, functionally graded nanocarbon tubes, isotropic metals, or maybe of a different kind due to imperfections or different geometry and different boundary conditions. Shells may be toroidal shells, spherical shells, cylindrical shells, or any kind of shell. It is shown that 18000 articles are divided, out of which 10000 articles are related to the buckling of cylindrical shells.

You can see that a very huge amount at least 50% of work is related to the buckling of cylindrical shells. Buckling of the cylindrical shell is a very important topic. Previously, in the case of bending, I said that a cylindrical shell is a most-simple shape, a lot of work has been done in that direction.

But when we talk about buckling, I would like to say that cylindrical shell is not simple when you compare the experimental results with the theoretical results there is a high mismatch. That is the reason a small imperfection in the cylinder may cause an entirely different state of buckling.

A major reason is that whatever we are predicting, let us say, the critical load of buckling, practically your actual situations happen very early as predicted. Buckling may be analysed in 3 steps, one is the pre-buckling (just before the buckling), 2^{nd} is buckling, and 3^{rd} is post-buckling (after the buckling).

I think in plates nobody studied about the post-buckling, very less work is related to post-buckling or pre-buckling, you have only heard about the buckling. But in the shells, these terms are very frequently used and a lot of paper is devoted to all 3 stages of buckling of the shell.

So, I think in plates nobody studied about the post-buckling, very less work is related to post buckling or pre-buckling kind of thing you have only heard about the buckling.

There may be some experimental or theoretical works are available, maybe some empirical formulas are used. After the 1980s or 1990s computer codes were used to study the buckling of shells.

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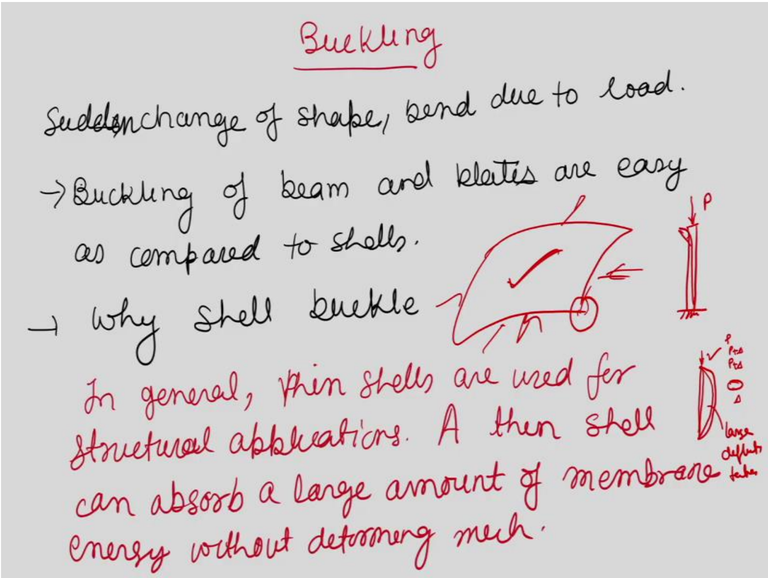

What do you mean by a buckling?

When we call a shell is buckled or a plate is buckled; I think at the undergraduate level the concept of column buckling is present, if a column is there and if you apply an axial pressure P at any boundary condition, it may try to bend like this.

let us say, this is a column and you are trying to put the axial pressure over there, you are increasing the pressure by delta. After some time, it reaches a point where it changes its shape. This means a large deflection takes place. A sudden change of a shape or a bending due to a load is called buckling.

Buckling of beams and plates are easy as compared to shell. In the case of a beam and plate buckling is comparatively very easy, only one type of buckling for compressive stress can be done. But in the shell, why the buckling in the shells is difficult, and why do shells buckle more frequently than the plate?

The reason behind that is thin shells are used for structural applications, these can absorb a large amount of membrane energy without deforming much. Because when we say that the shell is thin for membrane theory of shell, where the shell is subjected to in-plane stretching can store more membrane energy. Therefore, there will be less deformation.

But, due to some loading conditions, boundary conditions, and some geometrical parameters, if this membrane energy is converted into bending energy, it resists very little amount of bending energy. Due to the bending effect, large deflections may take place, but the shells cannot absorb more bending energy.

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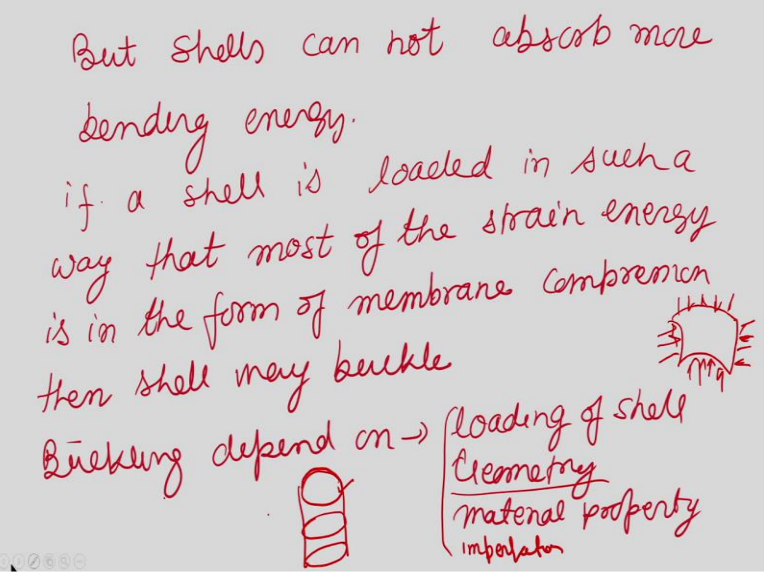

If a shell is loaded in such a way that most of the strain energy is in the form of membrane compression, then the shell may buckle, which means when you are going to subject a shell under the compressive stress, then the shell may buckle. It can convert the strain energy or the membrane energy into bending energy.

Buckling of a shell depends upon the loading of the shell, geometry of the shell, and the material properties of the shell. These are the important parameters. And further, we can say that in the shells during the manufacturing imperfection occur. Let us say, it is a circular cylinder, but along the length it may be slightly oval, it may not be a complete circle and this effect is known as imperfection causing a buckling. And the shell is thin, generally, they buckled more frequently.

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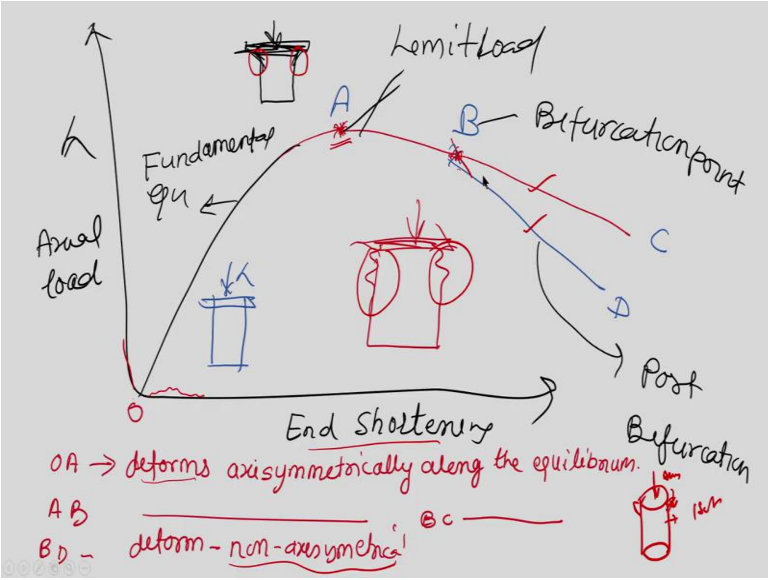

There is a graph of axial loading and the end shortening, let us say, we have a very thin cylindrical metal shell.

In this lecture first, I will discuss isotropic shells, how the buckling loads are affecting the end shortening or the behavior of the isotropic shells. Then, I shall discuss the behavior of composite shells. Let us say, a metal cylinder or an isotropic cylinder is subjected to axial stress.

If you increase the axial stress, slightly its length is decreasing at a point, it follows a fundamental equation of equilibrium. If you remove that, if you reach a point A that is known as limit load, there it starts buckling, slightly bulging will come near the edge.

Near the edge, at point, A bulging will take place. A state is reached that is known as limit load, and if you start further increasing, then a point B comes and that point is the bifurcation point. Why there is this bifurcation point? It may follow two routes one route is B to C another route is B to D.

Let us say, from O to A shell deforms axis symmetrically along the equilibrium. Then A to B deforms axis symmetrically, but B to C if it follows this path, it bifurcates and further deforms axis symmetrically.

But if it follows a path B to D, it may deform non-axis symmetrically and the behavior looks like this, these are no longer remain axis symmetry. Depending upon the imperfection and many other parameters, they may follow any path, but we can check through the experiment. From this point, they may go to two different routes.

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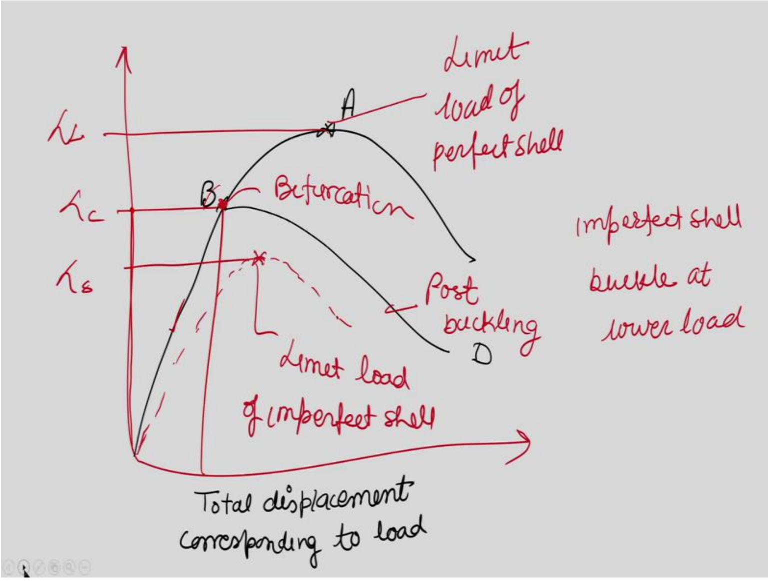

There is a graph of axial load limits and total displacement corresponding to the load. If a cylinder is a perfect shell, then a black curve will follow a point B, an early bifurcation point, and point A limit load comes later.

When we have an imperfection, which means instead of circular it is slightly oval or some defects are there, then it will follow a dotted line which is far below this black line. And lambda S is corresponding to limit load to the imperfection, and lambda C is bifurcation load, and lambda L is the limit load. This is the graph. Now, we can study both perfect and imperfect shells. In real life all shells are imperfect.

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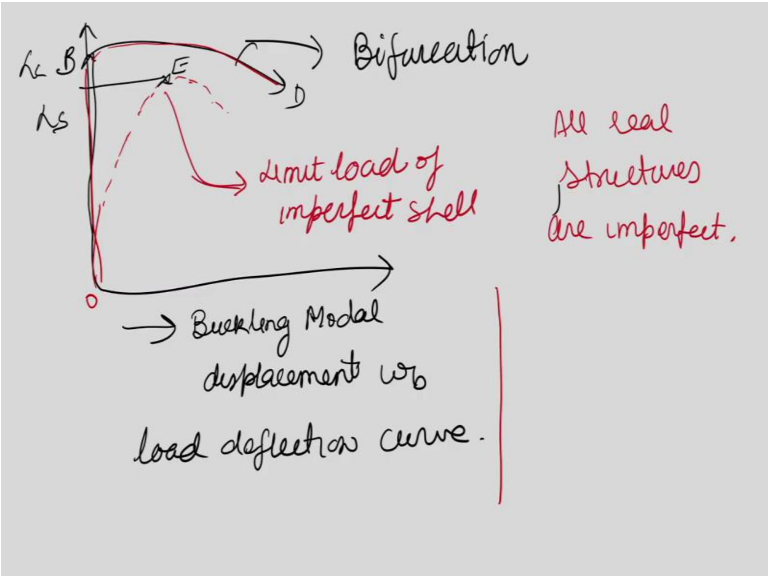

We can see this graph, when it is linear the initial displacements are very small. We can say that up to point B it is starting from here only because the deflection is very small. We can see that this graph is represented like 0 to B straight line and the red line B to D is for the imperfect shell. the red line.

From this graph, it is seen that for the imperfect shells the buckling load is smaller as compared to the theoretical perfect shell, that is why there is a mismatch between the theoretical and experimental results because theoretically, we assume it is a perfect cylinder, there are no imperfections or the boundary conditions are perfectly applied if you say clamped or free, but in a real situation, this may not be true.

For those cases, the buckling load may be different. And it has been experimentally observed that buckling load is very low as compared to the theoretical load.

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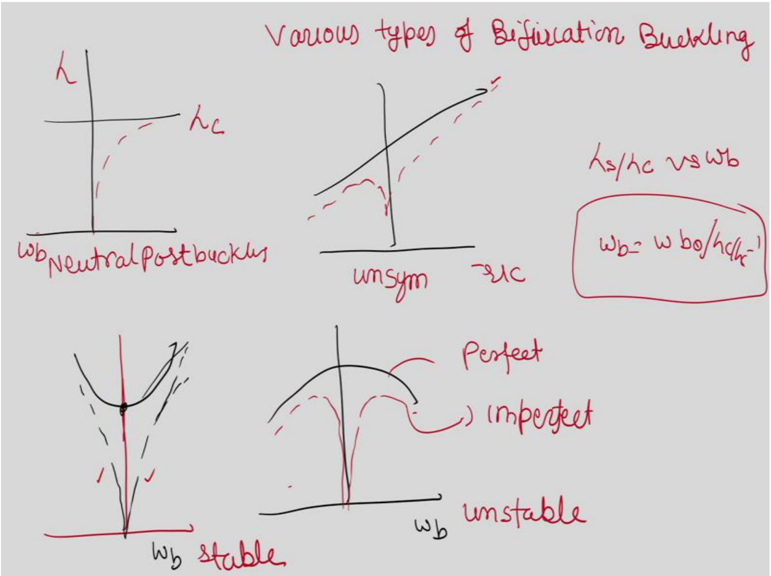

There are various types of bifurcations buckling; when we have a neutral post-buckling, then the graph will look like this for imperfect one black one is for the perfect case.

When we have unsymmetric post-buckling, then the black line is for a perfect cylinder and the dotted one is for the imperfect one. Then, we have stable, symmetric, and unstable. In this way, the different bifurcation processes are defined for the case of isotropic cylinders.

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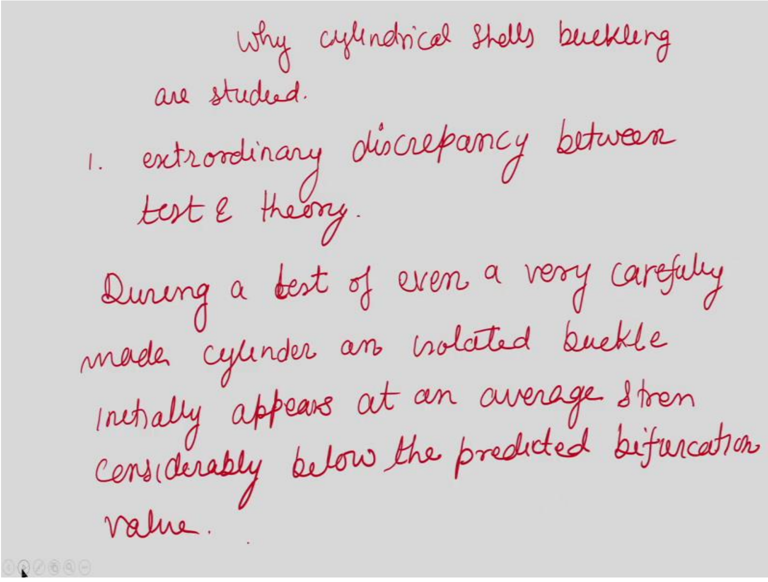

When we talk about the orthotropic or a laminated one, then we have to think because the composite laminates are brittle in nature, they are not like metal, elastic-plastic zone, or a local buckling effect. Before going there during a test of even a very carefully made cylinder on an isolated buckling initially appears at average stress considerably below the predicted bifurcation value.

There are many works presented in the literature, it has been done that they have developed a very perfect cylinder in the laboratory and tested on the ideal conditions, still, the results were far away from the theoretical one. It is not only the imperfections but still, there are some parameters which are governing the actual buckling of the shells.

Buckling of a shell is very important and most of the time very big cylindrical tanks, water tanks, LPG storage tanks, nuclear reactor tanks, all are very big tanks if we do not study the buckling effect very carefully, even at a very small load they may buckle and may cause hazardous effect when it is gasoline or a nuclear reactor, then it is harmful to nearby areas also.

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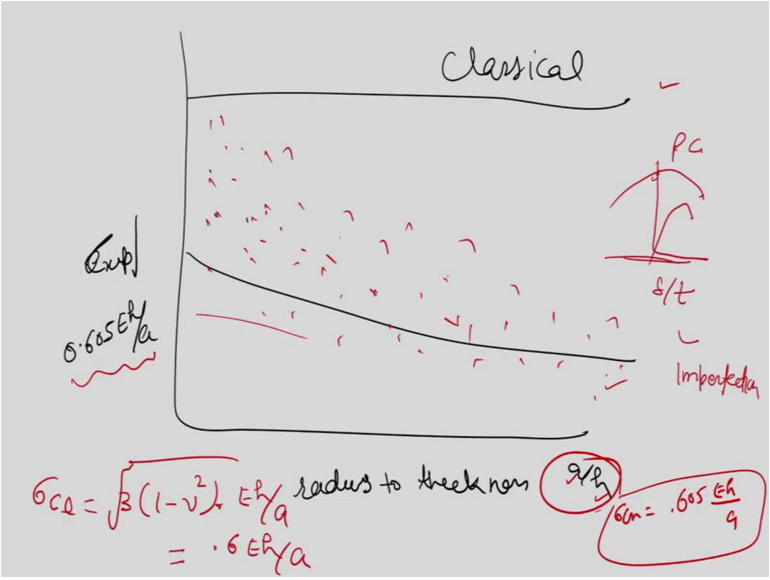

We can see in one of the graphs, I have taken from the David Bushnell paper the classical theory predicted this straight line whereas, the experimental results come here. You can see a very low range and then the experimental oblique theoretically is coming $\frac{0.605Eh}{a}$, where $\frac{a}{h}$ is the radius to thickness ratio, a is known as a radius. Critical load theoretically is predicted like this, it is coming very low.

In most of the books, if you see in the literature the experimental buckling of shells is discussed a lot. From the starting, I think the 1970s or 60s, even after the 1950s a lot of work has been done on the experimental buckling of the shells.

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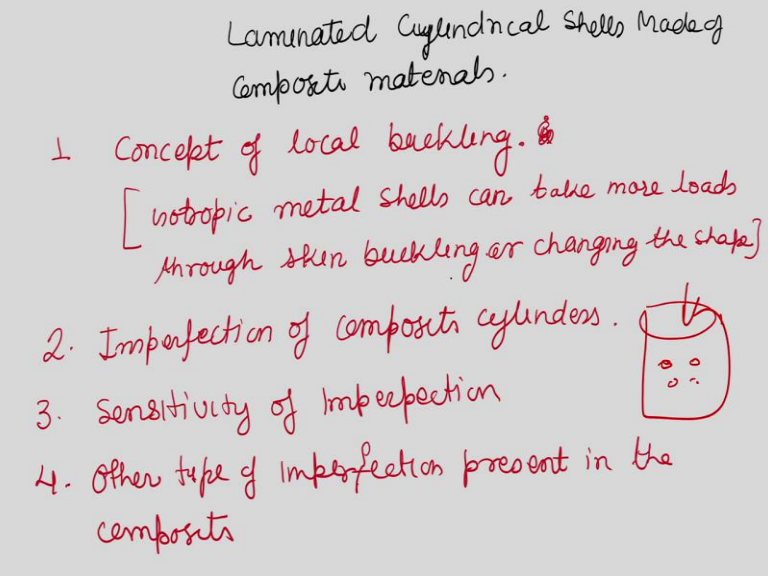

Now, I am going to discuss the composite laminated cylindrical shells which are made of composite materials.

In the metal or the isotropic materials because they are elastic materials the concept of local buckling comes into the picture. An isotropic metal or the shell can take more loads through skin buckling or the local buckling effect. It is not going to fail the structure, just a small dent appears if this is your cylinder. Because of this may be small wrinkles kind of thing may come up.

These may take more load as compared to the designed one and can even perform their functions without fail.

But as we know that the composite materials are not elastic or ductile, these cannot take more loads because the phenomena of local buckling may not happen in that case.

And then the concept of imperfections of the composite cylinders; in the case of metals, we know that types of imperfections may happen, but in the case of composite cylinders there may be different types of imperfections may present, due to the variability of angles or delamination or the voids or maybe some other effects.

Due to those, imperfections may come up, that need to be studied and a lot of work has been done in that direction. Then, the third concept is the sensitivity of imperfections. I have not discussed that for the case of isotropic material the imperfections play a very important role and specifically the cylindrical shell is very very sensitive to the imperfections a slight imperfection may lead to a critical buckling load.

In this way, some points need to be addressed, when we study the buckling of cylindrical shells. A lot of work has been done.

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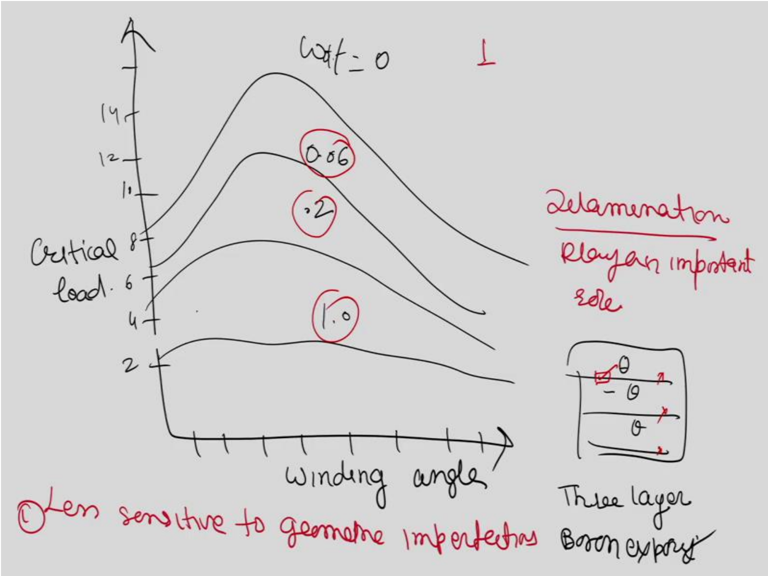

And through the course, it has been found that laminated shells or composite shells are less sensitive to imperfections. And there was a graph presented between a winding angle which means the fiber angle and the critical load the graph is plotted taking different imperfection factors.

And it is showing that these are less sensitive to geometric imperfection. But the concept of delamination between the layers. If there is small delamination that takes place in that area that may cause a loss in stiffness and further may be changed in the buckling load.

From the theoretical model, the way we have developed is similar there is not much difference, but experimentally, other factors can be analyzed and can be found. In the case of buckling semi-empirical formulas were designed, the designers can use those semi-empirical formulas to know the critical buckling loads to design the thickness of the shell or to analyze the shell under different loading boundary conditions.

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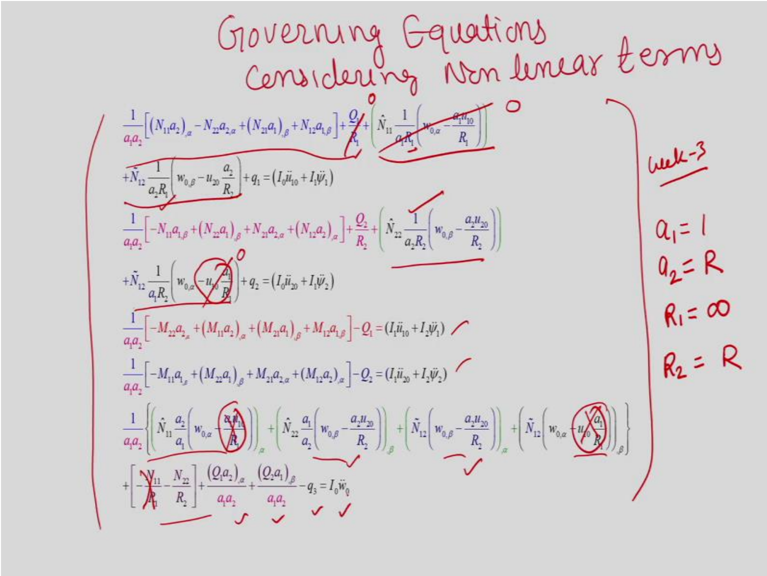

I will first discuss a standard formulation, partial differential equations:

$$\left[\left(N_{11}a_2\right)_{,\alpha}-N_{22}a_{2,\alpha}+\left(N_{21}a_1\right)_{,\beta}+N_{12}a_{1,\beta}\right]+\frac{Q_1}{R_1}+\left(N_{11}\frac{1}{a_1R_1}\left(w_{0},_{\alpha}-\frac{a_1u_{10}}{R_1}\right)\right)$$

$$+N_{12}\frac{1}{a_2R_1}\left(w_{0},_{\beta}-u_{20}\frac{a_2}{R_2}\right)+q_1=\left(I_0\ddot{u}_{10}+I_1\ddot{\psi}_1\right)$$

$$\frac{1}{a_1a_2}-N_{11}a_{1,\beta}+\left(N_{22}a_1\right)_{,\beta}+N_{21}a_{2,\alpha}+\left(N_{12}a_2\right)_{,\alpha}+\frac{Q_2}{R_2}+\left(N_{22}\frac{1}{a_2R_2}\left(w_{0},_{\beta}-\frac{a_2u_{20}}{R_2}\right)\right)$$

$$+N_{12}\frac{1}{a_1R_2}\left(w_{0},_{\beta}-u_{10}\frac{a_1}{R_1}\right)+q_2=\left(I_0\ddot{u}_{20}+I_1\ddot{\psi}_2\right)$$

$$\frac{1}{a_1a_2}\left[-M_{22}a_{2,\alpha}+\left(M_{11}a_2\right)_{,\alpha}+\left(M_{21}a_1\right)_{,\beta}+M_{12}a_{1,\beta}\right]-Q_1=\left(I_1\ddot{u}_{10}+I_2\ddot{\psi}_1\right)$$

$$\frac{1}{a_1a_2}\left[-M_{11}a_{1,\beta}+\left(M_{22}a_1\right)_{,\beta}+M_{21}a_{2,\alpha}+\left(M_{12}a_2\right)_{,\alpha}\right]-Q_2=\left(I_1\ddot{u}_{20}+I_2\ddot{\psi}_2\right)$$

$$\frac{1}{a_1a_2}\left[\begin{array}{l}\left(N_{11}\frac{a_2}{a_1}\left(w_{0},_{\alpha}-\frac{a_1u_{10}}{R_1}\right)\right)_{,\alpha}+\left(N_{22}\frac{a_1}{a_2}\left(w_{0},_{\beta}-\frac{a_2u_{20}}{R_2}\right)\right)_{,\beta}\\+\left(N_{12}\left(w_{0,\beta}-u_{20}\frac{a_2}{R_2}\right)\right)_{,\alpha}+\left(N_{12}\left(w_{0,\alpha}-u_{10}\frac{a_1}{R_1}\right)\right)_{,\beta}\end{array}\right]+$$

$$\left(-\frac{N_{11}}{R_1}-\frac{N_{22}}{R_2}\right)+\frac{\left(Q_1a_2\right)_{,\alpha}}{a_1a_2}+\frac{\left(Q_2a_1\right)_{,\beta}}{a_1a_2}-q_3=I_0\ddot{w}_0$$

And in lectures 4 and 5, we studied the bending-free vibration by considering only the linear terms. We did not consider non-linear terms we took those 0. But for the case of buckling, we are going to consider these non-linear terms.

In the first equation, we have two terms, in the second equation also we have two terms, in the third and fourth we do not have them, but in the fifth equation, 4 terms are there. These are the governing equations with non-linear terms.

If somebody is interested to study a generalized buckling behavior of a doubly curved shell then these equations are perfect and one can use that.

The cylindrical shells are studied mostly and for the present case, for explanation point of view, I will also use the cylindrical shell. For the cylindrical shell, these governing equations are reduced.

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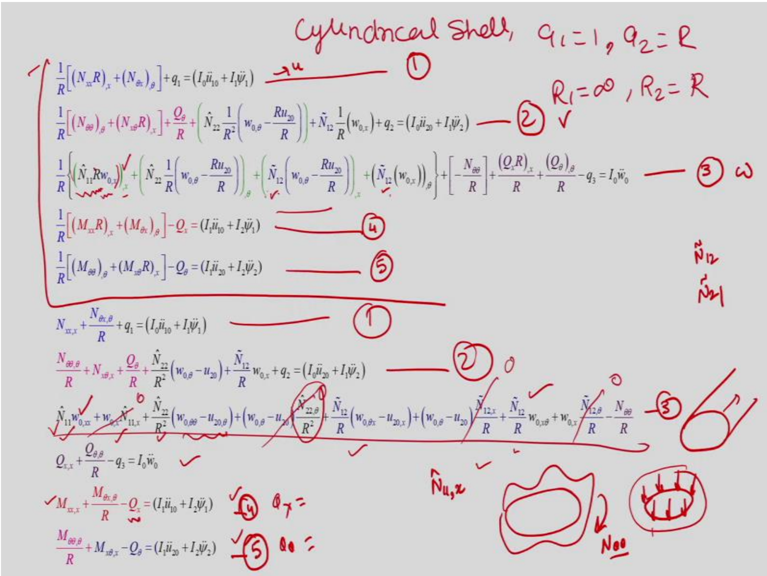

For the cylindrical shell, these governing equations are reduced. In the case of cylindrical shell lame parameter a_1 = 1, a_2 = R, R_1 = ∞, and R_2 = R.

If we follow this, the term $N_{11}\frac{1}{a_1R_1}\left(w_0,_\alpha - \frac{a_1u_{10}}{R_1}\right)$ will not contribute because it $\frac{a_1}{R_1}$.

Again, $N_{12}\frac{1}{a_2R_1}\left(w_0,_\beta - u_{20}\frac{a_2}{R_2}\right)$ will not contribute. In the first equation for the case of a cylindrical shell, there will be no contribution of non-linear terms as well as bending shear stress, but in the second equation $N_{22}\frac{1}{a_2R_2}\left(w_0,_\beta - \frac{a_2u_{20}}{R_2}\right)$ will contribute and

$u_{10}\frac{a_1}{R_1}$ will not contribute.

Similarly, in the fifth equation $\frac{a_1u_{10}}{R_1}$ will not contribute, $\frac{a_2u_{20}}{R_2}$ will contribute, $u_{20}\frac{a_2}{R_2}$ will contribute, $u_{10}\frac{a_1}{R_1}$ will not contribute, and $\frac{N_{11}}{R_1}$ will not contribute. By following this procedure, the final governing equations are represented like this:

$$\frac{1}{R}\left[\left(N_{xx}R\right)_{,x}+\left(N_{\theta x}\right)_{,\theta}\right]+q_1=I_0\ddot{u}_{10}+I_1\ddot{\psi}_1$$

$$\frac{1}{R}\left[\left(N_{\theta\theta}\right)_{,\theta}+\left(N_{x\theta}R\right)_{,x}\right]+\frac{Q_\theta}{R}+\left(\frac{N_{22}}{R^2}\left(w_0,_\theta-u_{20}\right)\right)+N_{12}\frac{1}{R}\left(w_0,_x\right)+q_2=\left(I_0\ddot{u}_{20}+I_1\ddot{\psi}_2\right)$$

$$\frac{1}{R}\left[\left(N_{11}Rw_{0,x}\right)_{,x}+\left(N_{22}\frac{1}{R}\left(w_0,_\theta-\frac{Ru_{20}}{R}\right)\right)_{,\theta}+\left(N_{12}\left(w_{0,\theta}-\frac{Ru_{20}}{R}\right)\right)_{,x}+\left(N_{11}Rw_{0,x}\right)_{,\theta}\right]$$

$$+\left(-\frac{N_{\theta\theta}}{R}\right)+\frac{\left(Q_xR\right)_{,x}}{R}+\frac{\left(Q_\theta\right)_{,\theta}}{R}-q_3=I_0\ddot{w}_0$$

$$\frac{1}{R}\left[\left(M_{xx}R\right)_{,x}+\left(M_{x\theta}\right)_{,\theta}\right]-Q_1=I_1\ddot{u}_{10}+I_2\ddot{\psi}_1$$

$$\frac{1}{R}M_{\theta\theta,\theta}+\left(M_{\theta x}\right)_{,x}-Q_\theta=I_1\ddot{u}_{20}+I_2\ddot{\psi}_2$$

I have done some changes here, in literature in most of the books the equation number fifth in the present case is kept at third position. The first equation is corresponding to u, the second equation is corresponding to v, the third equation is corresponding to w, the fourth is corresponding to ψ_1, and the fifth equation is corresponding to ψ_2, rotation variables. You can see in the third equation $\left(N_{11}Rw_{0,x}\right)_{,x}$ is the whole derivative with respect to x.

If we open it, R will get canceled it is not a function of x then it will be:

$N_{11}w_0,_{xx} + w_0,_x N_{11,x}$.

Mathematically, this term $w_0,_x N_{11,x}$ will exist, but in most practical applications, we take the axial load constant over the circumference or an external uniform pressure. These are not a function and can be 0. But if you say that even over the circumference, they may follow some variation loading then you have to consider those. It is a very generalized case, but in most of the books in which buckling of the shell is studied, only the first term is considered because we consider the axial load constant, it is not a function of x.

When it is not a function of x that is 0. Similarly, the second term $N_{22}\frac{1}{R}\left(w_0,_{\theta}-\frac{Ru_{20}}{R}\right)$ is a function of θ.

It will be: $\frac{N_{22}}{R^2}\left(w_0,_{\theta\theta}-u_{20,\theta}\right)+\left(w_0,_{\theta}-u_{20}\right)\frac{N_{22,\theta}}{R^2}$, the first function as it is differentiation of the second function. And then $N_{22,\theta}$ the circumferential in-plane stress resultant is a function of θ, for the case of buckling we also do not consider this term $\left(w_0,_{\theta}-u_{20}\right)\frac{N_{22,\theta}}{R^2}$.

I am telling you this term because graduate students or postgraduate students when they do it mathematically can open it like this and have these terms, but when we go to a general article or a book, we do not find these terms. Therefore, we must know why we are putting these as 0.

Similarly, N_{12} and N_{21} are same, therefore we have kept it same, and ultimately the contribution of N_{12} are presented here. Q_x and Q_θ can be represented in terms of the moment.

In most of the cases; the following 5 equations are converted into 3 equations:

$$N_{xx,x}+\frac{N_{\theta x,\theta}}{R}+q_1=I_0\ddot{u}_{10}+I_1\ddot{\psi}_1 \quad equation(1)$$

$$\frac{N_{\theta\theta,\theta}}{R}+N_{x\theta,x}+\frac{Q_\theta}{R}+\frac{N_{22}}{R^2}\left(w_0,_{\theta}-u_{20}\right)+\frac{N_{12}}{R}\left(w_0,_{x}\right)+q_2=\left(I_0\ddot{u}_{20}+I_1\ddot{\psi}_2\right) \quad equation(2)$$

$$N_{11}w_0,_{xx}+w_0,_{x}N_{11,x}+\frac{N_{22}}{R^2}\left(w_0,_{\theta\theta}-u_{20,\theta}\right)+\left(w_0,_{\theta}-u_{20}\right)\frac{N_{22,\theta}}{R^2}+\frac{N_{12}}{R}\left(w_0,_{\theta x}-u_{20,x}\right)$$
$$+\left(w_0,_{\theta}-u_{20}\right)\frac{N_{12,x}}{R}+\frac{N_{12}}{R}w_0,_{x\theta}+w_0,_{x}\frac{\widehat{N_{12,\theta}}}{R}-\frac{N_{\theta\theta}}{R}Q_{x,x}+\frac{Q_{\theta,\theta}}{R}-q_3=I_0\ddot{w}_0 \quad equation(3)$$

$$M_{xx,x}+\frac{M_{\theta x,\theta}}{R}-Q_x=I_1\ddot{u}_{10}+I_2\ddot{\psi}_1 \quad equation(4)$$

$$\frac{M_{\theta\theta,\theta}}{R}+M_{x\theta,x}-Q_\theta=I_1\ddot{u}_{20}+I_2\ddot{\psi}_2 \quad equation(5)$$

Because, when you have 5 equations then you have to solve 5 equations altogether. We know that Q_x and Q_θ can be expressed like this. Therefore, you can directly substitute it there. Then, you will have to solve only 3 equations. In this way, it may reduce to the classical shell theory.

In most of the cases though it was a general application, when we go for buckling Q_x

and Q_θ theta are expressed in terms of M_x and M_θ and substitute it into equation (3).

But for the present case; I tried to keep all 5 equations together because when we do so the boundary conditions need to be reduced. For that purpose, I kept all 5 equations. For us, it does not matter whether we solve 3 equations together or 5 equations together.

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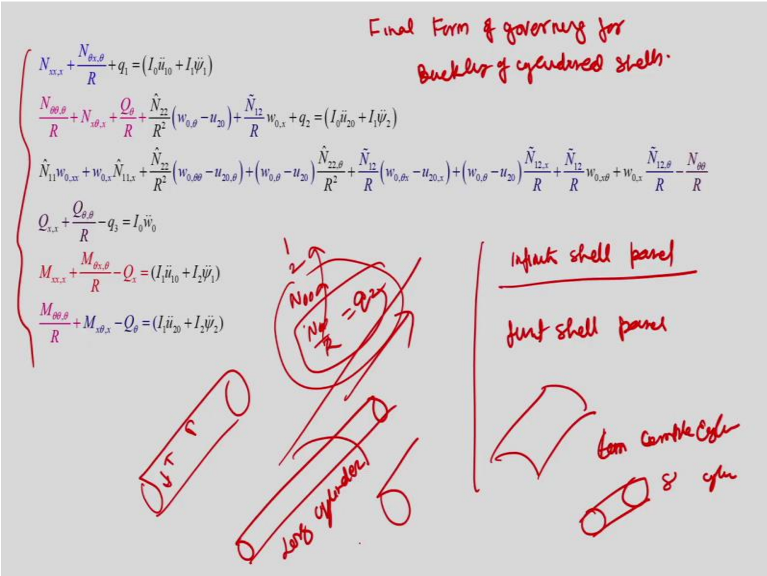

Following are the final form of governing equations for buckling of cylindrical shell:

$$N_{xx,x} + \frac{N_{\theta x,\theta}}{R} + q_1 = I_0 \ddot{u}_{10} + I_1 \ddot{\psi}_1 \quad equation(1)$$

$$\frac{N_{\theta\theta,\theta}}{R} + N_{x\theta,x} + \frac{Q_\theta}{R} + \frac{N_{22}}{R^2}\left(w_{0,\theta} - u_{20}\right) + \frac{N_{12}}{R}\left(w_{0,x}\right) + q_2 = \left(I_0 \ddot{u}_{20} + I_1 \ddot{\psi}_2\right) \quad equation(2)$$

$$N_{11} w_{0,xx} + \frac{N_{22}}{R^2}\left(w_{0,\theta\theta} - u_{20,\theta}\right) + \frac{N_{12}}{R}\left(w_{0,\theta x} - u_{20,x}\right) + \frac{N_{12}}{R} w_{0,x\theta} - \frac{N_{\theta\theta}}{R} + Q_{x,x}$$

$$+ \frac{Q_{\theta,\theta}}{R} - q_3 = I_0 \ddot{w}_0 \quad equation(3)$$

$$M_{xx,x} + \frac{M_{\theta\theta,\theta}}{R} - Q_x = I_1 \ddot{u}_{10} + I_2 \ddot{\psi}_1 \quad equation(4)$$

$$\frac{M_{\theta\theta,\theta}}{R} + M_{x\theta,x} - Q_\theta = I_1 \ddot{u}_{20} + I_2 \ddot{\psi}_2 \quad equation(5)$$

In week- 05, I gave the concept of infinite shell panel and finite shell panel. After this step, these equations may be further reduced to infinite shell panel which means when a shell is very long along x-direction finite shell panel. When a shell is having a shell panel is of finite length or maybe for the case of buckling complete cylindrical shell complete shell is like this.

For that case, there may be two varieties: short cylinders and long cylinders.

These equations are valid, we can convert using further approximation. When we talk about a complete cylinder when we say that it is axis symmetry, in that case, derivative θ will also vanish. Equations will be further simplified.

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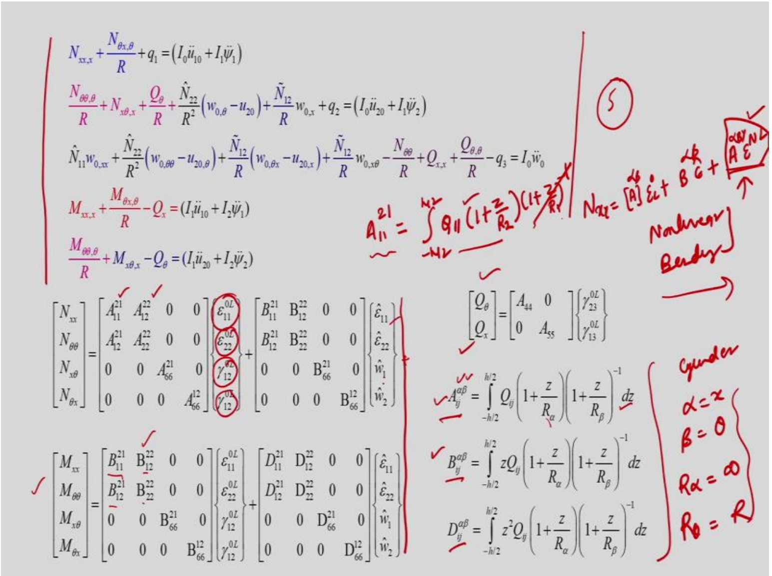

In the case of bending, when I did special cases like a cylinder under internal pressure, I used only 3 equations. From here, $\dfrac{N_{\theta\theta,\theta}}{R} = q_z$ and subjecting back to second and third we can solve the stress resultants finally.

But when we are interested in buckling though it may be a cylinder under external pressure, still we cannot directly use these in the terms of stress resultant, we have to first convert it into a primary displacement form. For that purpose, these 5 equations are important.

When we go for buckling, the non-linear terms in N_{xx} definition and M_{xx} definition are not considered. Initially, $N_{xx} = [A]^{\alpha\beta}\varepsilon_L^0 + [A]^{\alpha\beta}\varepsilon_L^0 + A^{\alpha\beta\gamma}\varepsilon^{NL}$.

This term $A^{\alpha\beta\gamma}\varepsilon^{NL}$ we are not considering.

But if you want to do a non-linear study like load-deflection curve, bifurcation curves, and all these things then we may consider these non-linear terms also. For that purpose, we completely open up to non-linear terms and going to solve non-linear bending vibration in that way we consider all these things and solve them completely.

But for the case of buckling:

$$\begin{bmatrix} N_{xx} \\ N_{\theta\theta} \\ N_{x\theta} \\ N_{\theta x} \end{bmatrix} = \begin{bmatrix} A_{11}^{21} & A_{12}^{22} & o & o \\ A_{12}^{21} & A_{22}^{22} & o & o \\ o & o & A_{66}^{21} & o \\ o & o & o & A_{66}^{12} \end{bmatrix} \begin{bmatrix} \varepsilon_{11}^{0L} \\ \varepsilon_{22}^{0L} \\ \gamma_{12}^{0L} \\ \gamma_{-12}^{0L} \end{bmatrix} + \begin{bmatrix} B_{11}^{21} & B_{12}^{22} & o & o \\ B_{12}^{21} & B_{22}^{22} & o & o \\ o & o & B_{66}^{21} & o \\ o & o & o & B_{66}^{12} \end{bmatrix} \begin{bmatrix} \hat{\varepsilon}_{11} \\ \hat{\varepsilon}_{22} \\ \hat{\omega}_{1} \\ \hat{\omega}_{2} \end{bmatrix}$$

$$\begin{bmatrix} M_{xx} \\ M_{\theta\theta} \\ M_{x\theta} \\ M_{\theta x} \end{bmatrix} = \begin{bmatrix} B_{11}^{21} & B_{12}^{22} & o & o \\ B_{12}^{21} & B_{22}^{22} & o & o \\ o & o & B_{66}^{21} & o \\ o & o & o & B_{66}^{12} \end{bmatrix} \begin{bmatrix} \varepsilon_{11}^{0L} \\ \varepsilon_{22}^{0L} \\ \gamma_{12}^{0L} \\ \gamma_{12}^{0L} \end{bmatrix} + \begin{bmatrix} D_{11}^{21} & D_{12}^{22} & o & o \\ D_{12}^{21} & D_{22}^{22} & o & o \\ o & o & D_{66}^{21} & o \\ o & o & o & D_{66}^{12} \end{bmatrix} \begin{bmatrix} \hat{\varepsilon}_{11} \\ \hat{\varepsilon}_{22} \\ \hat{\omega}_{1} \\ \hat{\omega}_{2} \end{bmatrix}$$

$$\begin{bmatrix} Q_{\theta} \\ Q_{x} \end{bmatrix} = \begin{bmatrix} A_{44} & o \\ o & A_{55} \end{bmatrix} \begin{bmatrix} \gamma_{23}^{0} \\ \gamma_{13}^{0} \end{bmatrix}$$

The definition of $A_{ij}^{\alpha\beta} = \int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{ij} \left(1+\frac{\varsigma}{R_\alpha}\right)\left(1+\frac{\varsigma}{R_\beta}\right)^{-1} d\varsigma$

$$B_{ij}^{\alpha\beta} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \varsigma Q_{ij} \left(1+\frac{\varsigma}{R_\alpha}\right)\left(1+\frac{\varsigma}{R_\beta}\right)^{-1} d\varsigma$$

$$D_{ij}^{\alpha\beta} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \varsigma^2 Q_{ij} \left(1+\frac{\varsigma}{R_\alpha}\right)\left(1+\frac{\varsigma}{R_\beta}\right)^{-1} d\varsigma .$$

For the case of a cylinder α = x, $\beta = \theta$, $R_\alpha = \infty$, and $R_\beta =$ R.

We can say:

$$A_{11}^{21} = \int_{\frac{-h}{2}}^{\frac{h}{2}} Q_{11} \left(1+\frac{\varsigma}{R_2}\right)\left(1+\frac{\varsigma}{R_1}\right)^{-1} d\varsigma .$$

$R_1 = \infty$, $\left(1+\frac{\varsigma}{R_1}\right)^{-1}$ will not exist and $\left(1+\frac{\varsigma}{R_2}\right)$ will exist.

In lecture 05, I explicitly did more terms A_{11}, A_{12}, A_{13}, but in a combined form, we expressed in the lecture-04, we can write A_{ij} and finally, evaluate a number and put it there or in some expression form. It is a very general expression and a very useful expression for making the program.

We have written all the indexes properly, putting the proper α β you can find these numbers B_{11}^{21} B_{12}^{22}, and so on, it will be a complete form. The fewer numbers of unknowns will be there in the final governing equation.

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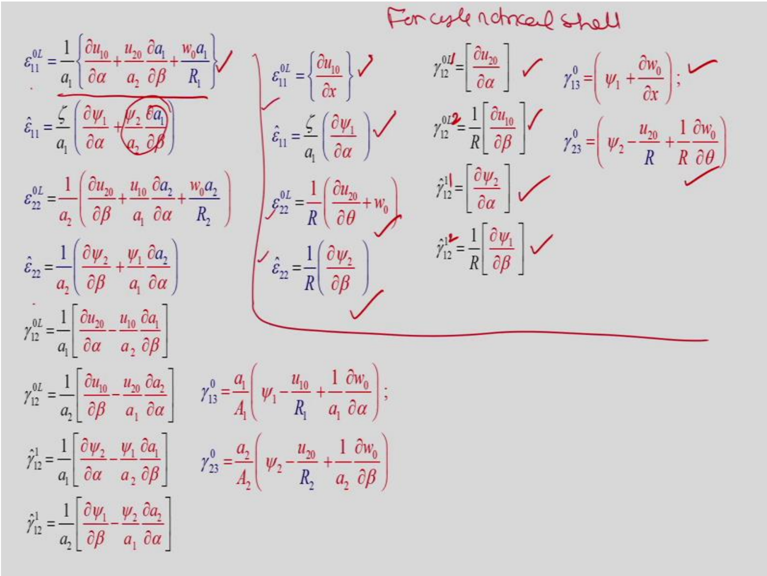

If you use this concept and substitute it into the above governing equations, before going there we must know what are ε_{11}^{0L} , $\hat{\varepsilon}_{11}$, ε_{22}^{0L} , $\hat{\varepsilon}_{22}$ and so on for the present case.

$\varepsilon_{11}^{0L} = \dfrac{1}{a_1}\left(\dfrac{\partial u_{10}}{\partial \alpha} + \dfrac{u_{20}}{a_2}\dfrac{\partial a_1}{\partial \beta} + \dfrac{w_0 a_1}{R_1}\right)$ for a general shell. But for the present case cylindrical shell is reduced to $\left\{\dfrac{\partial u_{10}}{\partial \alpha}\right\}$.

Similarly, $\hat{\varepsilon}_{11} = \dfrac{\varsigma}{a_1}\left(\dfrac{\partial \psi_1}{\partial \alpha} + \dfrac{\psi_2}{a_2}\dfrac{\partial a_1}{\partial \beta}\right)$, now it is $\left\{\dfrac{\partial \psi_1}{\partial \alpha}\right\}$.

For the present case: $\varepsilon_{22}^{0L} = \dfrac{1}{R}\left(\dfrac{\partial u_{20}}{\partial \theta} + w_0\right)$

$$\hat{\varepsilon}_{22} = \frac{1}{R}\left(\frac{\partial \psi_2}{\partial \beta}\right)$$

$$\gamma_{12}^{011} = \left(\frac{\partial u_{20}}{\partial \alpha}\right)$$

$$\gamma_{12}^{012} = \frac{1}{R}\left(\frac{\partial u_{10}}{\partial \beta}\right)$$

$$\gamma_{12}^{-11} = \left\{\frac{\partial \psi_2}{\partial \alpha}\right\}$$

$$\gamma_{12}^{-12} = \frac{1}{R}\left\{\frac{\partial \psi_1}{\partial \beta}\right\}$$

$$\gamma_{13}^{0} = \left(\psi_1 + \frac{\partial w_0}{\partial x}\right)$$

$$\gamma_{23}^{0} = \left(\psi_2 - \frac{u_{20}}{R_2} + \frac{1}{R}\frac{\partial w_0}{\partial \theta}\right).$$

If somebody is giving a write-up for a cylindrical shell the first step is to give the strain displacement relations for a cylindrical shell.

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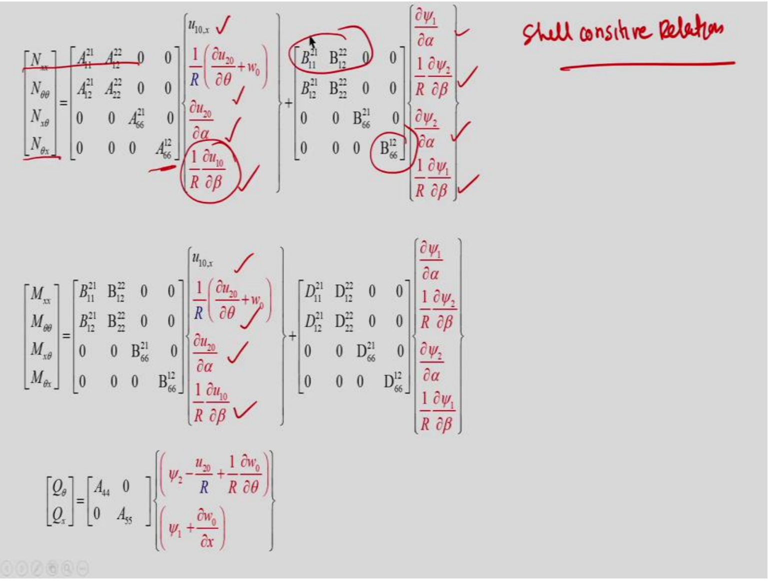

Writing ε_{11}^{0L} in terms of these things, that is the most appropriate form for working. If we write the shell constitutive relations like this:

$$\begin{bmatrix} N_{xx} \\ N_{\theta\theta} \\ N_{x\theta} \\ N_{\theta x} \end{bmatrix} = \begin{bmatrix} A_{11}^{21} & A_{12}^{22} & o & o \\ A_{12}^{21} & A_{22}^{22} & o & o \\ o & o & A_{66}^{21} & o \\ o & o & o & A_{66}^{12} \end{bmatrix} \begin{bmatrix} u_{10,x} \\ \frac{1}{R}\left(\frac{\partial u_{20}}{\partial \theta} + w_0\right) \\ \frac{\partial u_{20}}{\partial \alpha} \\ \frac{1}{R}\left(\frac{\partial u_{10}}{\partial \beta}\right) \end{bmatrix} + \begin{bmatrix} B_{11}^{21} & B_{12}^{22} & o & o \\ B_{12}^{21} & B_{22}^{22} & o & o \\ o & o & B_{66}^{21} & o \\ o & o & o & B_{66}^{12} \end{bmatrix} \begin{bmatrix} \frac{\partial \psi_1}{\partial \alpha} \\ \frac{1}{R}\left(\frac{\partial \psi_2}{\partial \beta}\right) \\ \frac{\partial \psi_2}{\partial \alpha} \\ \frac{1}{R}\left\{\frac{\partial \psi_1}{\partial \beta}\right\} \end{bmatrix}$$

$$\begin{bmatrix} M_{xx} \\ M_{\theta\theta} \\ M_{x\theta} \\ M_{\theta x} \end{bmatrix} = \begin{bmatrix} B_{11}^{21} & B_{12}^{22} & o & o \\ B_{12}^{21} & B_{22}^{22} & o & o \\ o & o & B_{66}^{21} & o \\ o & o & o & B_{66}^{12} \end{bmatrix} \begin{bmatrix} u_{10,x} \\ \frac{1}{R}\left(\frac{\partial u_{20}}{\partial \theta} + w_0\right) \\ \frac{\partial u_{20}}{\partial \alpha} \\ \frac{1}{R}\left(\frac{\partial u_{10}}{\partial \beta}\right) \end{bmatrix} + \begin{bmatrix} D_{11}^{21} & D_{12}^{22} & o & o \\ D_{12}^{21} & D_{22}^{22} & o & o \\ o & o & D_{66}^{21} & o \\ o & o & o & D_{66}^{12} \end{bmatrix} \begin{bmatrix} \frac{\partial \psi_1}{\partial \alpha} \\ \frac{1}{R}\left(\frac{\partial \psi_2}{\partial \beta}\right) \\ \frac{\partial \psi_2}{\partial \alpha} \\ \frac{1}{R}\left\{\frac{\partial \psi_1}{\partial \beta}\right\} \end{bmatrix}$$

$$\begin{bmatrix} Q_{\theta} \\ Q_{x} \end{bmatrix} = \begin{bmatrix} A_{44} & o \\ o & A_{55} \end{bmatrix} \begin{bmatrix} \left(\psi_2 - \frac{u_{20}}{R_2} + \frac{1}{R}\frac{\partial w_0}{\partial \theta}\right) \\ \left(\psi_1 + \frac{\partial w_0}{\partial x}\right) \end{bmatrix}$$

If we write in terms of ε_{11}^{0L}, then we have to convert it into a displacement form and use that. This is the most convenient form of shell constitutive relations.

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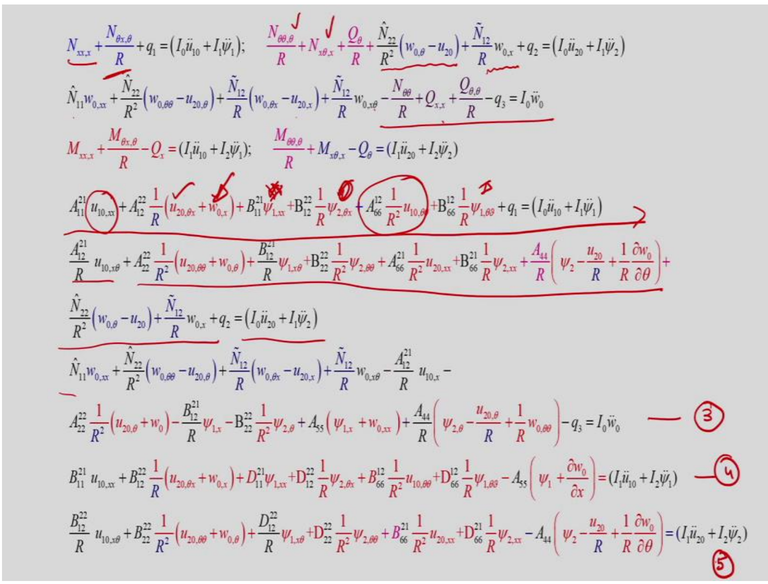

In the first equation:

$N_{xx,x} + \frac{N_{\theta x,\theta}}{R} + q_1 = I_0 \ddot{u}_{10} + I_1 \ddot{\psi}_1$, we can write that in terms of

$$A_{11}^{21} u_{10,xx} + A_{12}^{22} \frac{1}{R}\left(u_{20,\theta x} + w_{0,x}\right) + B_{11}^{21}\psi_{1,xx} + B_{12}^{22}\frac{1}{R}\psi_{2,\theta x} + A_{66}^{12}\frac{1}{R^2}u_{10,\theta\theta}$$
$$+B_{66}^{12}\frac{1}{R}\psi_{1,\theta\theta} + q_1 = \left(I_0\ddot{u}_{10} + I_1\ddot{\psi}_1\right)$$

In this way, the first equation is represented in terms of primary displacement variables.

Same way in the second equation:

$$\frac{N_{\theta\theta,\theta}}{R} + N_{x\theta,x} + \frac{Q_\theta}{R} + \frac{N_{22}}{R^2}\left(w_0,_\theta - u_{20}\right) + \frac{N_{12}}{R}\left(w_0,_x\right) + q_2 = \left(I_0\ddot{u}_{20} + I_1\ddot{\psi}_2\right)$$

If we substitute using the shell constitutive relations it becomes like this:

$$\frac{A_{12}^{21}}{R}u_{10,x\theta} + A_{22}^{22}\frac{1}{R^2}\left(u_{20,\theta\theta} + w_{0,\theta}\right) + \frac{B_{12}^{21}}{R}\psi_{1,x\theta} + B_{22}^{22}\frac{1}{R^2}\psi_{2,\theta\theta} + A_{66}^{21}\frac{1}{R^2}u_{20,xx} + B_{66}^{21}\frac{1}{R}\psi_{2,xx}$$

$$+\frac{A_{44}}{R}\left(\psi_2 - \frac{u_{20}}{R} + \frac{1}{R}\frac{\partial w_0}{\partial\theta}\right) + \frac{N_{22}}{R^2}\left(w_{0,\theta} - u_{20}\right) + \frac{N_{12}}{R}w_{0,x} + q_2 = \left(I_0\ddot{u}_{20} + I_1\ddot{\psi}_2\right).$$

Then, the third equation will be:

$$N_{11}w_{0,xx} + \frac{N_{22}}{R^2}\left(w_0,_{\theta\theta} - u_{20,\theta}\right) + \frac{N_{12}}{R}\left(w_0,_{\theta x} - u_{20,x}\right) + \frac{N_{12}}{R}w_0,_{x\theta} - \frac{A_{12}^{21}}{R}u_{10,x}$$
$$-A_{22}^{22}\frac{1}{R^2}\left(u_{20,\theta\theta} + w_{0,\theta}\right) + \frac{B_{12}^{21}}{R}\psi_{1,x} - B_{22}^{22}\frac{1}{R^2}\psi_{2,\theta} + A_{55}\left(\psi_{1,x} + w_{0,xx}\right)$$
$$+\frac{A_{44}}{R}\left(\psi_{2,\theta} - \frac{u_{20,\theta}}{R} + \frac{1}{R}w_0,_{\theta\theta}\right) - q_3 = I_0\ddot{w}_0$$

The fourth equation will be:

$$B_{11}^{21}u_{10,xx} + B_{12}^{22}\frac{1}{R}\left(u_{20,\theta x} + w_{0,x}\right) + D_{11}^{21}\psi_{1,xx} + D_{12}^{22}\frac{1}{R}\psi_{2,\theta x} + B_{66}^{12}\frac{1}{R^2}u_{10,\theta\theta} + D_{66}^{12}\frac{1}{R}\psi_{1,\theta\theta}$$
$$-A_{55}\left(\psi_1 + \frac{\partial w_0}{\partial x}\right) = \left(I_1\ddot{u}_{10} + I_2\ddot{\psi}_1\right)$$

And the fifth equation will be:

$$\frac{B_{12}^{22}}{R}u_{10,x\theta} + B_{22}^{22}\frac{1}{R^2}\left(u_{20,\theta\theta} + w_{0,\theta}\right) + \frac{D_{12}^{22}}{R}\psi_{1,x\theta} + D_{22}^{22}\frac{1}{R^2}\psi_{2,\theta\theta} + B_{66}^{21}\frac{1}{R^2}u_{20,xx} + D_{66}^{21}\frac{1}{R}\psi_{2,xx}$$
$$-\frac{A_{44}}{R}\left(\psi_2 - \frac{u_{20}}{R} + \frac{1}{R}\frac{\partial w_0}{\partial\theta}\right) = \left(I_1\ddot{u}_{20} + I_2\ddot{\psi}_2\right)$$

Now, you see we can write that in a matrix form so that working with these terms will be easy and even the coding will be easy.

We are trying to do it in such a way that arranging the matrix like coefficient of u_{10}, the coefficient of u_{20}, the coefficient of w_0, the coefficient of ψ_1, and the coefficient of ψ_2. If we do so, we can arrange in a matrix form.

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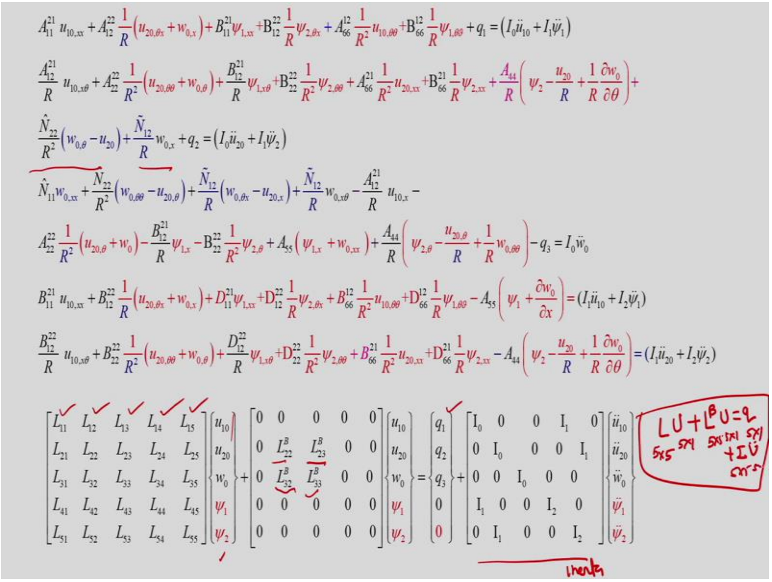

This matrix form is:

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} & L_{15} \\ L_{21} & L_{22} & L_{23} & L_{24} & L_{25} \\ L_{31} & L_{32} & L_{33} & L_{34} & L_{35} \\ L_{41} & L_{42} & L_{43} & L_{44} & L_{45} \\ L_{51} & L_{52} & L_{53} & L_{54} & L_{55} \end{bmatrix} \begin{bmatrix} u_{10} \\ u_{20} \\ w_0 \\ \psi_1 \\ \psi_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & L_{22}^B & L_{23}^B & 0 & 0 \\ 0 & L_{32}^B & L_{33}^B & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{10} \\ u_{20} \\ w_0 \\ \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} I_0 & 0 & 0 & I_1 & 0 \\ 0 & I_0 & 0 & 0 & I_1 \\ 0 & 0 & I_0 & 0 & 0 \\ I_1 & 0 & 0 & I_2 & 0 \\ 0 & I_1 & 0 & 0 & I_2 \end{bmatrix} \begin{bmatrix} \ddot{u}_{10} \\ \ddot{u}_{20} \\ \ddot{w}_0 \\ \ddot{\psi}_1 \\ \ddot{\psi}_2 \end{bmatrix}$$

These are the primary variables and these are the terms related to our non-linear buckling. For the cylindrical shell case, only the second and third equation contains terms related to buckling and these are the loading and this is our inertia matrix.

We can write:

$$LU + L^B U = q + I\ddot{U}$$

This is our general equation where L is a 5 by 5 matrix, U is 5 by 1 matrix, L^B is 5 by 5 matrix, U is 5 by 1 matrix, q is 5 by 1 matrix, I is 5 by 5 matrix, and $\ddot{U}$ is 5 by 1 matrix.

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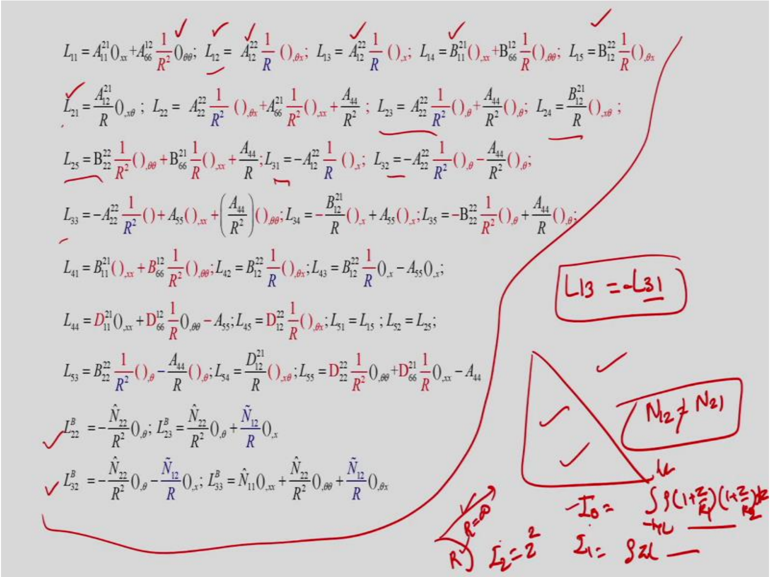

Now, what is L_{11} , L_{22}, and so on? Explicitly, these can be written as:

$$L_{11} = A_{11}^{21}(\ ),_{xx} + A_{66}^{12}\frac{1}{R^2}(\ ),_{\theta\theta};\ L_{12} = A_{12}^{22}\frac{1}{R}(\ ),_{\theta x};\ L_{13} = A_{12}^{22}\frac{1}{R}(\ ),_{x};\ L_{14} = B_{11}^{21}(\ ),_{xx} + B_{66}^{12}\frac{1}{R}(\ ),_{\theta\theta};$$

$$L_{15} = B_{12}^{22}\frac{1}{R}(\ ),_{\theta x};\ \ L_{21} = \frac{A_{11}^{21}}{R}(\ ),_{x\theta};\ L_{22} = A_{22}^{22}\frac{1}{R^2}(\ ),_{\theta x} + A_{66}^{21}\frac{1}{R^2}(\ ),_{xx} + \frac{A_{44}}{R^2};$$

$$L_{23} = A_{22}^{22}\frac{1}{R^2}(\ ),_{\theta} + \frac{A_{44}}{R^2}(\ ),_{\theta};\ L_{24} = \frac{B_{12}^{21}}{R}(\ ),_{x\theta};\ L_{25} = B_{22}^{22}\frac{1}{R^2}(\ ),_{\theta\theta} + B_{66}^{21}\frac{1}{R}(\ ),_{xx} + \frac{A_{44}}{R};$$

$$L_{31} = -A_{12}^{22}\frac{1}{R}(\ ),_{x};\ L_{32} = -A_{22}^{22}\frac{1}{R^2}(\ ),_{\theta} - \frac{A_{44}}{R^2}(\ ),_{\theta};\ L_{33} = -A_{22}^{22}\frac{1}{R^2}(\ ) + A_{55}(\ ),_{xx} + \frac{A_{44}}{R^2}(\ ),_{\theta\theta};$$

$$L_{34} = -\frac{B_{12}^{21}}{R}(\ ),_{x} + A_{55}(\ ),_{x};\ L_{35} = -B_{22}^{22}\frac{1}{R^2}(\ ),_{\theta} + \frac{A_{44}}{R}(\ ),_{\theta x};\ L_{41} = B_{11}^{21}(\ ),_{xx} + B_{66}^{12}\frac{1}{R^2}(\ ),_{\theta\theta};$$

$$L_{42} = B_{12}^{22}\frac{1}{R}(\ ),_{\theta x};\ \ L_{43} = B_{12}^{22}\frac{1}{R}(\ ),_{x} - A_{55}(\ ),_{x};\ \ L_{44} = D_{11}^{21}(\ ),_{xx} + D_{66}^{12}\frac{1}{R}(\ ),_{\theta\theta} - A_{55};$$

$$L_{45} = D_{12}^{22}\frac{1}{R}(\ ),_{\theta x};\ \ L_{51} = L_{15};\ L_{52} = L_{25};\ L_{53} = B_{22}^{22}\frac{1}{R^2}(\ ),_{\theta} - \frac{A_{44}}{R}(\ ),_{\theta};\ L_{54} = \frac{D_{12}^{21}}{R}(\ ),_{x\theta}$$

$$L_{55} = D_{22}^{22}\frac{1}{R^2}(\ ),_{\theta\theta} + D_{66}^{21}\frac{1}{R}(\ ),_{xx} - A_{44};\ L_{22}^{B} = -\frac{\hat{N}_{22}}{R^2}(\ ),_{\theta};\ L_{23}^{B} = \frac{\hat{N}_{22}}{R^2}(\ ),_{\theta} + \frac{\hat{N}_{12}}{R}(\ ),_{x}$$

$$L_{32}^{B} = -\frac{\hat{N}_{22}}{R^2}(\ ),_{\theta} + \frac{\hat{N}_{12}}{R}(\ ),_{x};\ L_{33}^{B} = \hat{N}_{11}(\ ),_{xx} + \frac{\hat{N}_{22}}{R^2}(\ ),_{\theta\theta} + \frac{\hat{N}_{12}}{R}(\ ),_{x}$$

Generally, these terms may be symmetric that $L_{12} = L_{21}$ or $L_{13} = L_{31}$, but for the present case for a cylindrical shell, it may be the same or with some minus sign. That is why we have written all the terms so that there will be no confusion that whether it is all symmetric or non-symmetric, but for the case of a plate there this matrix is completely symmetric there will be no ambiguity. Here the reason behind non-symmetry is that in one direction $R_1 = \infty$ and in another direction R_2 = R.

Due to that, there will be some mismatch like we can say that $N_{12} \neq N_{21}$, if these are not equal then $L_{21} \neq L_{12}$. This is a very important concept for developers that first find out these L_{11}, L_{12}, buckling and the inertia matrix where we know that:

$$I_0 = \int_{\frac{-h}{2}}^{\frac{h}{2}} \rho \left(1+\frac{\varsigma}{R_1}\right)\left(1+\frac{\varsigma}{R_2}\right) d\varsigma$$

$$I_1 \text{ is } \int_{\frac{-h}{2}}^{\frac{h}{2}} \rho\varsigma \left(1+\frac{\varsigma}{R_1}\right)\left(1+\frac{\varsigma}{R_2}\right) d\varsigma$$

$$I_2 \text{ is } \int_{\frac{-h}{2}}^{\frac{h}{2}} \rho\varsigma^2 \left(1+\frac{\varsigma}{R_1}\right)\left(1+\frac{\varsigma}{R_2}\right) d\varsigma .$$

In this way, we can find all the components and we can put them here.

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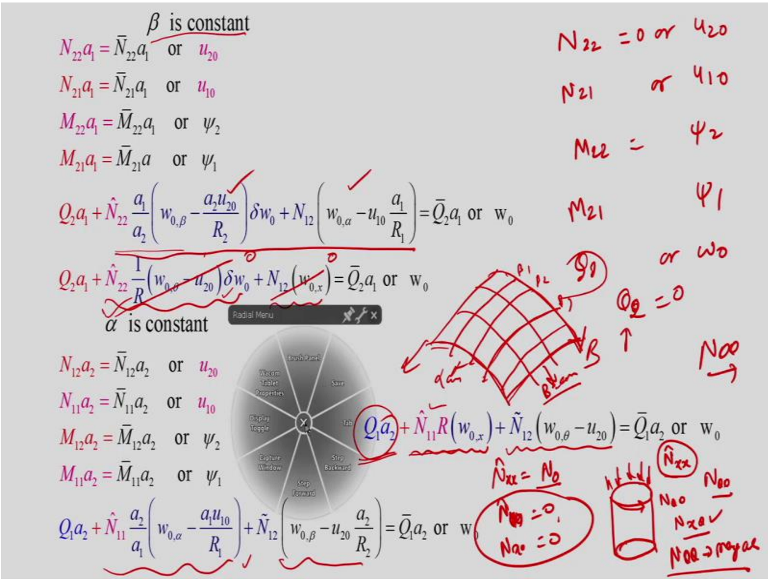

Now, what are the boundary conditions? Previously, when we considered a linear part, we neglected all this. Even in the boundary there are some non-linear terms exist you can ignore these things because when you go for a linear one then you can say:

Either N_{22} = 0 or u_{20} = 0;

Either N_{21} = 0 or u_{10} = 0;

Either M_{22} = 0 or ψ_2 = 0;

Either M_{21} = 0 or ψ_1 = 0;

Either Q_2 = 0 or w_0 = 0.

But for the case of nonlinear terms, whenever we are going to satisfy the free condition of a shell then it will not just Q_θ or Q_x = 0, we need to satisfy some more terms also. For that case, an edge where β is constant, if you say that these are β and α. The edge β is increasing.

The lines $\beta_1, \beta_2, \beta_3$ are corresponding to β constant. The edge where β is constant following variables need to be specified:

$$N_{22}a_1 = \bar{N}_{22}a_1 \quad or \quad u_{20}$$
$$N_{21}a_1 = \bar{N}_{22}a_1 \quad or \quad u_{10}$$
$$M_{22}a_1 = \bar{M}_{22}a_1 \quad or \quad \psi_2$$
$$M_{21}a_1 = \bar{M}_{21}a_1 \quad or \quad \psi_1$$
$$Q_2a_1 + N_{22}\frac{a_1}{a_2}\left(w_0,_\beta - \frac{a_2u_{20}}{R_2}\right)\partial w_0 + N_{12}\left(w_{0,\alpha} - u_{10}\frac{a_1}{R_1}\right)\partial w_0 = \bar{Q}_2a_1 \quad or \quad w_0$$

And for the present case the corresponding terms that N_{22} and N_{11} need to be considered.

The most important part is in most of the buckling cases, when we say that the cylindrical shell is subjected to the axial load that is corresponding to N_{xx} only, there will be no $N_{\theta\theta}$ and no circumferential loading. In that case, these terms may not come into the picture.

But if you say that this $N_{\theta\theta}$ may also be there, then N_θ, $N_{x\theta}$ are also combined, then you have to consider these terms. When you see the buckling papers or books you do not find these non-linear terms because they have assumed that shell or cylindrical shell is subjected to only the initial axial stress.

N_{xx} can be N_0, but $N_{\theta\theta}$ and $N_{x\theta}$ = 0. Due to that reason these terms are not contributing. But in actual when you say that there may be some circumferential stress, if you talk about a cylinder under external pressure or internal pressure in that case $N_{\theta\theta}$ may act.

For that case, there may be loading due to the external pressure or internal pressure, then the circumferential stress may exist, and ultimately, this $N_{\theta\theta}$ causes the bending into the shell.

Though the shell is subjected to external pressure, that external pressure is causing a $N_{\theta\theta}$ and due to that buckling may take place. Similarly, at an edge where α is constant, some more terms may come up:

$$N_{12}a_2 = \bar{N}_{12}a_2 \quad or \quad u_{20}$$
$$N_{11}a_2 = \bar{N}_{11}a_2 \quad or \quad u_{10}$$
$$M_{12}a_2 = \bar{M}_{12}a_2 \quad or \quad \psi_2$$
$$M_{11}a_2 = \bar{M}_{11}a_2 \quad or \quad \psi_1$$
$$Q_1a_2 + N_{11}\frac{a_2}{a_1}\left(w_0,_\alpha - \frac{a_1u_{10}}{R_1}\right) + N_{12}\left(w_{0,\beta} - u_{20}\frac{a_2}{R_2}\right)\partial w_0 = \bar{Q}_1a_2 \quad or \quad w_0$$

If we consider our present case, this term
$Q_1a_2 + N_{11}R\left(w_{0,x}\right) + N_{12}\left(w_{0,\theta} - u_{20}\right) = \bar{Q}_1a_2 \quad or \quad w_0$ will exist.

Along that edge x = 0 and x = a or α , these terms need to be satisfied + whatever initial stress you have. In this way, we have to see clearly that what we are going to be satisfied.

Boundary conditions need to be modified and need to consider the non-linear terms when we are going to study a buckling. Even the concept of buckling is done by the linearized buckling and non-linear buckling. For the present case, we are considering non-linear terms and Von Karman non-linearity is considered and the buckling effect is studied.

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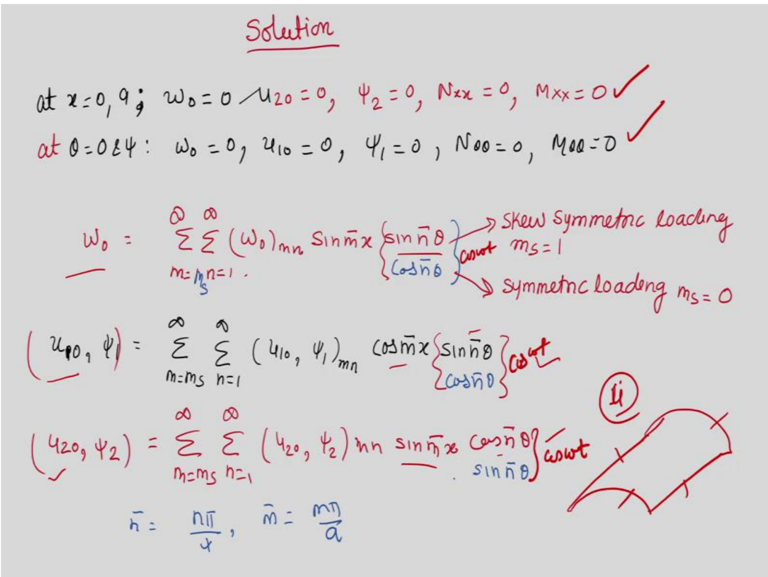

Now, the very general solution, if we talk about a finite cylindrical shell, when it is a closed boundary then only one edge x = 0 or x = L maybe there or there will be no edge, for that case, the solution is written. But when we talk about a finite shell panel when we have both edges free:

$$at\ x = 0, a:$$
$$w_0 = 0;\ u_{20} = 0;\ \psi_2 = 0; N_{xx} = 0;\ \ M_{xx} = 0$$
$$at\ \theta = 0, \alpha:$$
$$w_0 = 0;\ u_{10} = 0;\ \psi_1 = 0; N_{\theta\theta} = 0;\ \ M_{\theta\theta} = 0$$

If that is the case the cylindrical finite shell panel is subjected to all simply supported boundary conditions and for that case, the deflection is expressed into double Fourier sin series and in-plane displacement in cosine and sine series and u_{20} is sine and cosine series. In this way these are expressed:

$$w_0 = \sum_{m=ms}^{\infty}\sum_{n=1}^{\infty}(w_0)_{mn}\sin\bar{m}x\begin{Bmatrix}\sin\bar{n}\theta\\ \cos\bar{n}\theta\end{Bmatrix}\cos\omega t^2$$
$$(u_{10},\psi_1) = \sum_{m=ms}^{\infty}\sum_{n=1}^{\infty}(u_{10},\psi_1)_{mn}\cos\bar{m}x\begin{Bmatrix}\sin\bar{n}\theta\\ \cos\bar{n}\theta\end{Bmatrix}\cos\omega t$$
$$(u_{20},\psi_2) = \sum_{m=ms}^{\infty}\sum_{n=1}^{\infty}(u_{20},\psi_2)_{mn}\sin\bar{m}x\begin{Bmatrix}\cos\bar{n}\theta\\ \sin\bar{n}\theta\end{Bmatrix}\cos\omega t$$

Again, we have already taken the concept of time derivative. We can say that $\cos\omega t$ will also be there, the time will vary along $\cos\omega t$. If you substitute this expression into the previous equations L_{11}, L_{12}, L_{13}, these become a new constant and these constants are known as K_{11}, K_{12}, K_{13}.

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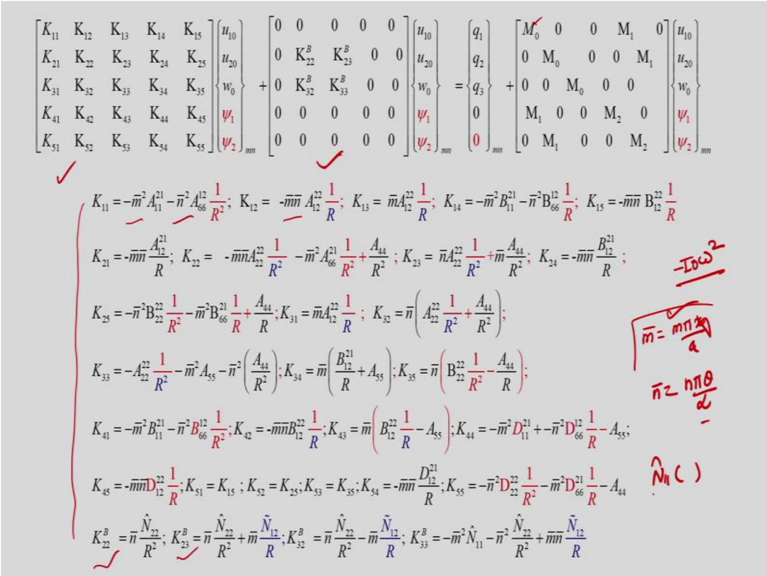

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} \end{bmatrix} \begin{bmatrix} u_{10} \\ u_{20} \\ w_0 \\ \psi_1 \\ \psi_2 \end{bmatrix}_{mn} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & K_{22}^B & K_{23}^B & 0 & 0 \\ 0 & K_{32}^B & K_{33}^B & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{10} \\ u_{20} \\ w_0 \\ \psi_1 \\ \psi_2 \end{bmatrix}_{mn} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ 0 \\ 0 \end{bmatrix}_{mn} +$$

$$\begin{bmatrix} M_0 & 0 & 0 & M_1 & 0 \\ 0 & M_0 & 0 & 0 & M_1 \\ 0 & 0 & M_0 & 0 & 0 \\ M_1 & 0 & 0 & M_2 & 0 \\ 0 & M_1 & 0 & 0 & M_2 \end{bmatrix} \begin{bmatrix} u_{10} \\ u_{20} \\ w_0 \\ \psi_1 \\ \psi_2 \end{bmatrix}_{mn}$$

These may contain mn. $\bar{m}$ is $m\pi$ by a and $\bar{n}$ is $n\pi$ by ψ. Now, this K is a constant matrix and expressed like this:

$$K_{11} = -\bar{m}^2 A_{11}^{21} - \bar{n}^2 A_{66}^{12} \frac{1}{R^2};\ K_{12} = -\bar{m}\bar{n} A_{12}^{22} \frac{1}{R};\ K_{13} = \bar{m} A_{12}^{22} \frac{1}{R};\ K_{14} = -\bar{m}^2 B_{11}^{21} - \bar{n}^2 B_{66}^{12} \frac{1}{R};$$

$$K_{15} = -\bar{m}\bar{n} B_{12}^{22} \frac{1}{R};\ \ K_{21} = -\bar{m}\bar{n} \frac{A_{11}^{21}}{R};\ K_{22} = -\bar{m}\bar{n} A_{22}^{22} \frac{1}{R^2} - \bar{m}^2 A_{66}^{21} \frac{1}{R^2} + \frac{A_{44}}{R^2};\ K_{23} = \bar{n} A_{22}^{22} \frac{1}{R^2} + \bar{m} \frac{A_{44}}{R^2};$$

$$K_{24} = -\bar{m}\bar{n} \frac{B_{12}^{21}}{R};\ K_{25} = -\bar{n}^2 B_{22}^{22} \frac{1}{R^2} - \bar{m}^2 B_{66}^{21} \frac{1}{R} + \frac{A_{44}}{R};\ K_{31} = \bar{m} A_{12}^{22} \frac{1}{R};\ K_{32} = \bar{n}\left(A_{22}^{22} \frac{1}{R^2} + \frac{A_{44}}{R^2} \right);$$

$$K_{33} = -A_{22}^{22} \frac{1}{R^2} - \bar{m}^2 A_{55} - \bar{n}^2 \left(\frac{A_{44}}{R^2} \right);\ K_{34} = \bar{m}\left(\frac{B_{12}^{21}}{R} + A_{55} \right);\ K_{35} = \bar{n}\left(B_{22}^{22} \frac{1}{R^2} + \frac{A_{44}}{R} \right);$$

$$K_{41} = -\bar{m}^2 B_{11}^{21} - \bar{n}^2 B_{66}^{12} \frac{1}{R^2};\ K_{42} = -\bar{m}\bar{n} B_{12}^{22} \frac{1}{R};\ K_{43} = \bar{m}\left(B_{12}^{22} \frac{1}{R} - A_{55} \right);$$

$$K_{44} = -\bar{m}^2 D_{11}^{21} - \bar{n}^2 D_{66}^{12} \frac{1}{R} - A_{55};\ K_{45} = -\bar{m}\bar{n} D_{12}^{22} \frac{1}{R};\ \ K_{51} = K_{15};\ K_{52} = K_{25};\ K_{53} = K_{35};$$

$$K_{54} = -\bar{m}\bar{n} \frac{D_{12}^{21}}{R};\ K_{55} = -\bar{n}^2 D_{22}^{22} \frac{1}{R^2} - \bar{m}^2 D_{66}^{21} \frac{1}{R} - A_{44};\ K_{22}^{B} = \bar{n} \frac{\hat{N}_{22}}{R^2};\ K_{23}^{B} = \bar{n} \frac{\hat{N}_{22}}{R^2} + \bar{m} \frac{\hat{N}_{12}}{R};$$

$$K_{32}^{B} = \bar{n} \frac{\hat{N}_{22}}{R^2} - \bar{m} \frac{\hat{N}_{12}}{R};\ K_{33}^{B} = -\bar{m}^2 \hat{N}_{11} - \bar{n}^2 \frac{\hat{N}_{22}}{R^2} + \bar{m}\bar{n} \frac{\hat{N}_{12}}{R}$$

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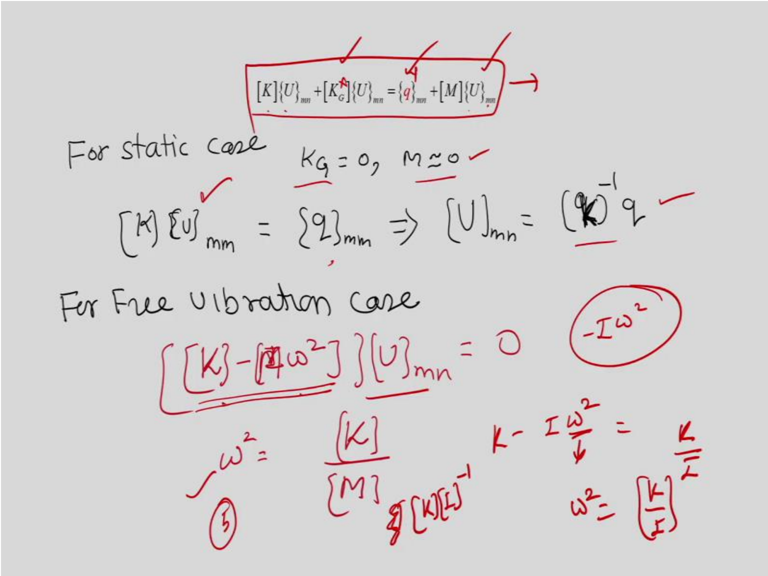

We can finally write the set of equations like this:
$[K][U]_{mn} + [K_G]\{U\}_{mn} = q_{mn} + [M]\{U\}_{mn}$ and this is a complete governing equation.

It contains mechanical loading, inertia terms, buckling loading. For the case of static, buckling and inertia terms are neglected.

Therefore, this equation reduces to $[K][U]_{mn} = q_{mn}$ and $\{U\}_{mn} = (K)^{-1} q$.

If we want to study a free vibration case of a shell then buckling will be 0, q = 0, and K U = M U. And M = $-I\omega^2$.

If we put this side, we can say that $\{U\}_{mn} \neq 0$, so this $\left[[K]-[I]\omega^2\right]\{U\}_{mn} = 0$.

ω is unknown to you, you do not know the natural frequency. $\frac{K}{I\omega^2}$ can be found or if it is a single one $\frac{K}{I}$ otherwise you write KI^{-1}. Five frequencies will be known for a simply supported cylindrical panel.

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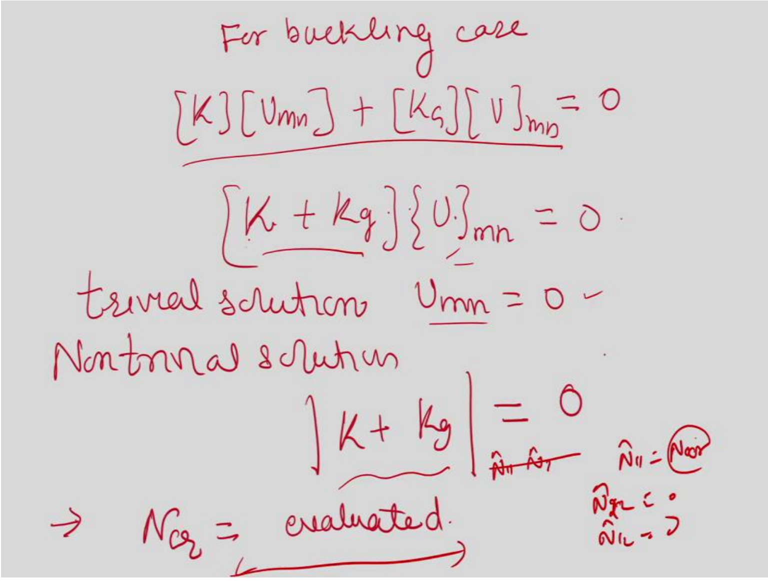

Now, for the case of buckling: q is equal to 0, and inertia terms are also considered 0. terms are considered 0. Now, our equation is reduced to $[K][U]_{mn} + [K_G]\{U\}_{mn}$, where K_G contains $\hat{N}_{11}$, $\hat{N}_{22}$, $\tilde{N}_{12}$, we have 3 terms.

If you put so, $[K + K_g]\{U\}_{mn}$ = 0, then the trivial solution $[U]_{mn}$ = 0, and the non-trivial solution $|K + K_g|$ = 0.

When we put this determinant as 0, it contains $\hat{N}_{11}$, $\hat{N}_{22}$, depending upon the case, we solve one case at a time. Let us say we consider a cylindrical shell is subjected to axial loading only, for that case $\hat{N}_{11}$ = N_0 and all $\hat{N}_{22}$, and $\tilde{N}_{12}$ = 0. Ultimately, a characteristic equation comes up and from there we can find N critical loading for the present case.

In the next lecture, I shall discuss buckling of shells for different cases under axial load, external pressure, thermal load, and under combined effect.

Thank you very much.